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WEAK NONCOMPACTNESS IN BANACH SEQUENCE SPACES AND ITS EXTRAPOLATION PROPERTIES

ANDRZEJ KRYCZKA

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Abstract. Explicit formulae in selected Banach sequence spaces are established for the measure of weak noncompactness based on James' criteria. Estimates of the deviation from weak compactness are given for bounded linear operators extrapolated by the Jawerth-Milman Σ_p and Δ_p methods for $1 < p < \infty$.

1. Introduction

A characterization of weakly compact sets in Banach spaces is a natural starting point for a quantitative approach to this property. The classical James' criteria of [13] referring to separated convex hulls enable to define measures of weak noncompactness with the use of sequences. This sequential character of measures is especially useful in applications when we deal with concrete spaces. A different concept is connected with Grothendieck's criterion which refers directly to weakly compact sets. In this paper we give explicit formulae in selected sequence spaces for the measure of weak noncompactness which joins a few well-known criteria.

The measures of weak noncompactness applied to operators have turned out to be useful in fixed point theory and the theories of integral and differential equations (see for example [1, 3]). One of the natural problems is the behavior of weak noncompactness of operators under interpolation. The measure we discuss here based on James' criteria has regular properties from this viewpoint. That is, for the related operator seminorm vanishing for weakly compact operators there exist logarithmically convex-type estimates corresponding to the norm estimates characteristic for real and complex interpolation (see [15, 17]). In contrast, this property is not shared by the measure based on Grothendieck's criterion (see [7]). In this paper, we study in some sense the converse problem. Using techniques similar to those elaborated in [17] for real interpolation, we will estimate the deviation from weak compactness of bounded linear operators extrapolated by the Jawerth-Milman Σ_p and Δ_p methods of [14] for $1 < p < \infty$. Again, as in the case of discrete real interpolation, the behavior of weak noncompactness in l_p vector-valued sequence spaces will play here a key role.

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2. Weak noncompactness of sets and operators

Let X be a Banach space, $\mathsf{B}(X)$ the open unit ball of X and $\mathscr{M}_X$ the family of all nonempty bounded subsets of X. Recall that by James' [13] criterion, weakly closed $A \in \mathscr{M}_X$ is not weakly compact if and only if there exist $\delta > 0$ and $(x_n) \subset A$ such that $\operatorname{dist}(\operatorname{conv}\{x_1, \ldots, x_r\}, \operatorname{conv}\{x_{r+1}, x_{r+2}, \ldots\}) \geqslant \delta$ for every r. It is convenient to use the following notions: (y_n) is said to be a sequence of *successive convex combinations*, or scc, for a sequence (x_n) in X if there exist integers $0 = r_1 < r_2 < \cdots$ such that $y_n \in \operatorname{conv}\{x_{r_n+1}, \ldots, x_{r_{n+1}}\}$ for every n; vectors u_1, u_2 are said to be a pair of scc for (x_n) if $u_1 \in \operatorname{conv}\{x_1, \ldots, x_r\}$ and $u_2 \in \operatorname{conv}\{x_{r+1}, x_{r+2}, \ldots\}$ for some integer $r \geqslant 1$. The *measure of weak noncompactness* γ defined in [17] is given for every $A \in \mathscr{M}_X$ by

$$\gamma(A) = \sup\{\operatorname{csep}(x_n) \colon (x_n) \subset A\},$$

where $\operatorname{csep}(x_n) = \inf \|u_1 - u_2\|$, the infimum being taken over all pairs u_1, u_2 of scc for (x_n). Clearly, $\gamma(A) = 0$ if and only if A is relatively weakly compact. Other properties of γ can be found in [17]. Here, let us only recall the following useful formulae:

$$\gamma(A) = \sup\left\{\lim_{m\to\infty}\lim_{n\to\infty} F_m(x_n) - \lim_{n\to\infty}\lim_{m\to\infty} F_m(x_n)\right\},$$

the supremum being taken over all sequences $(x_n) \subset A$, $(F_m) \subset \overline{\mathsf{B}}(X^*)$ (with X^* being taken for X over the real field) and such that all the limits exist;

$$\gamma(A) = \sup \operatorname{dist}(x^{**}, \operatorname{conv}\{x_n\}),$$

the supremum being taken over all sequences $(x_n) \subset A$ and all w^*-cluster points $x^{**} \in X^{**}$ of (x_n).

By Grothendieck's criterion, weakly closed $A \in \mathscr{M}_X$ is weakly compact if and only if for each $\varepsilon > 0$ there exists a weakly compact set $K_\varepsilon \subset X$ such that $A \subset K_\varepsilon + \varepsilon\overline{\mathsf{B}}(X)$. The corresponding measure of weak noncompactness ω was introduced by De Blasi [8]:

$$\omega(A) = \inf\{\varepsilon > 0 \colon A \subset K_\varepsilon + \varepsilon\overline{\mathsf{B}}(X),\ K_\varepsilon \subset X \text{ is weakly compact}\}.$$

The measures γ and ω are not equivalent in general. An example showing this can be found in [2].

In [17], the supremums in the above formulae for γ are taken over all sequences in $\operatorname{conv} A$. By a result on double limits recently proved in [10, Theorem 13] we can restrict this range to all sequences in A. Thus $\gamma(\operatorname{conv} A) = \gamma(A)$ for every $A \in \mathscr{M}_X$. This can be viewed as a quantitative version of the Krein-Šmulian theorem (if A is relatively weakly compact, then so is $\operatorname{conv} A$). It is worth noting that an analogous quantitative relation holds for ω (see [8]) and does not in another approach to weak noncompactness studied in [10] where $A \in \mathscr{M}_X$ is called ε*-weakly relatively compact* for $\varepsilon \geqslant 0$ if

$$\overline{A}^{w^*} \subset X + \varepsilon\overline{\mathsf{B}}(X^{**}).$$

Here, if A is ε-weakly relatively compact, then $\operatorname{conv} A$ is 2ε-weakly relatively compact [10]. In the general case, this constant cannot be improved [11].

Let $\mathscr{L}(X,Y)$ denote the space of all bounded linear operators acting between Banach spaces X and Y. Let $\mathscr{W}(X,Y)$ denote the subspace of $\mathscr{L}(X,Y)$ consisting of all weakly compact operators. To measure the deviation from weak compactness of $T \in \mathscr{L}(X,Y)$ we put $\Gamma(T) = \gamma(T(\mathsf{B}(X)))$. This gives a seminorm in $\mathscr{L}(X,Y)$ vanishing on $\mathscr{W}(X,Y)$. This seminorm is equivalent neither to the *weak essential norm* $\operatorname{dist}(T,\mathscr{W}(X,Y))$ nor to the *inner* and *outer measures* for $\mathscr{W}(X,Y)$ studied in [2] (see the problem of quantitative versions of Gantmacher's duality theorem in [15] and [20]).

3. Weak noncompactness in selected spaces

Let J_p, $1 < p < \infty$, denote the space of all real null sequences $x = (x(k))$ with finite

$$\|x\| = \sup \left(\sum_{l=1}^{n-1} |x(k_{l+1}) - x(k_l)|^p \right)^{1/p}, \tag{3.1}$$

where the supremum is taken over all finite sequences of integers $0 < k_1 < \cdots < k_n$ with $n \geqslant 2$. Recall that as in the case of the space introduced by James in [12], the space J_p has codimension 1 in its bidual. The vectors $e_n = (0,\dots,0,1,0,\dots)$ with 1 in the nth position form a basis of J_p. The bidual J_p^{**} can be identified with all real sequences $x^{**} = (x^{**}(k))$ with finite

$$\|x^{**}\| = \sup_n \left\| \sum_{k=1}^{n} x^{**}(k) e_k \right\|.$$

Since every such x^{**} is convergent and $(x^{**}(k) - \lim_{l\to\infty} x^{**}(l)) \in J_p$, we will identify J_p^{**} with $J_p \oplus \operatorname{span}\{(1,1,\dots)\}$.

For a sequence $x = (x(k))$ and $n = 1,2,\dots$ let

$$P_n x = (x(1),\dots,x(n),0,\dots), \quad R_n x = x - P_n x.$$

If we restrict P_n to J_p^{**}, then $\|P_n\| = 1$ for every n and $\lim_{n\to\infty} \|R_n x\| = 0$ for every $x \in J_p$. In the sequel a constant sequence $(\alpha,\alpha,\dots)$ will be also denoted by α.

LEMMA 3.1. ([16])*Let (x_n) be a bounded sequence in J_p. For every $\varepsilon > 0$ there exist a subsequence (x_{n_k}), an increasing sequence (m_k) of natural numbers and a constant α such that for $k = 1,2,\dots$ we have*

$$\left\| x_{n_k} - (P_{m_1} x_{n_1} + P_{m_{2k}} R_{m_1} \alpha + P_{m_{2k+1}} R_{m_{2k}} x_{n_k}) \right\| \leqslant \varepsilon.$$

THEOREM 3.2. *If $1 < p < \infty$ and $A \in \mathscr{M}_{J_p}$, then*

$$\gamma(A) = 2^{1/p} \sup \left\{ \lim_{k\to\infty} |x^{**}(k)| \right\}, \tag{3.2}$$

where the supremum is taken over all sequences (x_n) in A and all w^*-cluster points $x^{**} = (x^{**}(k)) \in J_p^{**}$ of (x_n).

Proof. Since γ is positively homogeneous, we can assume that $A \subset \overline{\mathsf{B}}(J_p)$. Let $\gamma'(A)$ denote the right-hand side of (3.2). Fix $\varepsilon > 0$, a sequence $(x_n) \subset A$ and its w^*-cluster point $x^{**} = (x^{**}(k)) \in J_p^{**}$. By passing to a subsequence we can assume that $x^{**} = w^*\text{-}\lim_{n\to\infty} x_n$. Then $x^{**}(k) = \lim_{n\to\infty} x_n(k)$ for $x_n = (x_n(k)) \in J_p^{**}$. Writing $\alpha = \lim_{k\to\infty} x^{**}(k)$ and passing to a subsequence once more, by Lemma 3.1 we can assume that $\|x_n - u_n\| \leqslant \varepsilon$ for all n, where $u_n = P_{m_1}x_1 + P_{m_{2n}}R_{m_1}\alpha + P_{m_{2n+1}}R_{m_{2n}}x_n$ and (m_k) is an increasing sequence of natural numbers. Moreover, we can assume that m_1 and x_1 are taken to satisfy $\|P_{m_1}(x^{**} - x_1)\| \leqslant \frac{1}{6}\varepsilon$ and $\|R_{m_1}(x^{**} - \alpha)\| \leqslant \frac{1}{6}\varepsilon$. Let $u^{**} = P_{m_1}x_1 + R_{m_1}\alpha$. Thus

$$\|x^{**} - u^{**}\| \leqslant \|P_{m_1}(x^{**} - x_1)\| + \|R_{m_1}(x^{**} - \alpha)\| \leqslant \frac{\varepsilon}{3}.$$

We first prove that $\gamma(A) \leqslant \gamma'(A)$. For $N = 1, 2, \ldots$ we have

$$\left\| u^{**} - \frac{1}{N}\sum_{n=1}^{N} u_n \right\| \leqslant \left\| \frac{1}{N}\sum_{n=1}^{N} R_{m_{2n}}\alpha \right\| + \left\| \frac{1}{N}\sum_{n=1}^{N} P_{m_{2n+1}}R_{m_{2n}}x_n \right\|.$$

Clearly, $\left\| \frac{1}{N}\sum_{n=1}^{N} R_{m_{2n}}\alpha \right\| = 2^{1/p}\,|\alpha|$. Applying $|a-b|^p \leqslant 2^{p-1}(|a|^p + |b|^p)$ to some terms realizing variations in the norm we obtain

$$\left\| \frac{1}{N}\sum_{n=1}^{N} P_{m_{2n+1}}R_{m_{2n}}x_n \right\|^p \leqslant \frac{2^{p-1}}{N^p}\sum_{n=1}^{N} \|P_{m_{2n+1}}R_{m_{2n}}x_n\|^p \leqslant \frac{2^{2p-1}}{N^{p-1}}.$$

We now choose N such that $(2^{2p-1}N^{1-p})^{1/p} \leqslant \frac{2}{3}\varepsilon$. Then

$$\begin{aligned} \left\| x^{**} - \frac{1}{N}\sum_{n=1}^{N} x_n \right\| &\leqslant \|x^{**} - u^{**}\| + \left\| u^{**} - \frac{1}{N}\sum_{n=1}^{N} u_n \right\| + \left\| \frac{1}{N}\sum_{n=1}^{N}(u_n - x_n) \right\| \\ &\leqslant 2^{1/p}\,|\alpha| + 2\varepsilon \leqslant \gamma'(A) + 2\varepsilon. \end{aligned}$$

It follows that $\operatorname{dist}(x^{**}, \operatorname{conv}\{x_n\}) \leqslant \gamma'(A) + 2\varepsilon$ and consequently, $\gamma(A) \leqslant \gamma'(A)$.

To prove the opposite inequality let us observe that in the case of (x_n) as above for every nonnegative $t_1, \ldots, t_k$ with sum 1 and each subset $\{n_1, \ldots, n_k\}$ of indices we have

$$2^{1/p}\,|\alpha| \leqslant \left\| u^{**} - \sum_{i=1}^{k} t_{n_i}u_{n_i} \right\| \leqslant \left\| x^{**} - \sum_{i=1}^{k} t_{n_i}x_{n_i} \right\| + \frac{4}{3}\varepsilon.$$

Hence

$$2^{1/p}\,|\alpha| \leqslant \operatorname{dist}(x^{**}, \operatorname{conv}\{x_n\}) + \frac{4}{3}\varepsilon \leqslant \gamma(A) + \frac{4}{3}\varepsilon$$

and finally, $\gamma'(A) \leqslant \gamma(A)$. □

Combining the above result with the evident formula $\operatorname{dist}(x^{**}, J_p) = \lim_{k\to\infty} |x^{**}(k)|$ for $x^{**} = (x^{**}(k)) \in J_p^{**}$, we get immediately the following corollary.

COROLLARY 3.3. *If* $1 < p < \infty$ *and* $A \in \mathscr{M}_{J_p}$, *then*

$$\gamma(A) = 2^{1/p} \sup \operatorname{dist}(x^{**}, J_p),$$

where the supremum is taken as in (3.2). In particular, $\gamma(\mathsf{B}(J_p)) = 2^{1/p}$.

Recall that if X is nonreflexive, then $\omega(\mathsf{B}(X)) = 1$ and $1 \leqslant \gamma(\mathsf{B}(X)) \leqslant 2$. The gap between 1 and 2 for γ is filled by $\mathsf{B}(J_p)$, $1 < p < \infty$, which is not the case for J_p endowed with the equivalent norm

$$\|x\|_0 = \sup \left(|x(k_n) - x(k_1)|^p + \sum_{l=1}^{n-1} |x(k_{l+1}) - x(k_l)|^p \right)^{1/p},$$

the supremum being taken as in (3.1), corresponding to the original one of [12] for $p = 2$. It is easily seen that $\|x\| \leqslant \|x\|_0 \leqslant 2^{1/p} \|x\|$. Let J_p^0 denote the space J_p with $\|\cdot\|_0$. Both Lemma 3.1 and Theorem 3.2 hold for J_p^0. Clearly, $\operatorname{dist}(x^{**}, J_p^0) = 2^{1/p} \lim_{k\to\infty} |x^{**}(k)|$. It follows that $\gamma(\mathsf{B}(J_p^0)) = 1$ for every $1 < p < \infty$.

Although nonequivalent in general, the measures γ and ω coincide up to a constant multiplier in some classical spaces. For example, in the Lebesgue space $L_1(\nu)$ with finite measure ν we have $\gamma = 2\omega$ (see [15]). In the space c_0 of null sequences we have $\gamma = \omega$ (see [17]). It is not difficult to prove that in l_1 we have $\gamma = 2\omega = 2\chi$, where χ is the Hausdorff measure of (strong) noncompactness. We show that in the space c of convergent real sequences with supremum norm γ and ω are equivalent but there is no constant a such that $\gamma = a\omega$. We identify c^{**} with l_∞, thus for the canonical image $x^{**} = (x^{**}(k)) \in l_\infty$ of $x = (x(k)) \in c$, we have $x^{**}(1) = \lim_{k\to\infty} x(k)$ and $x^{**}(k+1) = x(k)$ for $k \geqslant 1$.

THEOREM 3.4. *If $A \in \mathscr{M}_c$, then*

$$\gamma(A) = \sup \left\{ \limsup_{k\to\infty} |x^{**}(k) - x^{**}(1)| \right\}, \tag{3.3}$$

where the supremum is taken over all sequences (x_n) in A and all w^*-cluster points $x^{**} = (x^{**}(k)) \in c^{**}$ of (x_n).

Proof. We can assume that $A \subset \overline{\mathsf{B}}(c)$. Fix $\varepsilon > 0$, a sequence $(x_n) \subset A$ and its w^*-cluster point $x^{**} = (x^{**}(k)) \in c^{**}$. By passing to a subsequence we can assume that $x^{**} = w^*\text{-}\lim_{n\to\infty} x_n$. For $x_n = (x_n(k))$ we have $x^{**}(1) = \lim_{n\to\infty} \lim_{k\to\infty} x_n(k)$ and $x^{**}(k+1) = \lim_{n\to\infty} x_n(k)$ if $k \geqslant 1$. We can assume that $|x^{**}(1) - \lim_{k\to\infty} x_n(k)| \leqslant \varepsilon$ for every $n \geqslant 1$.

Let $\gamma'(A)$ denote the right-hand side of (3.3). We first prove that $\gamma(A) \leqslant \gamma'(A)$. Write $y_n(k) = x_n(k) - \lim_{l\to\infty} x_n(l)$ and $y^{**}(k) = x^{**}(k) - x^{**}(1)$ for all $k, n \geqslant 1$. Put $q = \limsup_{k\to\infty} |y^{**}(k)|$. Then $(y_n) \subset c_0$ and $y^{**}(k+1) = \lim_{n\to\infty} y_n(k)$ for $k \geqslant 1$. We now follow the case of c_0 in [17]. Fix natural N such that $N^{-1} \leqslant \varepsilon$. Choose $(y_{n_i})_{i=1}^N$ and integers $1 < k_1 < \cdots < k_{N+1}$ so that $|y^{**}(k)| \leqslant q + \varepsilon$ for $k > k_1$ and the following is satisfied: $\left|y_{n_i}^{**}(k) - y^{**}(k)\right| \leqslant \varepsilon$ for $1 \leqslant k \leqslant k_i$ and $\left|y_{n_i}^{**}(k)\right| \leqslant \varepsilon$ for $k > k_{i+1}$, where $i = 1, \ldots, N$. Let $u = \frac{1}{N} \sum_{i=1}^N y_{n_i}$. Then for $1 \leqslant k \leqslant k_1$,

$$|u^{**}(k) - y^{**}(k)| \leqslant \varepsilon,$$

and for $k > k_1$,

$$|u^{**}(k) - y^{**}(k)| \leqslant |y^{**}(k)| + \varepsilon + 2N^{-1} \leqslant q + 4\varepsilon.$$

For $v = \frac{1}{N}\sum_{i=1}^{N} x_{n_i} \in \operatorname{conv}\{x_n\}$ and $k \geqslant 1$ we have

$$|v^{**}(k) - x^{**}(k)| \leqslant \left| x^{**}(1) - \frac{1}{N}\sum_{i=1}^{N} \lim_{l\to\infty} x_{n_i}(l) \right| + q + 4\varepsilon \leqslant \gamma'(A) + 5\varepsilon,$$

which gives $\operatorname{dist}(x^{**}, \operatorname{conv}\{x_n\}) \leqslant \gamma'(A) + 5\varepsilon$ and consequently, $\gamma(A) \leqslant \gamma'(A)$.

To prove the opposite inequality from a given sequence take a subsequence (x_n) as in the beginning of our proof and $z \in \operatorname{conv}\{x_n\}$ such that $\|x^{**} - z\| \leqslant \operatorname{dist}(x^{**}, \operatorname{conv}\{x_n\}) + \varepsilon$. Then

$$\begin{aligned}\limsup_{k\to\infty} |x^{**}(k) - x^{**}(1)| &\leqslant \limsup_{k\to\infty} |x^{**}(k+1) - z(k)| + \left|\lim_{k\to\infty} z(k) - x^{**}(1)\right| \\ &\leqslant \|x^{**} - z\| + \varepsilon \leqslant \gamma(A) + 2\varepsilon.\end{aligned}$$

Finally, $\gamma(A) \geqslant \gamma'(A)$ and the proof is complete. □

THEOREM 3.5. *If $A \in \mathscr{M}_c$, then $\omega(A) \leqslant \gamma(A) \leqslant 2\omega(A)$.*

Proof. Clearly, $\gamma(A) \leqslant \gamma(\overline{\mathsf{B}}(c))\omega(A) = 2\omega(A)$. To prove the first inequality we proceed similarly to [17, Theorem 2.9]. For $q \geqslant 0$ let

$$r_q(\alpha) = \begin{cases} 0 & \text{if } |\alpha| \leqslant q \\ \alpha - q\alpha|\alpha|^{-1} & \text{if } |\alpha| > q. \end{cases}$$

Observe that r_q is continuous and $|\alpha - r_q(\alpha)| \leqslant q$ for every $\alpha \in \mathbb{R}$. Let the mapping $R_q : c \to c$ be given for every $x = (x(k))$ by

$$(R_q x)(k) = r_q\left(x(k) - \lim_{l\to\infty} x(l)\right) + \lim_{l\to\infty} x(l).$$

Let $A \in \mathscr{M}_c$ and $q = \gamma(A)$. We show that $\gamma(R_q(A)) = 0$. Let $(x_n) \subset A$. Assume that $w^*\text{-}\lim_{n\to\infty} x_n = x^{**}$. By Theorem 3.4, $\limsup_{k\to\infty} |x^{**}(k) - x^{**}(1)| \leqslant q$. Then

$$\begin{aligned}\limsup_{k\to\infty}\left|\left(w^*\text{-}\lim_{n\to\infty} R_q x_n\right)(k) - x^{**}(1)\right| &= \limsup_{k\to\infty} |r_q(x^{**}(k) - x^{**}(1))| \\ &= r_q\left(\limsup_{k\to\infty} |x^{**}(k) - x^{**}(1)|\right) = 0.\end{aligned}$$

By Theorem 3.4 we get $\gamma(R_q(A)) = 0$ and therefore $R_q(A)$ is relatively weakly compact. If $u = (u(k)) \in A$, then

$$\|u - R_q u\| = \sup_k \left| u(k) - r_q\left(u(k) - \lim_{l\to\infty} u(l)\right) - \lim_{l\to\infty} u(l)\right| \leqslant q.$$

Thus $A \subset R_q(A) + q\overline{\mathsf{B}}(c)$ and $\omega(A) \leqslant q = \gamma(A)$. □

The obtained estimates cannot be improved: $\gamma(\mathsf{B}(c)) = 2$ and $\omega(\mathsf{B}(c)) = 1$, on the other hand, $\gamma(\mathsf{B}(c_0)) = 1$ and $\omega(\mathsf{B}(c_0)) = 1$. The latter for $\mathsf{B}(c_0)$ as a subset of c can be verified directly (recall that ω may vary through linear isometries).

In the case of the Lions-Peetre discrete real interpolation a key role in interpolation of weak noncompactness plays certain class of operators acting between $l_p(X)$ vector-valued spaces (see [17]). For the Jawerth-Milman extrapolation methods the natural underlying space is $l_p(X_\nu)$ for some family $\{X_\nu\}_{\nu\in\mathbb{Z}}$ of Banach spaces. Here, $l_p(X_\nu)$, $1 \leqslant p \leqslant \infty$, denotes the Banach space of all families $x = \{x(\nu)\}_{\nu\in\mathbb{Z}}$, $x(\nu) \in X_\nu$, such that $\|x\|_{l_p(X_\nu)} < \infty$, where

$$\|x\|_{l_\infty(X_\nu)} = \sup_{\nu\in\mathbb{Z}} \|x(\nu)\|_{X_\nu}, \quad \|x\|_{l_p(X_\nu)} = \left(\sum_{\nu\in\mathbb{Z}} \|x(\nu)\|_{X_\nu}^p\right)^{1/p}, 1 \leqslant p < \infty.$$

THEOREM 3.6. *Let $\{X_\nu\}_{\nu\in\mathbb{Z}}$ and $\{Y_\nu\}_{\nu\in\mathbb{Z}}$ be families of Banach spaces. Let $\{T_\nu\}_{\nu\in\mathbb{Z}}$ be a family of operators such that $T_\nu \in \mathscr{L}(X_\nu, Y_\nu)$ and $\sup_{\nu\in\mathbb{Z}} \|T_\nu\| < \infty$. Suppose that $1 < p < \infty$ and $\overline{T} \in \mathscr{L}(l_p(X_\nu), l_p(Y_\nu))$ is given by $\overline{T}x = \{T_\nu x(\nu)\}_{\nu\in\mathbb{Z}}$ for every $x = \{x(\nu)\}_{\nu\in\mathbb{Z}} \in l_p(X_\nu)$. Then $\Gamma(\overline{T}) = \sup_{\nu\in\mathbb{Z}} \Gamma(T_\nu)$.*

The above result is an extension of Theorem 3.6 of [17] to arbitrary spaces $l_p(X_\nu)$ with $1 < p < \infty$. The proof with easy alterations can be adopted from [17].

In particular, if $A = \mathsf{B}(l_p(X_\nu))$ then $\gamma(A) = \sup_{\nu\in\mathbb{Z}} \gamma(P_\nu(A))$, where $P_\nu: l_p(X_\nu) \to X_\nu$ is the projection $P_\nu(x) = x(\nu)$, $x = \{x(\nu)\}_{\nu\in\mathbb{Z}}$. As in the case of ω (see [4]) such relation is not true for every bounded A. Indeed, let $X_\nu = c_0$ for every ν and let (e_n) be the standard basis of c_0. For all positive integers n, k let $x_{n,k} = \{x_{n,k}(\nu)\}_{\nu\in\mathbb{Z}}$ with

$$x_{n,k}(\nu) = \begin{cases} k^{-1/p}(e_1 + \cdots + e_n) & \text{if } 1 \leqslant \nu \leqslant k, \\ 0 & \text{otherwise.} \end{cases}$$

Then for $A_k = \{x_{n,k} : n \geqslant 1\}$ we have $\gamma(A_k) = 1$ and $\sup_{\nu\in\mathbb{Z}} \gamma(P_\nu(A_k)) = k^{-1/p}$.

4. Weak noncompactness of extrapolated operators

In this section we estimate Γ for operators extrapolated by the Σ_p and Δ_p Jawerth-Milman [14] methods for $1 < p < \infty$. These extrapolation methods enable to generate various scales of spaces, for example abstract *logarithmic spaces* $A_\theta(\log A)_{b,p}$ studied in [9], and to extend into these scales Yano's [21, 22] classical extrapolation result concerning $L(\log L)$, $L_{\exp}$ and L_p spaces.

Let us recall some terminology of [14]. A family $\{A_\theta\}_{\theta\in\Theta}$ of Banach spaces is said to be *strongly compatible* if there exist two Banach spaces Δ_α, Σ_α such that $\Delta_\alpha \hookrightarrow A_\theta \hookrightarrow \Sigma_\alpha$ (continuous embeddings) for every $\theta \in \Theta$. The norms of the inclusion maps $\Delta_\alpha \hookrightarrow A_\theta$ and $A_\theta \hookrightarrow \Sigma_\alpha$ will be denoted by $M_{\Delta_\alpha}(\theta)$ and $M_{\Sigma_\alpha}(\theta)$, respectively.

Let $\{A_\theta\}_{\theta\in\Theta}$ and $\{B_\theta\}_{\theta\in\Theta}$ be two strongly compatible families with spaces Δ_α, Σ_α and Δ_β, Σ_β, respectively. We write $T: \{A_\theta\}_{\theta\in\Theta} \xrightarrow{1} \{B_\theta\}_{\theta\in\Theta}$ if $T: \Sigma_\alpha \to \Sigma_\beta$ is a linear operator and for every $\theta \in \Theta$ the restriction of T to A_θ maps A_θ into B_θ with norm $\leqslant 1$. Banach spaces A and B are said to be *extrapolation spaces* (with respect to $\{A_\theta\}_{\theta\in\Theta}$ and $\{B_\theta\}_{\theta\in\Theta}$) if $\Delta_\alpha \hookrightarrow A \hookrightarrow \Sigma_\alpha$, $\Delta_\beta \hookrightarrow B \hookrightarrow \Sigma_\beta$ and if $T: \{A_\theta\}_{\theta\in\Theta} \xrightarrow{1} \{B_\theta\}_{\theta\in\Theta}$ implies $T: A \to B$, that is, the restriction of T to A maps boundedly A into B.

We deal with strongly compatible countable families $\{A_\theta\}_{\theta\in\mathbb{Z}}$. Let $1 < p < \infty$ and $1/p+1/q = 1$. The Σ_p extrapolation method is defined whenever $\sum_{\theta\in\mathbb{Z}}(M_{\Sigma_\alpha}(\theta))^q$ is finite. By the $\Sigma_p(A_\theta)$ space we mean all $a \in \Sigma_\alpha$ for which there exists a family $\{a(\theta)\}_{\theta\in\mathbb{Z}}$, $a(\theta) \in A_\theta$, such that $\sum_{\theta\in\mathbb{Z}}\|a(\theta)\|^p_{A_\theta}$ is finite and $\sum_{\theta\in\mathbb{Z}} a(\theta)$ is (absolutely) convergent to a in Σ_α. The norm in $\Sigma_p(A_\theta)$ is given by

$$\|a\|_{\Sigma_p(A_\theta)} = \inf\left(\sum_{\theta\in\mathbb{Z}}\|a(\theta)\|^p_{A_\theta}\right)^{1/p},$$

where the infimum is taken over all representations $\{a(\theta)\}_{\theta\in\mathbb{Z}}$ of a as above.

The Δ_q extrapolation method is defined whenever $\sum_{\theta\in\mathbb{Z}}(M_{\Delta_\alpha}(\theta))^q$ is finite. By the $\Delta_q(A_\theta)$ space we mean all $a \in \bigcap_{\theta\in\mathbb{Z}} A_\theta$ with $\sum_{\theta\in\mathbb{Z}}\|a\|^q_{A_\theta}$ finite. The norm in $\Delta_q(A_\theta)$ is given by

$$\|a\|_{\Delta_q(A_\theta)} = \left(\sum_{\theta\in\mathbb{Z}}\|a\|^q_{A_\theta}\right)^{1/q}.$$

THEOREM 4.1. *Let $1 < p < \infty$. Let $\{A_\theta\}_{\theta\in\mathbb{Z}}$ and $\{B_\theta\}_{\theta\in\mathbb{Z}}$ be strongly compatible families as for the Σ_p method. If $T\colon\{A_\theta\}_{\theta\in\mathbb{Z}} \xrightarrow{1} \{B_\theta\}_{\theta\in\mathbb{Z}}$, then*

$$\Gamma\left(T\colon\Sigma_p(A_\theta) \to \Sigma_p(B_\theta)\right) \leqslant \sup\left\{\Gamma\left(T\colon A_\theta \to B_\theta\right) : \theta \in \mathbb{Z}\right\}.$$

Proof. Let (a_n) be a sequence in $\mathsf{B}(\Sigma_p(A_\theta))$. Then for each n there exists a representation $\{a_n(\theta)\}_{\theta\in\mathbb{Z}} \in \mathsf{B}(l_p(A_\theta))$ of a_n. Write $b_n = Ta_n$ and $y_n = \{T_\theta a_n(\theta)\}_{\theta\in\mathbb{Z}}$, where T_θ is the restriction of T to A_θ. Clearly, $y_n \in l_p(B_\theta)$ and y_n is a representation of b_n.

Define the operator $\overline{T} \in \mathscr{L}(l_p(A_\theta), l_p(B_\theta))$ by $\overline{T}a = \{T_\theta a(\theta)\}_{\theta\in\mathbb{Z}}$ for every $a = \{a(\theta)\}_{\theta\in\mathbb{Z}} \in l_p(A_\theta)$. Thus $y_n \in \overline{T}(\mathsf{B}(l_p(A_\theta)))$. Fix $\varepsilon > 0$. By Theorem 2.1 in [17], there exists a sequence (y'_n) of scc for (y_n) such that

$$\|v_1 - v_2\|_{l_p(B_\theta)} \leqslant \operatorname{csep}(y'_n) + \varepsilon$$

for every pair v_1, v_2 of scc for (y'_n). Then $y'_n = \sum_{i=r_n+1}^{r_{n+1}} t_i y_i$ for some sequence of integers $0 = r_1 < r_2 < \cdots$ and nonnegative $t_{r_n+1}, \ldots, t_{r_{n+1}}$ with sum 1 for all n. Put $b'_n = \sum_{i=r_n+1}^{r_{n+1}} t_i b_i$. Then (b'_n) is a sequence of scc for (b_n). Therefore

$$\operatorname{csep}(b_n) \leqslant \operatorname{csep}(b'_n) \leqslant \|b'_1 - b'_2\|_{\Sigma_p(B_\theta)} \leqslant \|y'_1 - y'_2\|_{l_p(B_\theta)} \leqslant \operatorname{csep}(y'_n) + \varepsilon.$$

Since (y'_n) is a sequence in $\overline{T}(\mathsf{B}(l_p(A_\theta))$, we have

$$\operatorname{csep}(y'_n) \leqslant \Gamma(\overline{T}\colon l_p(A_\theta) \to l_p(B_\theta))$$

By Theorem 3.6 we get

$$\Gamma(\overline{T}\colon l_p(A_\theta) \to l_p(B_\theta)) = \sup\left\{\Gamma\left(T\colon A_\theta \to B_\theta\right) : \theta \in \mathbb{Z}\right\}.$$

An arbitrary choice of (a_n) and $\varepsilon > 0$ gives the assertion. $\square$

Arguments similar to those in the Σ_p method give immediately the next result.

THEOREM 4.2. *Let* $1 < q < \infty$. *Let* $\{A_\theta\}_{\theta\in\mathbb{Z}}$ *and* $\{B_\theta\}_{\theta\in\mathbb{Z}}$ *be strongly compatible families as for the* Δ_q *method. If* $T\colon \{A_\theta\}_{\theta\in\mathbb{Z}} \xrightarrow{1} \{B_\theta\}_{\theta\in\mathbb{Z}}$, *then*

$$\Gamma\left(T\colon \Delta_q(A_\theta) \to \Delta_q(B_\theta)\right) \leqslant \sup\left\{\Gamma\left(T\colon A_\theta \to B_\theta\right) : \theta \in \mathbb{Z}\right\}.$$

COROLLARY 4.3. *Under the assumptions of Theorem 4.1 and Theorem 4.2, respectively, if* $T\colon A_\theta \to B_\theta$ *is weakly compact for every* $\theta \in \mathbb{Z}$, *then so are* $T\colon \Sigma_p(A_\theta) \to \Sigma_p(B_\theta)$ *and* $T\colon \Delta_q(A_\theta) \to \Delta_q(B_\theta)$. *In particular, if* A_θ *is reflexive for every* $\theta \in \mathbb{Z}$, *then so are* $\Sigma_p(A_\theta)$ *and* $\Delta_q(A_\theta)$.

The above results cannot be extended into the Σ and Δ extrapolation methods of [14], which in the above notation can be viewed as, respectively, the Σ_1 and Δ_∞ methods for any Θ with usual alterations in the norms. Let Ω denote $[0,1]$ with Lebesgue measure. The Lions-Peetre interpolation space $(L_1(\Omega), L_\infty(\Omega))_{\theta,q}$ with $0 < \theta < 1$ and $1 \leqslant q \leqslant \infty$ is equal (up to an equivalent norm) to the Lorentz space $L_{p,q}(\Omega)$ with $1/p = 1-\theta$ (see [6]). The space $L_{p,q}(\Omega)$ (endowed with an equivalent norm if necessary) is reflexive whenever $1 < p,q < \infty$ (see [5]). According to [19, Example 8], we have

$$\Sigma(\theta^{-1}(L_1(\Omega), L_\infty(\Omega))_{\theta,q}) = L(\log L)(\Omega).$$

It follows that reflexivity is not extrapolated by the Σ method. A similar counterexample can be given for the Δ method with $L_{\exp}(\Omega)$ as a result [18, Example 22].

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Andrzej Kryczka
Institute of Mathematics
Maria Curie-Skłodowska University
20-031 Lublin
Poland
e-mail: andrzej.kryczka@umcs.lublin.pl