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MONOTONICITY RESULTS FOR THE POLYGAMMA FUNCTIONS

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Abstract. In this paper, the monotonic results of the functions $x^c|\psi^{(n)}(x+\beta)|$ and $x\psi^{(n+1)}(x+\beta)/\psi^{(n)}(x+\beta)$ are established. Several by-products are obtained. Moreover, we prove that the function $\frac{d\psi(\ln x)}{dx}$ is completely monotonic.

1. Introduction

The psi (or digamma) funtion is defined for all positive real numbers x as

$$\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}=-\gamma+\int_0^\infty \frac{e^{-t}-e^{-xt}}{1-e^{-t}}\,\mathrm{d}t=-\gamma-\frac{1}{x}+\sum_{i=1}^{\infty}\frac{x}{i(i+x)},$$

where Γ denotes Euler's gamma function and $\gamma=0.5772\ldots$ is Euler-Mascheroni constant. ψ and its derivatives are called polygamma functions.

The digamma and polygamma functions play a central role in the theory of special functions, and the have many applications in different branches, such as, mathematical physics and statistics.

There exists an extensive and rich literature on inequalities for ψ and its derivatives. For the recent developments in this area, we refer the reader to the articles [2, 3, 4, 5, 6, 7, 8, 18, 19, 20, 21] and the references therein. The aim of this paper is to continue these investigations and to prove some new inequalities for digamma and polygamma functions, which yield extensions and generalizations of known theorems.

First, we present the following theorem which generalize Alzer's result [5, Lemma 2.1].

THEOREM 1. *Let $n\geqslant 1$ be an integer, $c\in\mathbb{R}$, and $\beta>1/2$. The function*

$$g_n(x;c,\beta)=x^c|\psi^{(n)}(x+\beta)|$$

is strictly decreasing on $(0,\infty)$ if and only if $c\leqslant 0$. And, $g_n(x;c,\beta)$ is strictly increasing on $(0,\infty)$ if and only if $c\geqslant n$.

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In [6], the authors proved that the function $x\psi''(x+\beta)/\psi'(x+\beta)$ is strictly decreasing on $[0,\infty)$ with $\beta \geqslant 1/2$. Now, we will extend this result as follows.

THEOREM 2. *If $\beta \geqslant \frac{1}{2}$ is a real number and $n \geqslant 1$ is an integer, then the function*

$$f_n(x;\beta) = x\frac{\psi^{(n+1)}(x+\beta)}{\psi^{(n)}(x+\beta)}$$

is strictly decreasing on $[0,\infty)$.

THEOREM 3. *Let $k > n \geqslant 1$ be two integers, $\beta \geqslant \frac{1}{2}$ be a real number, then the function*

$$f_{k,n}(x;\beta) = x^{k-n}\frac{\psi^{(k)}(x+\beta)}{\psi^{(n)}(x+\beta)}$$

is strictly decreasing on $[0,\infty)$ if $k-n$ is odd; $f_{k,n}(x;\beta)$ is strictly increasing on $[0,\infty)$ if $k-n$ is even.

In [4], Alzer proved that the function $x\frac{\psi^{(n+1)}(x)}{\psi^{(n)}(x)}$ is strictly increasing on $\mathbb{R}_+$. With analogous proof method as Theorem 3, we get the following by-product.

COROLLARY 1. *Let $k > n \geqslant 1$ be two integers, then the function*

$$f_{k,n}(x) = x^{k-n}\frac{\psi^{(k)}(x)}{\psi^{(n)}(x)}$$

is strictly increasing on $(0,\infty)$ if $k-n$ is odd; $f_{k,n}(x)$ is strictly decreasing on $(0,\infty)$ if $k-n$ is even.

COROLLARY 2. *Let k and n be integers with $k > n \geqslant 1$, $\beta \geqslant 1/2$. Then we have for all $x > 0$:*

$$(n-1)_{k-n} < x^{k-n}\frac{|\psi^{(k)}(x+\beta)|}{|\psi^{(n)}(x+\beta)|} < (n)_{k-n},$$

where $(a)_m = a(a+1)\cdot\ldots\cdot(a+m-1)$. Both bounds are sharp.

2. Lemmas

LEMMA 1. ([1, 23]) *For $x > 0$ and $r > 0$,*

$$\frac{1}{x^r} = \frac{1}{\Gamma(r)}\int_0^\infty t^{r-1}e^{-xt}\,\mathrm{d}t. \tag{1}$$

LEMMA 2. ([17, p. 16]) *The derivatives ψ', ψ'', ψ''' are known as polygamma functions, which can be defined as*

$$\psi^{(n)}(x) = (-1)^{n+1}\int_0^\infty \frac{t^n}{1-e^{-t}}e^{-xt}\,\mathrm{d}t = (-1)^{k+1}k!\sum_{i=0}^\infty \frac{1}{(x+i)^{k+1}} \tag{2}$$

for $x > 0$ and $n \in \mathbb{N} := 1,2,\ldots$.

LEMMA 3. ([1, p. 260]) *Asymptotic expansions for* ψ *and* $\psi^{(n)}$ *can be expressed as*

$$\psi(x) \sim \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \cdots \quad (x \to \infty), \tag{3}$$

$$|\psi^{(n)}(x)| \sim \frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + \frac{(n+1)!}{12x^{n+2}} - \cdots \quad (x \to \infty; n = 1, 2, \dots). \tag{4}$$

LEMMA 4. ([24]) *Let* $f_i(t)$ *for* $i = 1, 2$ *be piecewise continuous in arbitrary finite intervals included in* $(0, \infty)$. *Suppose there exist some constants* $M_i > 0$ *and* $c_i \geqslant 0$ *such that* $|f_i(t)| \leqslant M_i e^{c_i t}$ *for* $i = 1, 2$, *then*

$$\int_0^\infty \left[\int_0^t f_1(u) f_2(t-u)\,\mathrm{d}u\right] e^{-st}\,\mathrm{d}t = \int_0^\infty f_1(u) e^{-su}\,\mathrm{d}u \int_0^\infty f_2(v) e^{-sv}\,\mathrm{d}v. \tag{5}$$

REMARK 1. Lemma 4 is the convolution theorem of Laplace transforms. It can be looked up in standard textbooks of integral transform.

LEMMA 5. ([8, Corollary 2.3]) *Let* $n \geqslant 2$ *be an integer. Then for all positive real numbers* x,

$$\frac{n-1}{n} < \frac{(\psi^{(n)}(x))^2}{\psi^{(n-1)}(x)\psi^{(n+1)}(x)} < \frac{n}{n+1}. \tag{6}$$

Both bounds are best possible.

Whereafter, we present two technical lemmas, which will help to prove our main results.

LEMMA 6. *If* $\beta > \frac{1}{2}$, *then the function*

$$\phi_\beta(x) = \frac{\beta x e^x - \beta x + x}{1 - e^x}$$

is strictly decreasing for $x > 0$.

Proof. The function ϕ_β is decreasing, which is equivalent to

$$\phi_\beta'(x) = -\beta + \frac{1 - e^x + xe^x}{(1 - e^x)^2} < 0.$$

This is

$$\beta > \frac{1 - e^x + xe^x}{(1 - e^x)^2} \triangleq \tau(x).$$

Taking logarithm and differentiation yields

$$\begin{aligned}
(\ln \tau(x))' &= \frac{xe^x}{1 - e^x + xe^x} + \frac{2e^x}{1 - e^x} \\
&= \frac{e^x(2e^x - xe^x - x - 2)}{(1 - e^x + xe^x)(e^x - 1)} \\
&\triangleq \frac{e^x \rho(x)}{(1 - e^x + xe^x)(e^x - 1)}.
\end{aligned}$$

It is quite easy to see that $\rho''(x) = -xe^x < 0$ and $\rho'(0) = \rho(0) = 0$. Then we obtain $\rho(x) < 0$ for $x > 0$. Ihis implies that $\tau(x)$ is strictly decreasing on $(0,\infty)$. From $\lim_{x\to 0+} \tau(x) = \frac{1}{2}$, we get $\beta > \frac{1}{2}$. □

LEMMA 7. *Let $a \in (0,1)$, if $\beta > \frac{1}{2}$, then the function*

$$\varphi_\beta(x) = \frac{e^{\beta(1+a)x} - e^{(\beta-1)(1+a)x}}{e^{\beta(1-a)x} - e^{(\beta-1)(1-a)x}}$$

is strictly increasing for $x > 0$.

Proof. The function φ_β is equivalent to $e^{2(\beta-1)}\frac{e^{(1+a)x}-1}{e^{(1-a)x}-1}$. Taking logarithm and differentiation yields

$$\begin{aligned}(\ln \varphi_\beta(x))' &= 2(\beta-1)a + \frac{(1+a)e^{(1+a)x}}{e^{(1+a)x}-1} - \frac{(1-a)e^{(1-a)x}}{e^{(1-a)x}-1}\\ &\triangleq 2(\beta-1)a + \mu(x).\end{aligned}$$

By standard argument, we have

$$\begin{aligned}&(e^{(1-a)x}-1)^2(e^{(1+a)x}-1)^2\mu'(x)\\ &\quad = e^{(1-a)x}\big[(e^{2x}-1)(e^{2ax}-1)(a^2+1) - 2a(1+e^{2x}+e^{2ax}+e^{2(1+a)x}-4e^{(1+a)x})\big]\\ &\quad \triangleq e^{(1-a)x}\nu(x),\\ &\nu'(x) = 2(1+a)\big[(a-1)^2e^{2(1+a)x} + 4ae^{(1+a)x} - (1+a)e^{2x} - a(1+a)e^{2ax}\big],\\ &\nu''(x) = 4(1+a^2)\big[(a-1)^2e^{2(1+a)x} + 2ae^{(1+a)x} - a^2e^{2ax} - e^{2x}\big].\end{aligned}$$

Since $0 < a < 1$, simple computation yields $\nu''(x) > 0$ and $\nu'(0) = \nu(0) = 0$. Then we conclude that $\mu(x)$ is strictly increasing for $x > 0$ and $0 < a < 1$. Moreover,

$$\mu(x) > \lim_{x\to 0+} \mu(x) = a.$$

If $2(\beta-1)a + a > 0$, which is equivalent to $\beta > \frac{1}{2}$, we obtain that $\varphi_\beta(x)$ is strictly increasing with $0 < a < 1$ and $x > 0$. This completes the proof. □

3. Proofs of theorems

Proof of Theorem 1. Differentiating $g_n(x;c,\beta)$ and applying (1) and (2) yields

$$\begin{aligned}\frac{g_n'(x;c,\beta)}{x^c} &= \frac{c}{x}|\psi^{(n)}(x+\beta)| - |\psi^{(n+1)}(x+\beta)|\\ &= c\int_0^\infty e^{-xt}\,\mathrm{d}t\int_0^\infty e^{-(x+\beta)t}\frac{t^n}{1-e^{-t}}\,\mathrm{d}t - \int_0^\infty e^{-(x+\beta)t}\frac{t^{n+1}}{1-e^{-t}}\,\mathrm{d}t\end{aligned}$$

Using Lemma 4 leads to

$$\frac{g_n'(x;c,\beta)}{x^c} = \int_0^\infty e^{-xt}h(t;c,\beta)\,\mathrm{d}t,$$

where

$$h(t;c,\beta) = c\int_0^\infty \frac{s^n e^{-s\beta}}{1-e^{-s}}\,\mathrm{d}s - \frac{t^{n+1}e^{-\beta t}}{1-e^{-t}}. \tag{7}$$

If $c \leqslant 0$, then we have $h(t;c,\beta) < 0$, or $g_n'(x;c,\beta) < 0$.

A simple calculation gives

$$p(t;c,\beta) \triangleq e^{(1+\beta)t}t^{-n}(1-e^{-t})^2 h'(t;c,\beta) = (e^t-1)(c-n-1+\beta t)+t. \tag{8}$$

It is clear that $p(t;c,\beta) > 0$ on $(0,\infty)$ is equivalent with

$$c-n-1 > \frac{\beta t e^t - \beta t + t}{1-e^t} = \phi_\beta(t).$$

From Lemma 6, we obtain that $\phi_\beta(t) < \lim_{x\to 0+}\phi_\beta(t) = -1$ with $\beta > \frac{1}{2}$. Thus, if $c \geqslant n$, then we have $p(t;c,\beta) > 0$ and $h'(t;c,\beta) > 0$ on $(0,\infty)$. Since $h(t;c,\beta)$ is increasing and $\lim_{t\to 0+} h(t;c,\beta)(t) = 0$, it is obtained that $h(t;c,\beta) > 0$ on $(0,\infty)$, which implies that $g_n'(x;c,\beta) > 0$ and $g_n(x;c,\beta)$ is strictly increasing for $x \in (0,\infty)$.

On the other hand, if $g_n(x;c,\beta)$ is strictly decreasing on $(0,\infty)$, then we obtain for $x > 0$ and $\beta > 1/2$:

$$\frac{g_n'(x;c,\beta)}{x^{c-1}} = c|\psi^{(n)}(x+\beta)| - x|\psi^{(n+1)}(x+\beta)| \leqslant 0.$$

This implies $c \leqslant 0$. Assume that $g_n(x;c,\beta)$ is strictly increasing, then we have for $x > 0$ and $\beta > 1/2$:

$$\frac{g_n'(x;c,\beta)}{x^{c-n-1}} = cx^n|\psi^{(n)}(x+\beta)| - x^{n+1}|\psi^{(n+1)}(x+\beta)| \geqslant 0.$$

Applying the asymptotic formula (4) we get

$$\lim_{x\to\infty} x^{n+1-c} g_n'(x;c,\beta) = c(n-1)! - n! \geqslant 0.$$

This implies $c \geqslant n$. The proof is complete. □

Proof of Theorem 2. Applying Lemma 4 we obtain for $x \in [0,\infty)$

$$-f_n(x;\beta) = -x\frac{\psi^{(n+1)}(x+\beta)}{\psi^{(n)}(x+\beta)} = \frac{|\psi^{(n+1)}(x+\beta)|}{\frac{1}{x}|\psi^{(n)}(x+\beta)|} = \frac{\int_0^\infty e^{-xt}P(t;\beta)\,\mathrm{d}t}{\int_0^\infty e^{-xt}Q(t;\beta)\,\mathrm{d}t},$$

where

$$P(t;\beta) = \frac{t^{n+1}e^{-\beta t}}{1-e^{-t}} \text{ and } Q(t;\beta) = \int_0^t \frac{s^n e^{-\beta s}}{1-e^{-s}}\,\mathrm{d}s.$$

Differentiating and applying convolution theorem of Laplace transform again, we get

$$\begin{aligned}
&-f_n'(x;\beta)\left(\int_0^\infty e^{-xt}Q(t;\beta)\,\mathrm{d}t\right)^2\\
&= \int_0^\infty e^{-xt}P(t;\beta)\,\mathrm{d}t\int_0^\infty e^{-xt}tQ(t;\beta)\,\mathrm{d}t - \int_0^\infty e^{-xt}tP(t;\beta)\,\mathrm{d}t\int_0^\infty e^{-xt}Q(t;\beta)\,\mathrm{d}t\\
&= \int_0^\infty e^{-xt}U(t;\beta)\,\mathrm{d}t,
\end{aligned}$$

where

$$U(t;\beta) = \int_0^\infty (t-2s)P(s;\beta)Q(t-s;\beta)\,\mathrm{d}s.$$

Let $s = \frac{t}{2}(1+y)$, then we have

$$U(t;\beta) = -\frac{t^2}{2}\int_{-1}^{1} yP\Big(\frac{p}{2}(1+y)\Big);\beta)Q\Big(\frac{p}{2}(1-y);\beta\Big)\,\mathrm{d}y. \tag{9}$$

Let $\delta \in \mathbb{R}_+$, we define for $y \in (0,1)$

$$v(y;\beta) = yP(\delta(1+y);\beta)Q(\delta(1-y);\beta) \text{ and } w(y;\beta) = v(y;\beta) + v(-y;\beta). \tag{10}$$

A simple calculation gives

$$\begin{aligned}\frac{1}{\delta^{n+1}y}w(y;\beta) &= \frac{(1+y)^{n+1}e^{-\beta\delta(1+y)}}{1-e^{-\delta(1+y)}}\int_0^{\delta(1-y)}\frac{s^n e^{-\beta s}}{1-e^{-s}}\,\mathrm{d}s\\ &\quad -\frac{(1-y)^{n+1}e^{-\beta\delta(1-y)}}{1-e^{-\delta(1-y)}}\int_0^{\delta(1+y)}\frac{s^n e^{-\beta s}}{1-e^{-s}}\,\mathrm{d}s.\end{aligned} \tag{11}$$

Using the substitution $s = \delta(1-y)z$ in the first integral and $s = \delta(1+y)z$ in the second integral, we get from (11)

$$\frac{1}{\delta^{2(n+1)}y(1-y^2)^{n+1}}w(y;\beta) = \int_0^1 \frac{z^k e^{-\beta\delta(1+y)z}e^{-\beta\delta(1+y)}(\Delta_\beta(z)-\Delta_\beta(1))}{(1-e^{-\delta(1+y)z})(1-e^{-\delta(1+y)})}\,\mathrm{d}z, \tag{12}$$

where

$$\Delta_\beta(z) = \frac{e^{\beta\delta(1+y)z}-e^{(\beta-1)\delta(1+y)z}}{e^{\beta\delta(1-y)z}-e^{(\beta-1)\delta(1-y)z}} \quad (y\in(0,1), \delta\in\mathbb{R}_+).$$

By Lemma 7, we obtain that the function $\Delta_\beta(z)$ is strictly increasing on $(0,1]$ with $\beta \geqslant \frac{1}{2}$. so we conclude from (12) that $w(y;\beta)$ is negative on $(0,1)$ with $\beta \geqslant \frac{1}{2}$. Hence, (9) and (10) imply

$$U(2\delta;\beta) = -2\delta^2\int_{-1}^{1} v(y;\beta)\,\mathrm{d}y = -2\delta^2\int_0^1 w(y;\beta)\,\mathrm{d}y > 0$$

for $\delta \in \mathbb{R}_+$. From $f'(x;\beta) < 0$, we know that the function $f_n(x;\beta)$ is strictly decreasing for $x \in [0,\infty)$ and $\beta \geqslant \frac{1}{2}$. This completes the proof.

Next, as the aim of researching psi function, we provide another proof of Theorem 2 as follows.

Differentiating $f_n(x;\beta)$ and applying (2), (6), we get

$$\begin{aligned}&\big(\psi^{(n)}(x+\beta)\big)^2 f_n'(x;\beta)\\ &= \psi^{(n+1)}(x+\beta)\psi^{(n)}(x+\beta) + x\Big[\psi^{(n+2)}(x+\beta)\psi^{(n)}(x+\beta) - \big(\psi^{(n+1)}(x+\beta)\big)^2\Big]\\ &< \psi^{(n+1)}(x+\beta)\psi^{(n)}(x+\beta) + \frac{x}{n}\big(\psi^{(n+1)}(x+\beta)\big)^2\\ &= (-1)^{n+2}x\int_0^\infty \frac{t^{n+1}e^{-(x+\beta)t}}{1-e^{-t}}\,\mathrm{d}t\Big[\frac{1}{x}\psi^{(n)}(x+\beta) + \frac{1}{n}\psi^{(n+1)}(x+\beta)\Big]\\ &\triangleq (-1)^{n+2}x\int_0^\infty \frac{t^{n+1}e^{-(x+\beta)t}}{1-e^{-t}}\,\mathrm{d}tH(t;\beta).\end{aligned}$$

Using Lemma 4 leads to

$$\begin{aligned} &H(t;\beta) \\ &= (-1)^{n+2}\left[-\int_0^\infty e^{-xt}\,\mathrm{d}t\int_0^\infty \frac{t^n e^{-(x+\beta)t}}{1-e^{-t}}\,\mathrm{d}t + \frac{1}{n}\int_0^\infty \frac{t^{n+1}e^{-(x+\beta)t}}{1-e^{-t}}\,\mathrm{d}t\right] \\ &\triangleq (-1)^{n+2}\int_0^\infty e^{-xt}G(t;\beta)\,\mathrm{d}t, \end{aligned}$$

where

$$G(t;\beta) = \frac{1}{n}\frac{t^{n+1}e^{-\beta t}}{1-e^{-t}} - \int_0^t \frac{s^n e^{-\beta s}}{1-e^{-s}}\,\mathrm{d}s.$$

A simple calculation shows that

$$\begin{aligned} e^{(\beta+1)t}t^{-n}(1-e^{-t})^2 nG'(t;\beta) &= (e^t-1)(1-\beta t)-t \\ &= \sum_{k=1}^\infty \frac{t^{k+1}[1-\beta(k+1)]}{(k+1)!}. \end{aligned}$$

If $\beta \geqslant \frac{1}{2}$, then we obtain $G'(t;\beta) < 0$. From $\lim_{x\to 0+} G(t;\beta) = 0$, we conclude $f_n'(x;\beta) < 0$. This implies that $f_n(x;\beta)$ is strictly decreasing for $x \in [0,\infty)$ and $\beta \geqslant \frac{1}{2}$. □

REMARK 2. The approach to the first proof of Theorem 2 is suggested by Alzer [4].

Proof of Theorem 3. Without loss of generality, we suppose $k-n=i$. From Theorem 2 we conclude that the function $f_n(x;\beta)$ is strictly decreasing for $x \in [0,\infty)$ and $\beta \geqslant \frac{1}{2}$. So we have

$$x\frac{\psi^{(k)}(x+\beta)}{\psi^{(k-1)}(x+\beta)} > y\frac{\psi^{(k)}(y+\beta)}{\psi^{(k-1)}(y+\beta)}$$

for $y > x > 0$. It is equivalent to

$$\frac{\psi^{(k)}(x+\beta)}{\psi^{(k)}(y+\beta)} < \frac{y\psi^{(k-1)}(x+\beta)}{x\psi^{(k-1)}(y+\beta)}.$$

Applying the same method, we obtain

$$\frac{\psi^{(k)}(x+\beta)}{\psi^{(k)}(y+\beta)} < \frac{y^i\psi^{(n)}(x+\beta)}{x^i\psi^{(n)}(y+\beta)}.$$

If $k-n$ is odd, we have

$$x^{k-n}\frac{\psi^{(k)}(x+\beta)}{\psi^{(n)}(x+\beta)} > y^{k-n}\frac{\psi^{(k)}(y+\beta)}{\psi^{(n)}(y+\beta)}$$

for $y > x > 0$. This implies that the function $f_{k,n}(x;\beta)$ is strictly decreasing for $x \in [0,\infty)$ and $\beta \geqslant \frac{1}{2}$. If $k-n$ is even, then we have

$$x^{k-n}\frac{\psi^{(k)}(x+\beta)}{\psi^{(n)}(x+\beta)} < y^{k-n}\frac{\psi^{(k)}(y+\beta)}{\psi^{(n)}(y+\beta)}$$

for $y > x > 0$. This implies that $f_{k,n}(x;\beta)$ is strictly increasing for $x \in [0,\infty)$ and $\beta \geqslant \frac{1}{2}$. □

Applying Theorem 3 and (4) we get upper and lower bounds for the ratio $|\psi^{(k)}(x+\beta)|/|\psi^{(n)}(x+\beta)|$ immediately.

4. A completely monotonic function

A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I which alternate successively in sign, that is

$$(-1)^n f^{(n)}(x) \geqslant 0 \quad (x \in I; n = 0,1,2,\cdots). \tag{13}$$

If the inequality (13) is strict, then f is said to be strictly completely monotonic on I.

Completely monotonic functions have remarkable applications in different branches. For instance, they play a role in potential theory [11], probability theory [12, 14, 16], physics [13], numerical and asymptotic analysis [15, 25], and combinatorics [9]. A detailed collection of the most important properties of completely monotonic functions can be found in [24, Chaptre IV], and in the abstract in [10].

Now, we establish completely monotonicity for polygamma function in the following theorem.

THEOREM 4. *The function* $\frac{d\psi(\ln x)}{dx}$ *is strictly completely monotonic on* $(1,\infty)$. *In other words,*

$$(-1)^{n+1}\frac{d^n\psi(\ln x)}{dx^n} > 0$$

for $x \in (1,\infty)$.

Proof. From the formula of higher derivatives of composite function [22, p. 197]: If the functions $y = f(u)$, $u = g(x)$ have lth derivative, then

$$\frac{d^n}{dx^n}\big(f(g(x))\big) = \sum_{\substack{1\leqslant i\leqslant n \\ \sum_l i_k = i \\ \sum_{k=1} k i_k = n}} \frac{n!f^{(i)}}{i_1!i_2!\cdots i_l!}\left(\frac{u^{(1)}}{1!}\right)^{i_1}\left(\frac{u^{(2)}}{2!}\right)^{i_2}\cdots\left(\frac{u^{(l)}}{l!}\right)^{i_l}, \tag{14}$$

where

$$f^{(i)} = \frac{d^if}{du^i}, \quad u^{(k)} = \frac{d^ku}{dx^k}.$$

and the formula

$$\ln^{(n)} x = (-1)^{n-1}(n-1)!\frac{1}{x^n},$$

we get for $x \in (1,\infty)$ and $n = 1,2,\ldots$

$$x^n \frac{d^n}{dx^n}\big(\psi(\ln x)\big) = x^n \sum_{\substack{1\leqslant i\leqslant n \\ \sum i_k = i \\ \sum_{k=1}^{l} k i_k = n}} \frac{n!\psi^{(i)}}{i_1!i_2!\cdots i_l!}\left(\frac{\ln^{(1)} x}{1!}\right)^{i_1}\left(\frac{\ln^{(2)} x}{2!}\right)^{i_2}\cdots\left(\frac{\ln^{(l)} x}{l!}\right)^{i_l}$$

$$= \sum \frac{n!\psi^{(i)}}{i_1!i_2!\cdots i_l!}\Big(-1\Big)^{i_1(1-1)}\Big(-\frac{1}{2}\Big)^{i_2(2-1)}\cdots\Big(-\frac{1}{l}\Big)^{i_l(l-1)}$$

$$= \sum \frac{n!\psi^{(i)}}{i_1!i_2!\cdots i_l!}\frac{(-1)^{i_1+2i_2+\cdots+li_l-(i_1+i_2+\cdots+i_l)}}{1\cdot 2^{i_2}\cdots i^{i_l(l-1)}}$$

$$= \sum \frac{n!\psi^{(i)}}{i_1!i_2!\cdots i_l!}\frac{(-1)^{n-i}}{1\cdot 2^{i_2}\cdots i^{i_l(l-1)}}.$$

Using (2), if n is odd,

$$\frac{d^n}{dx^n}\big(\psi(\ln x)\big)\begin{cases} >0, & i \text{ is odd,} \\ >0, & i \text{ is even.}\end{cases}$$

If n is even,

$$\frac{d^n}{dx^n}\big(\psi(\ln x)\big)\begin{cases} <0, & i \text{ is odd,} \\ <0, & i \text{ is even.}\end{cases}$$

These imply $\frac{d\psi(\ln x)}{dx}$ is strictly completely monotonic on $(1,\infty)$. This completes the proof. □

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