Mathematical
Inequalities
& Applications
Volume 11, Number 2 (2008), 317–325

APPROXIMATION BY A KANTOROVICH VARIANT OF THE BLEIMANN, BUTZER AND HAHN OPERATORS

XIAO-MING ZENG [1] , ULRICH ABEL AND MIRCEA IVAN

(communicated by Th. Rassias)

Abstract. We study the approximation properties of a Kantorovich variant of the Bleimann, Butzer and Hahn operators for locally bounded functions, and estimate their rate of convergence by some techniques of probability theory and analysis methods.

1. Introduction

In 1980, Bleimann, Butzer and Hahn [4] introduced a sequence of positive linear Bernstein-type operators (abbreviated in the following by BBH operators) defined on the space of real functions on the infinite interval $I = [0, \infty)$ by

$$L_n(f;x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n+1-k}\right), \quad x \in I,\ n \in \mathbb{N}, \tag{1}$$

where

$$b_{n,k}(x) = \binom{n}{k} x^k (1+x)^{-n}.$$

The approximation-theoretical properties of the operators L_n have been studied by several authors (cf. the references, in particular the book of Altomare and Campiti [3]. Recently, Abel and Ivan [2] introduced a Kantorovich variant of the BBH operators as an approximation process for locally integrable functions defined by

$$K_n(f;x) = (n+2) \sum_{k=0}^{n} b_{n,k}(x) \int_{\frac{k}{n+2-k}}^{\frac{k+1}{n+1-k}} \frac{f(t)}{(1+t)^2} dt \quad (n \in \mathbb{N}) \tag{2}$$

The operators K_n are called BBHK operators. Their basic approximation properties when applied to continuous or differentiable functions can be found in [2].

Mathematics subject classification (2000): 41A36, 41A25, 41A10.

Key words and phrases: Bleimann, Butzer and Hahn Operators, rate of convergence, probabilistic methods, Kantorovich variant, basis functions and moments.

[1] The first author was supported by NSFC under Grant 10571145.

In this paper we consider the rate of convergence of the operators K_n for the following class of locally bounded functions of exponential growth:

$$E = \{ f : I \to \mathbb{R} \mid f \text{ is locally bounded on } I \text{ and, for a constant } A, f(t) = O(e^{At}) \text{ as } t \to \infty \} .$$

For $f \in E$, $x \in I$ and $\lambda \geqslant 0$, set

$$\omega_x(f,\lambda) = \sup_{t\in[x-\lambda,x+\lambda]} |f(t) - f(x)|.$$

It is clear that

(i) $\omega_x(f,\lambda)$ is monotone increasing with respect to λ,

(ii) $\lim_{\lambda\to 0} \omega_x(f,\lambda) = 0$, if f is continuous at the point x,

(iii) if f is of bounded variation on $[a,b]$, and $\bigvee_a^b(f)$ denotes the total variation of f on $[a,b]$, then $\omega_x(f,\lambda) \leqslant \bigvee_{x-\lambda}^{x+\lambda}(f)$.

For further properties of $\omega_x(f,\lambda)$ we refer to Zeng and Cheng [14].

In the case that for $f \in E$ and $x \in (0,\infty)$ both one-sided limits $f(x+)$, $f(x-)$ exist the function f_x is defined as

$$f_x(t) = \begin{cases} f(t) - f(x+), & x < t \leqslant 1, \\ 0, & t = x, \\ f(t) - f(x-), & 0 \leqslant t < x. \end{cases} \tag{3}$$

Now let us state our main result as follows:

THEOREM 1. *Let $f \in E$ with $|f(t)| \leqslant Me^{At}$ on I. Assume that $f(x+)$, $f(x-)$ exist at a fixed point $x \in (0,\infty)$. Then for all $n \geqslant 1$ we have*

$$\begin{aligned} \left| K_n(f,x) - \frac{1}{2}(f(x+) + f(x-)) \right| \leqslant{} & \frac{x^2 + 4(1+x)^4}{nx^2} \sum_{k=1}^{n} \omega_x\left(f_x, x/\sqrt{k}\right) \\ & + \frac{1+2x}{\sqrt{nx}} (|f(x+) - f(x-)| + R_n(x), \end{aligned} \tag{4}$$

where

$$R_n(x) = 2Me^{-A}(1+x)^2 \exp\left(-(n+2)\left(\frac{x}{16(1+x)^2} - A \right) \right). \tag{5}$$

If f is of polynomial growth with $|f(t)| \leqslant M(1+t)^\gamma$ on I for some $\gamma \geqslant 0$ we can put

$$R_n(x) = 2M(n+2)^\gamma (1+x)^2 \exp\left(\frac{-(n+2)x}{16(1+x)^2} \right). \tag{6}$$

REMARK 1. The term $R_n(x)$ in estimate (5) decays exponentially fast with $n \to \infty$ for each $x > 0$ with

$$\frac{x}{16(1+x)^2} > A.$$

According to Eq. (6) this is always the case if f is of polynomial growth.

2. Auxiliary Results

We first give several auxiliary results, which mainly are some estimates concerning the basis functions and moments of BBHK operators. Some results and techniques of probability theory play important roles in this section.

Throughout this paper we define

$$H_n(x,t) = (n+2)\sum_{k=0}^{n} b_{n,k}(x)\,\frac{\varphi_{n,k}(t)}{(1+t)^2}, \tag{7}$$

where $\varphi_{n,k}$ denotes the characteristic function of the interval $I_k = [k/(n+2-k), (k+1)/(n+1-k)]$ with respect to $I = [0,\infty)$. With H_n as kernel function the BBHK operators can be written in the form

$$K_n(f,x) = \int_0^\infty f(t)H_n(x,t)dt.$$

LEMMA 2. *For* $0<y<x$, *there holds*

$$\int_0^y H_n(x,t)dt \leqslant \frac{(1+x)^4}{n(x-y)^2}.$$

Proof. From Lemma 3.1 of [2] it is known that

$$K_n((t-x)^2,x) \leqslant \frac{x(1+x)^2}{n+1} + \frac{(1+x)^4}{(n+1)(n+3)} \leqslant \frac{(1+x)^4}{n}.$$

Thus

$$\int_0^y H_n(x,t)dt \leqslant \int_0^y \frac{(x-t)^2}{(x-y)^2}H_n(x,t)dt \leqslant \frac{K_n((x-t)^2,x)}{(x-y)^2} \leqslant \frac{(1+x)^4}{n(x-y)^2}.$$

□

LEMMA 3. *Let* $\{\xi_k\}_{k=1}^\infty$ *be a sequence of independent random variables with the same two-point distribution*

$$P(\xi_i = j) = \left(\frac{x}{1+x}\right)^j \left(\frac{1}{1+x}\right)^{1-j}, \qquad j=0,1, \qquad x\in[0,\infty).$$

Then

$$E\xi_1 = \frac{x}{1+x},\ \ E(\xi_1-E\xi_1)^2 = \frac{x}{(1+x)^2},\ \ E(\xi_1-E\xi_1)^4 = \frac{x(x^2-x+1)}{(1+x)^4}, \tag{8}$$

and

$$E|\xi_1-E\xi_1|^3 \leqslant \frac{x\sqrt{x^2-x+1}}{(1+x)^3}.$$

Proof. Direct calculation derives Eq. (8), and by Hölder inequality we get

$$E|\xi_1-E\xi_1|^3 \leqslant \sqrt{E(\xi_1-E\xi_1)^4E(\xi_1-E\xi_1)^2} \leqslant \frac{x\sqrt{x^2-x+1}}{(1+x)^3}.$$

□

The following lemma is the well-known Berry-Esseen bound for the central limit theorem of probability theory [7, 12].

LEMMA 4. *Let* $\{\xi_k\}_{k=1}^{\infty}$ *be a sequence of independent and identically distributed random variables with the expectation* $E\xi_1$, *the variance* $E(\xi_1 - E\xi_1)^2 = \sigma^2 > 0$, $E|\xi_1 - E\xi_1|^3 = \rho < \infty$, *and let* F_n *stand for the distribution function of* $\sum_{k=1}^{n}(\xi_k - E\xi_1)/(\sigma\sqrt{n})$. *Then there exists an absolute constant* C, $1/\sqrt{2\pi} \leqslant C < 0.8$, *such that for all t and n*

$$\left|F_n(t) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-u^2/2}du\right| < \frac{C\rho}{\sigma^3\sqrt{n}}.$$

LEMMA 5. *If* $x \in \left[\frac{k'}{n+2-k'}, \frac{k'+1}{n+1-k'}\right)$ *for some nonnegative integer* $k' \leqslant n$, *then*

$$\left|\sum_{k=k'+1}^{n} b_{n,k}(x) - \frac{1}{2}\right| \leqslant \frac{1+2x}{\sqrt{nx}}.$$

Proof. Let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence of independent random variables with the same two-point distribution

$$P(\xi_i = j) = \left(\frac{x}{1+x}\right)^{j}\left(\frac{1}{1+x}\right)^{1-j}, \qquad j = 0, 1, \qquad x \in [0, \infty).$$

and let $\eta_n = \sum_{i=1}^{n}\xi_i$. Then the probability distribution of the random variable η_n is

$$P(\eta_n = k) = \binom{n}{k} x^k(1+x)^{-n} = b_{n,k}(x).$$

Thus,

$$\begin{aligned}\sum_{k=k'+1}^{n} b_{n,k}(x) &= \sum_{\frac{(n+2)x}{1+x}<k\leqslant n} b_{n,k}(x) = P\left(\eta_n > \frac{(n+2)x}{1+x}\right)\\ &= 1 - P\left(\frac{\eta_n - nx/(1+x)}{\sqrt{nx}/(1+x)} \leqslant \frac{2\sqrt{x}}{\sqrt{n}}\right) = 1 - F_n\left(\frac{2\sqrt{x}}{\sqrt{n}}\right).\end{aligned}$$

Hence by Lemma 3, Lemma 4 and direct computations we obtain

$$\begin{aligned}\left|\sum_{k=k'+1}^{n} b_{n,k}(x) - \frac{1}{2}\right| &= \left|\frac{1}{2} - F_n\left(\frac{2\sqrt{x}}{\sqrt{n}}\right)\right|\\ &\leqslant \frac{C\rho}{\sigma^3\sqrt{n}} + \frac{1}{\sqrt{2\pi}}\int_0^{2\sqrt{x}/\sqrt{n}} e^{-u^2/2}du\\ &< \frac{0.8\sqrt{x^2-x+1}}{\sqrt{nx}} + \frac{1}{\sqrt{2\pi}}\frac{2x}{\sqrt{nx}} \leqslant \frac{1+2x}{\sqrt{nx}}.\end{aligned}$$

□

LEMMA 6. [9, Lemma 2.3] *Let* $x > 0,\ \delta > 0$ *and put*

$$d_\delta(x) = \frac{\delta^2}{16x(1+x)^2}.$$

Then there holds

$$\sum_{|k/(n-k)-x| \geqslant \delta} b_{n,k}(x) \leqslant 2e^{-nd_\delta(x)} \qquad (0 < \delta \leqslant x,\ n \in \mathbb{N}).$$

3. Proof of Theorem 1

For any $f \in E$, if $f(x+)$ and $f(x-)$ exist at x, Bojanic-Cheng decomposition yields

$$\begin{aligned} f(t) = \frac{f(x+)+f(x-)}{2} &+ f_x(t) + \frac{f(x+)-f(x-)}{2}\operatorname{sgn}_x(t) \\ &+ \delta_x(t)\left[f(x) - \frac{f(x+)+f(x-)}{2}\right], \end{aligned}$$

where f_x is defined in (3) and

$$sgn_x(t) = \begin{cases} 1, & t > x, \\ 0, & t = x, \\ -1, & t < x, \end{cases} \qquad \delta_x(t) = \begin{cases} 1, & t = x, \\ 0, & t \neq x. \end{cases}$$

Obviously

$$K_n(\delta_x, x) = 0.$$

Hence it follows that

$$\left|K_n(f,x) - \frac{f(x+)+f(x-)}{2}\right| \leqslant |K_n(f_x,x)| + \left|\frac{f(x+)-f(x-)}{2}K_n(\operatorname{sgn}_x,x)\right|. \tag{9}$$

We need to estimate $|K_n(\operatorname{sgn}_x, x)|$ and $|K_n(f_x,x)|$.

Let $x \in [k'/(n+2-k'),\ (k'+1)/(n+1-k'))$ for some k'. Then by Lemma 5 combining some direct computations, we have

$$\begin{aligned} K_n(\operatorname{sgn}_x, x) &= -\sum_{k=0}^{k'-1} b_{n,k}(x) + \sum_{k=k'+1}^{n} b_{n,k}(x) + b_{n,k'}(x)\int_{I_{k'}} \frac{\operatorname{sgn}_x(t)}{(1+t)^2}dt \\ &\leqslant \left|2\sum_{k=k'+1}^{n} b_{n,k}(x) - 1\right| + 2b_{n,k'}(x) \\ &\leqslant \frac{1+2x}{\sqrt{nx}} + \frac{2+2x}{\sqrt{2enx}} < \frac{2+4x}{\sqrt{nx}}. \end{aligned} \tag{10}$$

Now it is clear from (9) and (10) that Theorem 1 will be proved if we establish that

$$|K_n(f_x,x)| \leqslant \frac{x^2+4(1+x)^4}{nx^2}\sum_{k=1}^{n}\omega_x(f_x,x\sqrt{k})+R_n\,(x)\,. \tag{11}$$

Recalling the Lebesgue-Stieltjes integral representation we have

$$K_n(f_x,x)=\int_{[0,\infty)} f_x(t)H_n(x,t)dt,$$

where H_n is as defined in (7). We divide $K_n(f_x,x)$ into four parts

$$K_n(f_x,x)=\sum_{j=1}^{4}\int_{I_j} f_x(t)H_n(x,t)dt,$$

where

$$I_1 := [0,\ x-x/\sqrt{n}], \quad I_2 := (x-x/\sqrt{n},\ x+x/\sqrt{n}],$$
$$I_3 := (x+x/\sqrt{n},\ 2x], \quad I_4 := (2x,\ \infty).$$

Firstly, note that $f_x(x)=0$. Thus

$$|\int_{I_2} f_x(t)H_n(x,t)dt| \leqslant \omega_x(f_x,x/\sqrt{n}). \tag{12}$$

Next we estimate $|\int_{I_1} f_x(t)H_n(x,t)dt|$. Note that $|f_x(t)| \leqslant \omega_x(f_x,\ x-t)$, it follows that

$$\left|\int_{I_1} f_x(t)H_n(x,t)dt\right| \leqslant \int_{I_1} \omega_x(f_x,x-t)H_n(x,t)dt.$$

Integration by parts with $y=x-x/\sqrt{n}$ and using Lemma 2, we have

$$\begin{aligned}
&\int_{I_1} \omega_x(f_x,x-t)H_n(x,t)dt\\
&\qquad \leqslant \omega_x(f_x,x-y)\int_0^y H_n(x,u)du+\int_0^y [\int_0^t H_n(x,u)du]d_t(-\omega_x(f_x,x-t))\\
&\qquad \leqslant \omega_x(f_x,x-y)\frac{(1+x)^4}{n(x-y)^2}+\frac{(1+x)^4}{n}\int_0^y \frac{d_t(-\omega_x(f_x,x-t))}{(x-t)^2}.
\end{aligned} \tag{13}$$

Since

$$\int_0^y \frac{d(-\omega_x(f_x,x-t))}{(x-t)^2}=-\frac{\omega_x(f_x,x-y)}{(x-t)^2}+\frac{\omega_x(f_x,x)}{x^2}+\int_0^y \frac{2\omega_x(f_x,x-t)}{(x-t)^3}dt.$$

So from (13) we have

$$\left|\int_{I_1} f_x(t)H_n(x,t)dt\right| \leqslant \frac{(1+x)^4}{nx^2}\omega_x(f_x,x)+\frac{(1+x)^4}{n}\int_{I_1}\frac{2\omega_x(f_x,x-t)}{(x-t)^3}dt.$$

Putting $t=x-x/\sqrt{u}$ in the last integral we obtain

$$\int_0^{x-x/\sqrt{n}} \omega_x(f_x,x-t)\frac{2}{(x-t)^3}dt=\frac{1}{x^2}\int_1^n \omega_x(f_x,x/\sqrt{u})du.$$

Consequently

$$\left|\int_{I_1} f_x(t)H_n(x,t)dt\right| \leqslant \frac{(1+x)^4}{nx^2}\left(\omega_x(f_x,x)+\int_1^n \omega_x(f_x,x/\sqrt{u})du\right).$$

$$\leqslant \frac{2(1+x)^4}{nx^2}\sum_{k=1}^{n}\omega(f_x,x/\sqrt{k}). \tag{14}$$

Using the similar method we get

$$\left|\int_{I_3} f_x(t)H_n(x,t)dt\right| \leqslant \frac{2(1+x)^4}{nx^2}\sum_{k=1}^{n}\omega(f_x,x/\sqrt{k}) \tag{15}$$

Finally, we estimate

$$\left|\int_{I_4} f_x(t)H_n(x,t)dt\right|.$$

Since $|f\ (t)| \leqslant Me^{At}$ on I and

$$\int_{\frac{k}{n+2-k}}^{\frac{k+1}{n+1-k}} (1+t)^{-2}dt = (n+2)^{-1}$$

we conclude that

$$\left|\int_{I_4} f_x(t)H_n(x,t)dt\right| \leqslant Me^{A(n+1)} \sum_{\substack{0\leqslant k\leqslant n\\ k/(n+2-k)\geqslant 2x}} b_{n,k}(x).$$

Using the relation

$$b_{n,k}(x) = \frac{(n+2-k)(n+1-k)}{(n+2)(n+1)}(1+x)^2 b_{n+2,k}(x) \leqslant (1+x)^2 b_{n+2,k}(x) \tag{16}$$

we obtain

$$\left|\int_{I_4} f_x(t)H_n(x,t)dt\right| \leqslant Me^{A(n+1)}(1+x)^2 \sum_{k/(n+2-k)\geqslant 2x} b_{n+2,k}(x).$$

An application of Lemma 6 with $\delta = x$ yields

$$\left|\int_{I_4} f_x(t)H_n(x,t)dt\right| \leqslant 2Me^{-A}(1+x)^2 \exp\left(-(n+2)\left(\frac{x}{16(1+x)^2}-A\right)\right). \tag{17}$$

In a similar way, we prove the estimate for functions of polynomial growth satisfying $|f\ (t)| \leqslant M(1+t)^{\gamma}$ on I for some $\gamma \geqslant 0$. An application of the mean value theorem for integrals yields

$$\int_{\frac{k}{n+2-k}}^{\frac{k+1}{n+1-k}} (1+t)^{\gamma-2}dt = \frac{n+2}{(n+2-k)(n+1-k)}(1+\xi)^{\gamma-2}$$

with $\frac{k}{n+2-k} < \xi < \frac{k+1}{n+1-k}$. For $\frac{k}{n+2-k} \geqslant 2x$, it is easily to verify that

$$(1+\xi)^{\gamma-2} \leqslant (n+2)^{\gamma-1}(n+1).$$

Hence, by Eq. (16) we conclude that

$$\begin{aligned}(n+2)\, b_{n,k}(x) \int_{\frac{k}{n+2-k}}^{\frac{k+1}{n+1-k}} \frac{|f(t)|}{(1+t)^2} dt &\leqslant \frac{(n+2)^2(1+\xi)^{\gamma-2}}{(n+2-k)(n+1-k)} M b_{n,k}(x) \\ &= \frac{n+2}{n+1}(1+\xi)^{\gamma-2} M\,(1+x)^2\, b_{n+2,k}(x)\end{aligned}$$

and we obtain,

$$\left| \int_{I_4} f_x(t) H_n(x,t)\,dt \right| \leqslant \frac{n+2}{n+1} M\,(1+x)^2 \sum_{k/(n+2-k)\geqslant 2x} (1+\xi)^{\gamma-2} b_{n+2,k}(x).$$

Finally, an application of Lemma 6 with $\delta = x$ yields

$$\left| \int_{I_4} f_x(t) H_n(x,t)\,dt \right| \leqslant 2M(n+2)^{\gamma}\,(1+x)^2 \exp\left(\frac{-(n+2)\,x}{16\,(1+x)^2} \right). \tag{18}$$

Collecting the estimates (12), (14), (15), and (17) resp. (18), we obtain

$$|K_n(f_x,x)| \leqslant \frac{x^2+4(1+x)^4}{nx^2} \sum_{k=1}^{n} \omega\left(f_x, x/\sqrt{k}\right) + R_n(x),$$

with $R_n(x)$ as defined in (5) and (6), respectively. □

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(Received October 1, 2006)

Xiao-Ming Zeng
Department of Mathematics
Xiamen University
Xiamen 361005
P. R. China
e-mail: xmzeng@xmu.edu.cn

Ulrich Abel
Fachhochschule Giessen-Friedberg
University of Applied Sciences
Fachbereich MND
Wilhelm-Leuschner-Strasse 13
61169 Friedberg
Germany
e-mail: Ulrich.Abel@mnd.fh-friedberg.de

Mircea Ivan
Department of Mathematics
Technical University of Cluj-Napoca
Str. C. Daicoviciu 15
400020 Cluj-Napoca
Romania
e-mail: Mircea.Ivan@math.utcluj.ro