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SOME PROPERTIES OF THE JUNG–KIM–SRIVASTAVA INTEGRAL OPERATOR

A. A. ATTIYA

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Abstract. The purpose of the present paper is to derive an important inequality of the integral operator $I^{\sigma}(f)$ which was introduced by Jung, Kim and Srivastava. [*J. Math. Anal. Appl. 176(1993), 138–147*]. Using the technique of differential subordination, an interesting property of $I^{\sigma}(f)$ is also obtained.

1. Introduction

Let A denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

A function $f(z)$ in the class A is said to be in the class $S^*(\alpha)$ of *starlike functions of order* α, if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha (z \in \mathbb{U}), \tag{1.2}$$

for some α $(0 \leqslant \alpha < 1)$. Also, we write $S(0) = S^*$, the class of starlike functions in $\mathbb{U}$.

For $f(z) \in A$ and $z \in \mathbb{U}$, let the integral operators $L(f)$ be defined as

$$L(f)(z) = \frac{2}{z}\int_0^z f(t)\,dt. \tag{1.3}$$

The operator $L(f)$ is said to be Libera operator which was introduced earlier by Libera [3].

Jung *et al.* [2] introduced the following integral operator:

$$I^{\sigma}(f)(z) = \frac{2^{\sigma}}{z\Gamma(\sigma)}\int_0^z \left(\log\left(\frac{z}{t}\right)\right)^{\sigma-1} f(t)\,dt \quad (\sigma > 0,\ f(z) \in A). \tag{1.4}$$

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They showed that

$$I^{\sigma}(f)(z) = z + \sum_{k=2}^{\infty} \left(\frac{2}{k+1} \right)^{\sigma} a_k z^k. \tag{1.5}$$

The operator $I^{\sigma}(f)$ is closely related to multiplier transformations studied earlier by Flett [1], see also ([4], [5], [9]).

For our purpose we introduce:

DEFINITION 1. Let H be the set of complex valued function $h(r,s,t) : \mathbb{C}^3 \to \mathbb{C}$, such that

(i) $h(r,s,t)$ is continuous in a domain $D \subset \mathbb{C}^3$,
(ii) $(0,0,0) \in D$ and $|h(0,0,0)| < 1$,
(iii) $\left|h\left(e^{i\theta}, \frac{1}{2}(1+m)e^{i\theta}, \frac{1}{4}\left[(1+3m)e^{i\theta}+L\right]\right)\right| > 1$, whenever

$$\left(e^{i\theta}, \frac{1}{2}(1+m)e^{i\theta}, \frac{1}{4}\left[(1+3m)e^{i\theta}+L\right]\right) \in \mathbb{D},$$

where $\mathrm{Re}\,(e^{-i\theta}L) \geqslant m(m-1)$ for real θ and real $m \geqslant 1$.

Also, we shall need the following definitions:

DEFINITION 2. Let $f(z)$ and $F(z)$ be analytic functions. The function $f(z)$ is said to be *subordinate* to $F(z)$, written $f(z) \prec F(z)$, if there exists a function $w(z)$ analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| \leqslant 1$, and such that $f(z) = F(w(z))$. If $F(z)$ is univalent, then $f(z) \prec F(z)$ *if and only if* $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$.

DEFINITION 3. Let $\Psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ be analytic in a domain $\mathbb{D}$, and let $h(z)$ be univalent in $\mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$ with $(p(z), zp'(z)) \in \mathbb{D}$ when $z \in \mathbb{U}$, then we say that $p(z)$ satisfies a first order differential subordination if:

$$\Psi(p(z), zp'(z); z) \prec h(z) (z \in \mathbb{U}). \tag{1.6}$$

The univalent function $q(z)$ is called *domainant* of the differential subordination (1.6), if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.6), if $\tilde{q}(z) \prec q(z)$ for all dominant of (1.6), then we say that $\tilde{q}(z)$ is the best domainant of (1.6).

2. An inequality for $I^{\sigma}(f)$

To show our result for the operator $I^{\sigma}(f)$, we need the following lemma by Miller and Mocanu [6].

LEMMA 1. *Let $w(z) \in A$ with $w(z) \neq 0$ in $\mathbb{U}$. If $z_0 = r_0 e^{i\theta_0}$ $(0 < r_0 < 1)$ and $|w(z_0)| = \max_{|z| \leqslant |z_0|} |w(z)|$, then*

$$z_0 w'(z_0) = m w(z_0) \tag{2.1}$$

and

$$\mathrm{Re}\left\{1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right\} \geqslant m, \tag{2.2}$$

where m is real and $m \geqslant 1$.

THEOREM 1. *Let $h(r,s,t) \in H$, and let the function $f(z)$ belonging to A satisfy*

$$\left(I^{\sigma+1}(f)(z), I^{\sigma}(f)(z), I^{\sigma-1}(f)(z)\right) \in \mathbf{D} \subset \mathbb{C}^3 \tag{2.3}$$

and

$$\left|h\left(I^{\sigma+1}(f)(z), I^{\sigma}(f)(z), I^{\sigma-1}(f)(z)\right)\right| < 1, \tag{2.4}$$

for $\sigma > 1$ and $z \in \mathbb{U}$. Then we have

$$\left|I^{\sigma+1}(f)(z)\right| < 1 \quad (z \in \mathbb{U}). \tag{2.5}$$

Proof. Using the identity

$$z(I^{\sigma+1}(f)(z))' = 2I^{\sigma}((f)(z)) - I^{\sigma+1}((f)(z)), \tag{2.6}$$

we define

$$w(z) = I^{\sigma+1}(f)(z). \tag{2.7}$$

Then, we have $w(z) \in A$, $w(0) = 0$ and $w(z) \neq 0 (z \in \mathbb{U})$.

Note that

$$I^{\sigma}(f)(z) = \frac{1}{2}\left(w(z) + zw'(z)\right) \tag{2.8}$$

and

$$I^{\sigma-1}(f)(z) = \frac{1}{4}\left(w(z) + 3zw'(z) + z^2 w''(z)\right). \tag{2.9}$$

If $z_0 = r_0 e^{i\theta_0}$ $(0 < r_0 < 1)$ and

$$|w(z_0)| = \max_{|z| \leqslant |z_0|} |w(z)| = 1, \tag{2.10}$$

using (2.1) and (2.10), we see that

$$I^{\sigma+1} f(z_0) = e^{i\theta}, \tag{2.11}$$

$$I^{\sigma} f(z_0) = \frac{1}{2}(1+m)e^{i\theta} \tag{2.12}$$

and

$$I^{\sigma-1} f(z_0) = \frac{1}{4}\left[(1+3m)e^{i\theta} + D\right], \tag{2.13}$$

where $D = z_0^2 w''(z_0)$. Furthermore, (2.2) implies

$$\operatorname{Re}\left\{\frac{z_0 w''(z_0)}{w'(z_0)}\right\} = \operatorname{Re}\left\{\frac{z_0^2 w''(z_0)}{m e^{i\theta}}\right\} \geqslant m - 1, \tag{2.14}$$

that is, that

$$\operatorname{Re}(e^{-i\theta} D) \geqslant m(m-1). \tag{2.15}$$

Therefore, it follows from $h(r,s,t) \in H$ that

$$\begin{aligned} &\left|h\left(I^{\sigma+1}(f)(z), I^{\sigma}(f)(z), I^{\sigma-1}(f)(z)\right)\right| \\ &= \left|h\left(e^{i\theta}, \frac{1}{2}(1+m)e^{i\theta}, \frac{1}{4}\left[(1+3m)e^{i\theta} + D\right]\right)\right| > 1 \end{aligned} \tag{2.16}$$

which contradicts the condition (2.4). This proves that $|w(z)| = \left|I^{\sigma+1}(f)(z)\right| < 1$, for all $z \in \mathbb{U}$. □

Let $h(r,s,t) = h_1(r,s,t) = s$. It is obvious that $h_1(r,s,t) \in H$ with $\mathbb{D} = \mathbb{C}^3$. By iteration of Theorem 1 applied to $h_1(r,s,t)$, we obtain the following corollary

COROLLARY 1. *Let the function $f(z) \in A$, satisfy.* $|\, I^{\sigma}(f)(z)| < 1$. *Then*

$$\left|\, I^{\sigma+1}(f)(z)\right| < 1 \quad (z \in \mathbb{U};\ \sigma > 1). \tag{2.17}$$

3. Differential subordination with $I^{\sigma}(f)$

We require the following lemma due to Miller and Mocanu [7], see also [8, p. 132].

LEMMA 2. *Let $q(z)$ be univalent in $\mathbb{U}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{U})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{U})$. Set $Q(z) = zq'(z)\phi[q(z)]$, $h(z) = \theta[q(z)] + Q(z)$ and suppose that either*
(i) $h(z)$ is convex, or
(ii) $Q(z)$ is starlike.
In addition, assume that
(iii) $\mathrm{Re}\left\{\dfrac{zh'(z)}{Q(z)}\right\} > 0.$
If $p(z)$ is analytic in $\mathbb{U}$, with $p(0) = q(0)$, $p(\mathbb{U}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zQ'(z)\phi[q(z)] = h(z), \tag{3.1}$$

then $p(z) \prec q(z)$, and $q(z)$ is the best dominant of (3.1).

Now, we prove the following theorem

THEOREM 2. *Let $\alpha \in [0,1)$ and $\sigma > 0$. Also, let*

$$\frac{z(I^{\sigma}(f)(z))'}{I^{\sigma}(f)(z)} \prec h(z) \quad (z \in \mathbb{U}), \tag{3.2}$$

for all $f \in A$ satisfies $\dfrac{I^{\sigma+i}(f)(z)}{z} \neq 0$ $(i = 0,1)$. Then $I^{\sigma+1}(f)(z) \in S^{*}(\alpha)$, α is the best possible, where

$$h(z) = \frac{1 + (1-2\alpha)z}{1-z} + \frac{(1-\alpha)z}{(1-z)(1-\alpha z)}. \tag{3.3}$$

Proof. We choose $p(z) = \dfrac{z(I^{\sigma+1}(f)(z))'}{I^{\sigma+1}(f)(z)}$, then (2.6) becomes

$$(1 + p(z))\, I^{\sigma+1}(f)(z) = 2\, I^{\sigma}(f)(z). \tag{3.4}$$

Then, from identity (2.6) and (3.4), we have

$$\frac{z(I^{\sigma}(f)(z))'}{I^{\sigma}(f)(z)} = \left(p(z) + \frac{zp'(z)}{p(z)+1}\right). \tag{3.5}$$

Therefore, (3.5) becomes

$$p(z) + \frac{zp'(z)}{p(z)+1} \prec h(z), \quad (z \in \mathbb{U}). \tag{3.6}$$

Let $q(z) = \dfrac{1+(1-2\alpha)z}{1-z}$, $\theta(w) = w$ and $\phi(w) = \dfrac{1}{w+1}$. Then $\theta(w)$ and $\phi(w)$ are analytic with domain $\mathbb{D} = \mathbb{C}\backslash\{-1\}$ which contains $q(\mathbb{U})$, $q(0) = 1$ and $\phi(w) \neq 0$ when $w \in q(\mathbb{U})$.

Also, we define the function $Q(z)$ by

$$\begin{aligned} Q(z) &= zQ'(z)\phi(q(z)) \\ &= \frac{(1-\alpha)z}{(1-z)(1-\alpha z)}. \end{aligned} \tag{3.7}$$

Since

$$\theta[q(z)] + Q(z) = \frac{1+(1-2\alpha)z}{1-z} + \frac{(1-\alpha)z}{(1-z)(1-\alpha z)} = h(z), \tag{3.8}$$

$$\frac{zQ'(z)}{Q(z)} = \frac{1}{1-z} + \frac{\alpha z}{1-\alpha z} \tag{3.9}$$

and

$$\phi[q(z)] = \frac{1-z}{2(1-\alpha z)}. \tag{3.10}$$

It follows from (3.8) and (3.9) that, for $z \in \mathbb{U}$, $Q(z)$ is starlike and

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{\frac{1}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)}\right\}. \tag{3.11}$$

Since $\operatorname{Re}\,(\phi(q(z))) > 0$, then we have $\operatorname{Re}\left\{\dfrac{zh'(z)}{Q(z)}\right\} > 0$. Also, the condition $\dfrac{I^{\sigma+1}(f)(z)}{z} \neq 0$ gives that the function $p(z)$ is analytic in $\mathbb{U}$, $p(0) = q(0) = 1$, and the condition $\dfrac{I^{\sigma}(f)(z)}{z} \neq 0$ gives that $-1 \notin p(\mathbb{U})$, therefore $p(\mathbb{U}) \subset \mathbb{D}$. By Lemma 2, we deduce $p(z) \prec q(z)$, i.e., $I^{\sigma+1}((f)(z)) \in S^*(\alpha)$, and $q(z)$ is the best dominant of (3.6), therefore the constant α is the best possible. □

Putting $\alpha = 0$, in Theorem 2, we have

COROLLARY 2. *Let the function $f(z) \in A$. satisfy $\dfrac{I^{\sigma+i}(f)(z)}{z} \neq 0$ $(i = 0,1)$. If*

$$\left|\frac{\dfrac{z(I^{\sigma}(f)(z))'}{I^{\sigma}(f)(z)} - 1}{\dfrac{z(I^{\sigma}(f)(z))'}{I^{\sigma}(f)(z)} + 2}\right| < 1 \quad (z \in \mathbb{U};\ \sigma > 0), \tag{3.12}$$

then $I^{\sigma+1}(f)(z) \in S^*$, this result is the best possible.

COROLLARY 3. *Let the function* $f(z) \in A$ *satisfy the conditions* $\frac{L(f)(z)}{z} \neq 0$ *and* $\frac{2}{z^2}\int_0^z L(f)(t)\,dt \neq 0$. *If*

$$\frac{z(L(f)(z))'}{L(f)(z)} \prec h(z) \quad (z \in \mathbb{U}), \tag{3.13}$$

then $\frac{2}{z}\int_0^z L(f)(t)\,dt \in S^*(\alpha)$ $(0 \leqslant \alpha < 1)$, *the constant* α *is the best possible, where* $h(z)$ *is given by (3.3).*

Proof. Since

$$\begin{aligned} I^1(f)(z) &= z + \sum_{k=2}^{\infty}\left(\frac{2}{k+1}\right) a_k z^k \\ &= z + \frac{2}{z}\int_0^z \left(\sum_{k=2}^{\infty} a_k t^k\right) dt \\ &= \frac{2}{z}\int_0^z f(t)\,dt = L(f)(z), \end{aligned} \tag{3.14}$$

using the identity (2.6), we have the first order linear differential equation. Therefore

$$I^2(f)(z) = \frac{2}{z}\int_0^z L(f)(t)\,dt. \tag{3.15}$$

The corollary is obtained by taking $\sigma = 1$ in Theorem 2. □

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A. A. Attiya
Department of Mathematics
Teachers' College in Abha
King Khalid University
Abha
P. O. Box 249
Saudi Arabia
and
Department of Mathematics
Faculty of Science
University of Mansoura
Mansoura 35516
Egypt
e-mail: aattiy@mans.edu.eg