Mathematical
Inequalities
& Applications
Volume 11, Number 2 (2008), 371–380

GENERALIZED HYERS—ULAM STABILITY OF MAPPINGS ON NORMED LIE TRIPLE SYSTEMS

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Dedicated to the memory of Professor S. M. Ulam and Professor D. H. Hyers in admiration

(*communicated by H. Srivastava*)

Abstract. We prove the generalized Hyers–Ulam stability of mappings on normed spaces for the Pexiderized Cauchy–Jensen additive mapping

$$f\left(\frac{x+y}{2}+z\right)+g\left(\frac{x-y}{2}-z\right)=h(x).$$

Then we apply the results for investigating the stability of homomorphisms and derivations on normed Lie triple systems.

1. Introduction

The stability problem of functional equations originated from a question of S. Ulam [34], posed in 1940, concerning the stability of group homomorphisms. In 1941, D. H. Hyers [13] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. In 1950, a generalized version of Hyers' theorem for approximate additive mappings was given by T. Aoki [2] (see also [22]). In 1978, Th. M. Rassias [29] extended the theorem of Hyers by considering the unbounded Cauchy difference inequality

$$\|f(x+y)-f(x)-f(y)\| \leqslant \varepsilon(\|x\|^p+\|y\|^p), \quad (\epsilon>0, p\in[0,1)). \tag{1.1}$$

Th. M. Rassias [29] was the first who proved the stability of the linear mappings between Banach spaces. In 1990, Th. M. Rassias [30] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geqslant 1$. In 1991, Z. Gajda [9] following the same approach as in Th. M. Rassias [29], gave an affirmative solution to this question for $p > 1$. It was proved by Z. Gajda [9], as well as by Th. M. Rassias and P. Semrl [33] that one cannot prove a Th. M. Rassias' type theorem when $p = 1$. Th. M. Rassias' Theorem for the stability of the linear mappings between Banach spaces provided some influence for the development

Mathematics subject classification (2000): 39B82, 16W25, 17A40, 39B52, 47Jxx.

Key words and phrases: Pexiderized Cauchy–Jensen additive mapping; normed Lie system; homomorphism; derivation; generalized Hyers–Ulam stability.

The first author was in part supported by a grant from IPM (No. 85390031).

Paper MIA-11-28

of the concept of generalized Hyers–Ulam stability, a fact which rekindled interest in the subject of stability of functional equations. This concept is known today as generalized Hyers–Ulam stability or Hyers–Ulam–Rassias stability of functional equations; cf. [7, 14, 17, 32]. Several mathematicians following the spirit of the approach in the paper of Th. M. Rassias [29] for the unbounded Cauchy difference obtained various results. For example in 1982, J. M. Rassias [26] obtained an analogous stability theorem in which he replaced the factor $\|x\|^p+\|y\|^p$ by $\|x\|^p\cdot\|y\|^q$ for $p,q\in\mathbb{R}$ with $p+q\neq 1$. In 1994, P. Găvruta [10] provided a further generalization of Th. M. Rassias' theorem in which he replaced the bound $\varepsilon(\|x\|^p+\|y\|^p)$ in (1.1) by a general control function $\varphi(x,y)$. In 1996, G. Isac and Th. M. Rassias [16] applied the generalized Hyers–Ulam stability theory to prove fixed point theorems and obtained some new applications in Nonlinear Analysis. During the last decades several stability problems of functional equations have been investigated by a number of mathematicians; cf. [6, 8, 20, 21, 18, 31] and the references therein.

One of the interesting functional equations is Cauchy–Jensen additive type equation

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x-y}{2}-z\right)=f(x),$$

where f is a mapping between linear spaces. It is easy to see that a function f satisfies the above Cauchy–Jensen additive type equation if and only if it is additive. For other types of Cauchy–Jensen equation see [26, 27].

A normed (Banach) Lie triple system is a normed (Banach) space $(\mathscr{A};\|.\|)$ with a trilinear mapping $(x,y,z)\mapsto[x,y,z]$ from $\mathscr{A}\times\mathscr{A}\times\mathscr{A}$ to $\mathscr{A}$ satisfying the following axioms

$$\begin{aligned}
&[x,y,z]=-[y,x,z],\\
&[x,y,z]+[y,z,x]+[z,x,y]=0,\\
&[u,v,[x,y,z]]=[[u,v,x],y,z]+[x,[u,v,y],z]+[x,y,[u,v,z]],\\
&\|[x,y,z]\|\leqslant\|x\|\,\|y\|\,\|z\|,
\end{aligned}$$

for all $u,v,x,y,z\in\mathscr{A}$. The concept of Lie triple system was first introduced by W. G. Lister [19] (see also [12]).

Let $\mathscr{A}$ and $\mathscr{B}$ be normed Lie triple systems. A $\mathbb{C}$-linear mapping $H:\mathscr{A}\to\mathscr{B}$ is said to be a homomorphism if $H([x,y,z])=[H(x),H(y),H(z)]$ for all $x,y,z\in\mathscr{A}$. A $\mathbb{C}$-linear mapping $D:\mathscr{A}\to\mathscr{B}$ is called a derivation if $D([x,y,z])=[D(x),y,z]+[x,D(y),z]+[x,y,D(z)]$ for all $x,y,z\in\mathscr{A}$. The third identity asserts that the mappings $D_{u,v}:x\mapsto[u,v,x]$ are (inner) derivations of $\mathscr{A}$.

Clearly, every Lie algebra is at the same time a Lie triple system via $[x,y,z]:=[[x,y],z]$, and our definition of a homomorphism (derivation) coincides with that of a prehomomorphism (prederivation) on a Lie algebra [5]. Also, If U is an involutive automorphism of a Lie algebra $(\mathscr{L},[\ ,\ ])$, then the eigenspace $E_{-1}(U)$ is a Lie triple system. Lie triple systems are important since they give the structure of the tangent space of a symmetric space, see [11]. Also some applications of Lie triple systems may be found in Nambuòs approach to modifying the Heisenberg equation of motion [23].

In this paper, we shall establish the generalized Hyers–Ulam stability of Pexiderized Cauchy–Jensen additive type equation

$$f\left(\frac{x+y}{2}+z\right)+g\left(\frac{x-y}{2}-z\right)=h(x),$$

and then apply our results to study stability of homomorphisms and derivations associated to Pexiderized Cauchy–Jensen additive mapping in normed Lie triple systems, which can be regarded as ternary structures. The reader is referred to [3, 24, 25] for some other related results. Some applications of our results may be hopefully found in other areas of research working with ternary (and Lie) structures (see also [1]).

Throughout this paper, suppose that $\mathscr{A}$ is a normed Lie triple system with norm $\|\cdot\|_{\mathscr{A}}$ and that $\mathscr{B}$ is a Banach Lie triple system with norm $\|\cdot\|_{\mathscr{B}}$. For given mappings $f,g,h:\mathscr{A}\to\mathscr{B}$ and given subset $\mathbb{E}$ of $\mathbb{C}$, we define

$$J_\lambda(f,g,h)(x,y,z):=f\left(\frac{\lambda x+\lambda y}{2}+\lambda z\right)+\lambda g\left(\frac{x-y}{2}-z\right)-\lambda h(x),$$

for all $\lambda\in\mathbb{E}$ and all $x,y,z\in\mathscr{A}$. We write $J_\lambda(f)(x,y,z)$ for $J_\lambda(f,f,f)(x,y,z)$.

2. Stability of mappings in normed spaces

Throughout this section $\mathscr{X}$ denotes a normed space and $\mathscr{Y}$ is a Banach space. We aim to prove the generalized Hyers–Ulam stability of Cauchy–Jensen equation of Pexider type.

THEOREM 2.1. *Suppose $f,g,h:\mathscr{X}\to\mathscr{Y}$ are mappings with $f(0)=g(0)=h(0)=0$ for which there exists a function $\varphi:\mathscr{X}\times\mathscr{X}\times\mathscr{X}\to[0,\infty)$ such that*

$$\widetilde{\varphi}(x,y,z):=\frac{1}{2}\sum_{n=0}^{\infty}2^{-n}\varphi(2^nx,2^ny,2^nz)<\infty \tag{2.1}$$

$$\|J_\lambda(f,g,h)(x,y,z)\|\leqslant\varphi(x,y,z) \tag{2.2}$$

for all $\lambda\in\mathbb{T}:=\{z\in\mathbb{C}:|z|=1\}$ and all $x,y,z\in\mathscr{X}$. Then there exists a unique linear mapping $T:\mathscr{X}\to\mathscr{Y}$ such that

$$\|f(x)-T(x)\|\leqslant\widetilde{\varphi}(x,x,x)+\widetilde{\varphi}(x,x,0)+\widetilde{\varphi}(0,0,x),$$
$$\|h(x)-T(x)\|\leqslant\varphi(x,x,0)+\widetilde{\varphi}(x,x,x)+\widetilde{\varphi}(x,x,0)+\widetilde{\varphi}(0,0,x),$$
$$\|g(x)-T(x)\|\leqslant\varphi(2x,0,0)+2\widetilde{\varphi}(x,x,x)+2\widetilde{\varphi}(x,x,0)+2\widetilde{\varphi}(0,0,x)+\varphi(x,x,0),$$

for all $x\in\mathscr{X}$.

Proof. Put $y=x$ in (2.2), to get

$$\|f(\lambda x+\lambda z)+\lambda g(-z)-\lambda h(x)\|\leqslant\varphi(x,x,z), \tag{2.3}$$

for all $x,z\in\mathscr{X}$ and all $\lambda\in\mathbb{T}$. Set $z=0$ in (2.3) to obtain

$$\|f(\lambda x)-\lambda h(x)\|\leqslant\varphi(x,x,0), \tag{2.4}$$

for all $x \in \mathscr{X}$ and all $\lambda \in \mathbb{T}$. Putting $x = 0$ in (2.3), we have

$$\|f(\lambda z) + \lambda g(-z)\| \leqslant \varphi(0,0,z), \tag{2.5}$$

for all $z \in \mathscr{X}$ and all $\lambda \in \mathbb{T}$. It follows from (2.3), (2.4) and (2.5) with $\lambda = 1$ that

$$\begin{aligned} \|f(x+z) - f(x) - f(z)\| &\leqslant \|f(x+z) + g(-z) - h(x)\| + \|h(x) - f(x)\| \\ &\quad + \| - f(z) - g(-z)\| \\ &\leqslant \varphi(x,x,z) + \varphi(x,x,0) + \varphi(0,0,z), \end{aligned} \tag{2.6}$$

for all $x, z \in \mathscr{X}$. Set $z = x$ in (2.6) and put $\psi(x) := \varphi(x,x,x) + \varphi(x,x,0) + \varphi(0,0,x)$ to get

$$\|f(2x) - 2f(x)\| \leqslant \psi(x),$$

for all $x \in \mathscr{X}$. One can use induction to show that

$$\|2^{-n}f(2^n x) - 2^{-m}f(2^m x)\| \leqslant \frac{1}{2}\sum_{k=m}^{n-1} 2^{-k}\psi(2^k x), \tag{2.7}$$

for all nonnegative integers $n > m$ and all $x \in \mathscr{X}$. It follows from the convergence of the series (2.1) and (2.7) that the sequence $\left\{\frac{f(2^n x)}{2^n}\right\}$ is a Cauchy sequence. Due to the completeness of $\mathscr{X}$, this sequence is convergent. Set

$$T(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

By replacing x, y by $2^n x, 2^n y$, respectively, in (2.6), we get

$$\|2^{-n}f(2^n(x+y)) - 2^{-n}f(2^n x) - 2^{-n}f(2^n y)\| \leqslant 2^{-n}\psi(2^n x).$$

Taking the limit as $n \to \infty$ we obtain

$$T(x+y) = T(x) + T(y),$$

for all $x, y \in \mathscr{X}$. It follows from (2.4) that

$$\begin{aligned} \|f(\lambda x) - \lambda f(x)\| &\leqslant \|f(\lambda x) - \lambda h(x)\| + \|\lambda f(x) - \lambda h(x)\| \\ &\leqslant (1 + |\lambda|)\varphi(x,x,0), \end{aligned} \tag{2.8}$$

for all $x \in \mathscr{X}$ and all $\lambda \in \mathbb{T}$. Obviously, $T(0x) = 0 = 0T(x)$. Next, let $\mu \in \mathbb{C}$ $(\mu \neq 0)$, and let M be a natural number greater than $|\mu|$. By an easily geometric argument, one can conclude that there exist two numbers $\lambda_1, \lambda_2 \in \mathbb{T}$ such that $2\frac{\mu}{M} = \lambda_1 + \lambda_2$. By the additivity of T we get $T\left(\frac{1}{2}x\right) = \frac{1}{2}T(x)$ for all $x \in \mathscr{X}$. Therefore

$$\begin{aligned} T(\mu x) &= T\left(\frac{M}{2}\cdot 2\cdot\frac{\mu}{M}x\right) = MT\left(\frac{1}{2}\cdot 2\cdot\frac{\mu}{M}x\right) = \frac{M}{2}T\left(2\cdot\frac{\mu}{M}x\right) \\ &= \frac{M}{2}T(\lambda_1 x + \lambda_2 x) = \frac{M}{2}\left(T(\lambda_1 x) + T(\lambda_2 x)\right) \\ &= \frac{M}{2}(\lambda_1 + \lambda_2)T(x) = \frac{M}{2}\cdot 2\cdot\frac{\mu}{M} = \mu T(x) \end{aligned}$$

for all $x \in \mathscr{X}$, so that T is a $\mathbb{C}$-linear mapping. Set $m = 0$ in (2.7) and let n tend to infinity to get

$$\|f(x) - T(x)\| \leqslant \widetilde{\varphi}(x,x,x) + \widetilde{\varphi}(x,x,0) + \widetilde{\varphi}(0,0,x). \tag{2.9}$$

It follows from (2.9) and (2.4) that

$$\begin{aligned} \|h(x) - T(x)\| &\leqslant \|h(x) - f(x)\| + \|f(x) - T(x)\| && (2.10)\\ &\leqslant \varphi(x,x,0) + \widetilde{\varphi}(x,x,x) + \widetilde{\varphi}(x,x,0) + \widetilde{\varphi}(0,0,x). && (2.11) \end{aligned}$$

Now put $y = z = 0$ and $\lambda = 1$ and replace x by $2x$ in (2.2) to obtain

$$\|f(x) + g(x) - h(2x)\| \leqslant \varphi(2x,0,0). \tag{2.12}$$

for all $x \in \mathscr{X}$. Now we deduce from (2.12), (2.9) and (2.10) that

$$\begin{aligned} \|g(x) - T(x)\| &\leqslant \|f(x) + g(x) - h(2x)\| + \|T(x) - f(x)\| + \|-2T(x) + h(2x)\|\\ &\leqslant \varphi(2x,0,0) + 2\widetilde{\varphi}(x,x,x) + 2\widetilde{\varphi}(x,x,0) + 2\widetilde{\varphi}(0,0,x) + \varphi(x,x,0), \end{aligned}$$

for all $x \in \mathscr{X}$.

For establishing the uniqueness assertion, let T' be another additive mapping satisfying

$$\|f(x) - T'(x)\| \leqslant \widetilde{\varphi}(x,x,x) + \widetilde{\varphi}(x,x,0) + \widetilde{\varphi}(0,0,x).$$

Foe any $x \in \mathscr{A}$, we have

$$\begin{aligned} \|T(x) - T'(x)\| &\leqslant 2^{-n}\|T(2^n x) - T'(2^n x)\|\\ &\leqslant 2^{-n}\|T(2^n x) - f(2^n x)\| + 2^{-n}\|f(2^n x) - T'(2^n x)\|\\ &\leqslant 2.2^{-n}\widetilde{\varphi}(2^n x, 2^n x, 2^n x) + 2.2^{-n}\widetilde{\varphi}(2^n x, 2^n x, 0)\\ &\quad +2.2^{-n}\widetilde{\varphi}(0,0,2^n x)\\ &\leqslant 2^{-n}\sum_{k=0}^{\infty} 2^{-k}\psi(2^k 2^n x)\\ &= \sum_{k=n}^{\infty} 2^{-k}\varphi(2^k x) \to 0, \end{aligned}$$

as $n \to \infty$. Hence $T = T'$. □

The next result is a dual to the previous theorem in some sense.

THEOREM 2.2. *Suppose $f, g, h : \mathscr{X} \to \mathscr{Y}$ are mappings with $f(0) = g(0) = h(0) = 0$ for which there exists a function $\varphi : \mathscr{X} \times \mathscr{X} \times \mathscr{X} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x,y,z) := \frac{1}{2}\sum_{n=1}^{\infty} 2^n \varphi(2^{-n}x, 2^{-n}y, 2^{-n}z) < \infty$$
$$\|J_\lambda(f,g,h)(x,y,z)\| \leqslant \varphi(x,y,z)$$

for $\lambda = 1, \mathbf{i}$ and all $x, y, z \in \mathscr{X}$. Assume that for each fixed $x \in \mathscr{A}$ the function

$t \mapsto f(tx)$ is continuous on $\mathbb{R}$. Then there exists a unique linear mapping $T : \mathscr{X} \to \mathscr{Y}$ such that

$$\|f(x) - T(x)\| \leqslant \widetilde{\varphi}(x,x,x) + \widetilde{\varphi}(x,x,0) + \widetilde{\varphi}(0,0,x),$$
$$\|h(x) - T(x)\| \leqslant \varphi(x,x,0) + \widetilde{\varphi}(x,x,x) + \widetilde{\varphi}(x,x,0) + \widetilde{\varphi}(0,0,x),$$
$$\|g(x) - T(x)\| \leqslant \varphi(2x,0,0) + 2\widetilde{\varphi}(x,x,x) + 2\widetilde{\varphi}(x,x,0) + 2\widetilde{\varphi}(0,0,x) + \varphi(x,x,0),$$

for all $x \in \mathscr{X}$.

Proof. Applying the same argument as in the proof of 3.1 one can deduce the existence of a unique additive mapping $T : \mathscr{X} \to \mathscr{Y}$ given by

$$T(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

satisfying the required inequalities. By the same reasoning as in the proof of the theorem of [29], the mapping T is $\mathbb{R}$-linear.

It follows from (2.8) that $\|f(\mathbf{i}x) - \mathbf{i}f(x)\| \leqslant 2\varphi(x,x,0)$, for all $x \in \mathscr{A}$. Hence $2^n\|f(2^{-n}\mathbf{i}x) - \mathbf{i}f(2^{-n}x)\| \leqslant 2\varphi(2^{-n}x, 2^{-n}x, 0)$ for all $n \in \mathbb{N}$ and all $x \in \mathscr{A}$. The right hand side tends to zero as $n \to \infty$, so that

$$\begin{aligned} T(\mathbf{i}x) &= \lim_{n\to\infty} 2^n f(\frac{\mathbf{i}x}{2^n}) \\ &= \lim_{n\to\infty} \mathbf{i}2^n f(\frac{x}{2^n}) \\ &= \mathbf{i}T(a) \end{aligned}$$

for all $a \in \mathscr{A}$. For each $\mu \in \mathbb{C}, \mu = \lambda_1 + \mathbf{i}\lambda_2 \ (\lambda_1, \lambda_2 \in \mathbb{R})$. Hence

$$\begin{aligned} T(\mu x) &= T(\lambda_1 x + \mathbf{i}\lambda_2 x) = \lambda_1 T(x) + \lambda_2 T(\mathbf{i}x) \\ &= \lambda_1 T(x) + \mathbf{i}\lambda_2 T(x) = (\lambda_1 + \mathbf{i}\lambda_2) T(x) \\ &= \mu T(x). \end{aligned}$$

Thus T is $\mathbb{C}$-linear. □

3. Stability of homomorphisms in normed Lie triple systems

In this section, we prove the stability of homomorphisms in normed Lie triple systems associated with the Cauchy–Jensen additive mapping.

THEOREM 3.1. *Let θ be a positive real number, let $r < 1$ and let $f : \mathscr{A} \to \mathscr{B}$ be a mapping with $f(0) = 0$ such that*

$$\|J_\lambda(f)(x,y,z)\|_{\mathscr{B}} \leqslant \theta(\|x\|^r_{\mathscr{A}} + \|y\|^r_{\mathscr{A}} + \|z\|^r_{\mathscr{A}}),$$
$$\|f([x,y,z]) - [f(x), f(y), f(z)]\|_{\mathscr{B}} \leqslant \theta(\|x\|^r_{\mathscr{A}} + \|y\|^r_{\mathscr{A}} + \|z\|^r_{\mathscr{A}}), \tag{3.1}$$

for all $\lambda \in \mathbb{T}$ and all $x, y, z \in \mathscr{A}$. Then there exists a unique homomorphism $H : \mathscr{A} \to \mathscr{B}$ such that

$$\|f(x) - H(x)\|_{\mathscr{B}} \leqslant \frac{3\theta}{1 - 2^{r-1}} \|x\|^r_{\mathscr{A}} \tag{3.2}$$

for all $x \in \mathscr{A}$.

Proof. First let us assume that $\|0\|^r = \infty$ for $r < 0$. Put $\varphi(x,y,z) = \theta(\|x\|^r_{\mathscr{A}} + \|y\|^r_{\mathscr{A}} + \|z\|^r_{\mathscr{A}})$ in Theorem 2.1, to get a unique $\mathbb{C}$-linear mapping H given by $H(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ satisfying (3.2). It follows from (3.1) that

$$\begin{aligned}
&\|H([x,y,z]) - [H(x),H(y),H(z)]\|_{\mathscr{B}} \\
&\quad = \lim_{n\to\infty} 2^{3n} \left\| f\left(\frac{[x,y,z]}{2^{3n}}\right) - \left[f\left(\frac{x}{2^n}\right), f\left(\frac{y}{2^n}\right), f\left(\frac{z}{2^n}\right)\right] \right\|_{\mathscr{B}} \\
&\quad \leqslant \lim_{n\to\infty} \frac{2^{3n}\theta}{2^{nr}} (\|x\|^r_{\mathscr{A}} + \|y\|^r_{\mathscr{A}} + \|z\|^r_{\mathscr{A}}) = 0
\end{aligned}$$

for all $x,y,z \in \mathscr{A}$. So

$$H([x,y,z]) = [H(x),H(y),H(z)]$$

for all $x,y,z \in \mathscr{A}$. □

The following is a modification of Example 2.3 of [3] for Lie triple systems.

EXAMPLE 3.2. *Let $L : \mathscr{A} \to \mathscr{B}$ be a norm one homomorphism between normed Lie triple systems, let $f : \mathscr{A} \to \mathscr{B}$ be defined by*

$$f(x) = \begin{cases} L(x) & \|x\| < 1 \\ 0 & \|x\| \geqslant 1 \end{cases},$$

let $r = 0$ and $\theta = 3$. Then

$$\|J_\lambda(f)(x,y,z)\|_{\mathscr{B}} \leqslant 3 = \theta,$$

and

$$\|f([x,y,z]) - [f(x),f(y),f(z)]\|_{\mathscr{B}} \leqslant 2 \leqslant \theta,$$

for all $\lambda \in \mathbb{T}$ and all $x,y,z \in \mathscr{A}$. Note also that f is not linear.

By 3.1 there is a homomorphism H given by $H(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$. Further, $H(0) = \lim_{n\to\infty} \frac{f(0)}{2^n} = 0$ and for $x \neq 0$ we have

$$H(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} = \lim_{n\to\infty} \frac{0}{2^n} = 0,$$

since for sufficiently large n, $\|2^n x\| \geqslant 1$. Thus H is identically zero and

$$\|f(x) - H(x)\| \leqslant 1 \leqslant 6\theta,$$

for all $x \in \mathscr{A}$.

THEOREM 3.3. *Let θ be a positive real number, let $r > 1$ and let $f : \mathscr{A} \to \mathscr{B}$ be a mapping such that*

$$\|J_\lambda(f)(x,y,z)\|_{\mathscr{B}} \leqslant \theta(\|x\|^r_{\mathscr{A}} + \|y\|^r_{\mathscr{A}} + \|z\|^r_{\mathscr{A}}), \tag{3.3}$$

$$\|f([x,y,z]) - [f(x), f(y), f(z)]\|_{\mathscr{B}} \leqslant \theta(\|x\|_{\mathscr{A}}^r + \|y\|_{\mathscr{A}}^r + \|z\|_{\mathscr{A}}^r),$$

for $\lambda = 1, \mathbf{i}$ *and all* $x, y, z \in \mathscr{X}$. *Assume that for each fixed* $x \in \mathscr{A}$ *the function* $t \mapsto f(tx)$ *is continuous on* $\mathbb{R}$. *Then there exists a unique homomorphism* $H : \mathscr{A} \to \mathscr{B}$ *such that*

$$\|f(x) - H(x)\|_{\mathscr{B}} \leqslant \frac{3\theta}{2^{r-1} - 1} \|x\|_{\mathscr{A}}^r$$

for all $x \in \mathscr{A}$.

Proof. Put $\varphi(x,y,z) = \theta(\|x\|_{\mathscr{A}}^r + \|y\|_{\mathscr{A}}^r + \|z\|_{\mathscr{A}}^r)$ in Theorem 2.2 and note that inequality (3.3) implies that $f(0) = 0$. The reminder is similar to 3.1. □

4. Stability of derivations on normed Lie triple systems

We prove the stability of derivations in normed Lie triple systems associated with the Cauchy–Jensen additive mapping.

THEOREM 4.1. *Let* θ *be a positive real number, let* $r, s, t \in \mathbb{R}$ *with* $r + s + t > 3$, *and let* $f : \mathscr{A} \to \mathscr{B}$ *be a mapping such that*

$$\|J_\lambda(f)(x,y,z)\|_{\mathscr{B}} \leqslant \theta(\|x\|_{\mathscr{A}}^r \cdot \|y\|_{\mathscr{A}}^s \cdot \|z\|_{\mathscr{A}}^t),$$

$$\begin{aligned} \|f([x,y,z]) - [f(x), y, z] &- [x, f(y), z] - [x, y, f(z)]\|_{\mathscr{A}} \\ &\leqslant \theta \cdot \|x\|_{\mathscr{A}}^r \cdot \|y\|_{\mathscr{A}}^s \cdot \|z\|_{\mathscr{A}}^t \end{aligned} \tag{4.1}$$

for $\lambda = 1, \mathbf{i}$ *and all* $x, y, z \in \mathscr{X}$. *Assume that for each fixed* $x \in \mathscr{A}$ *the function* $t \mapsto f(tx)$ *is continuous on* $\mathbb{R}$. *Then there exists a unique derivation* $D : \mathscr{A} \to \mathscr{A}$ *such that*

$$\|f(x) - D(x)\|_{\mathscr{A}} \leqslant \frac{\theta \|x\|^{r+s+t}}{2^{r+s+t} - 2} \tag{4.2}$$

for all $x \in \mathscr{A}$.

Proof. Put $\varphi(x,y,z) = \theta(\|x\|_{\mathscr{A}}^r \cdot \|y\|_{\mathscr{A}}^s \cdot \|z\|_{\mathscr{A}}^t)$ in Theorem 2.2, to get a unique $\mathbb{C}$-linear mapping H satisfying (4.2). The mapping $D : \mathscr{A} \to \mathscr{A}$ is defined by

$$D(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in \mathscr{A}$. It follows from (4.1) that

$$\begin{aligned} &\|D([x,y,z]) - [D(x), y, z] - [x, D(y), z] - [x, y, D(z)]\|_{\mathscr{A}} \\ &= \lim_{n\to\infty} 2^{3n} \left\| f\left(\frac{[x,y,z]}{2^{3n}}\right) - \left[f\left(\frac{x}{2^d}\right), \frac{y}{2^n}, \frac{z}{2^n}\right] \right. \\ &\qquad \left. - \left[\frac{x}{2^n}, f\left(\frac{y}{2^n}\right), \frac{z}{2^n}\right] - \left[\frac{x}{2^n}, \frac{y}{2^n}, f\left(\frac{z}{2^d}\right)\right] \right\|_{\mathscr{A}} \\ &\leqslant \lim_{n\to\infty} \frac{2^{3n}\theta}{2^{(r+s+t)n}} \|x\|_{\mathscr{A}}^r \cdot \|y\|_{\mathscr{A}}^s \cdot \|z\|_{\mathscr{A}}^t = 0 \end{aligned}$$

for all $x,y,z \in \mathscr{A}$. So

$$D([x,y,z]) = [D(x),y,z] + [x,D(y),z] + [x,y,D(z)]$$

for all $x,y,z \in \mathscr{A}$. $\square$

THEOREM 4.2. *Let θ be a positive real number, let $r,s,t \in \mathbb{R}$ with $r+s+t < 1$, and let $f : \mathscr{A} \to \mathscr{B}$ be a mapping with $f(0) = 0$ such that*

$$\|J_\lambda(f)(x,y,z)\|_{\mathscr{B}} \leqslant \theta(\|x\|^r_{\mathscr{A}} \cdot \|y\|^s_{\mathscr{A}} \cdot \|z\|^t_{\mathscr{A}}),$$

$$\begin{aligned} \|f([x,y,z]) - [f(x),y,z] &- [x,f(y),z] - [x,y,f(z)]\|_{\mathscr{A}} \\ &\leqslant \theta \cdot \|x\|^r_{\mathscr{A}} \cdot \|y\|^s_{\mathscr{A}} \cdot \|z\|^t_{\mathscr{A}} \end{aligned}$$

for all $\lambda \in \mathbb{T}$ and all $x,y,z \in \mathscr{A}$. Then there exists a unique derivation $D : \mathscr{A} \to \mathscr{A}$ such that

$$\|f(x) - D(x)\|_{\mathscr{A}} \leqslant \frac{\theta\|x\|^{r+s+t}}{2 - 2^{r+s+t}}$$

for all $x \in \mathscr{A}$.

Proof. First let us assume that $\|0\|^p = \infty$ for $p < 0$. Put $\varphi(x,y,z) = \theta(\|x\|^r_{\mathscr{A}} \cdot \|y\|^s_{\mathscr{A}} \cdot \|z\|^t_{\mathscr{A}})$ in Theorem 2.1. The reminder is similar to 4.1. $\square$

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(Received July 27, 2006)

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