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HARDY INEQUALITY WITH THREE MEASURES ON MONOTONE FUNCTIONS

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Abstract. Characterization of $L^p_\nu[0,\infty)-L^q_\mu[0,\infty)$ boundedness of the general Hardy operator $(H_sf)(x)=\left(\int_{[0,x]}f^s u d\lambda\right)^{\frac{1}{s}}$ restricted to monotone functions $f\geqslant 0$ for $0<p,q,s<\infty$ with positive Borel σ-finite measures λ,μ and ν is obtained.

1. Introduction

Let $\mathfrak{M}^+$ be the class consisting of all Borel functions $f\colon[0,\infty)\to[0,+\infty]$ and $\mathfrak{M}\downarrow$ ($\mathfrak{M}\uparrow$) be a subclass of $\mathfrak{M}^+$ which consists of all non-increasing (non-decreasing) functions $f\in\mathfrak{M}^+$. Suppose that λ, μ and ν are positive Borel σ-finite measures on $[0,\infty)$ and $u,v,w\in\mathfrak{M}^+$ are weight functions.

For $0<p,q,s<\infty$ we study the problem when the Hardy inequality of the form

$$\left(\int_{[0,\infty)}(H_sf)^q v d\mu\right)^{\frac{1}{q}}\leqslant C\left(\int_{[0,\infty)}f^p w d\nu\right)^{\frac{1}{p}},\tag{1.1}$$

holds for all $f\in\mathfrak{M}\downarrow$ or for all $f\in\mathfrak{M}\uparrow$, where

$$(H_sf)(x):=\left(\int_{[0,x]}f^s u d\lambda\right)^{\frac{1}{s}}.\tag{1.2}$$

Since by the substitution $f^s\to f$ the inequality (1.1) can be reduced to the equivalent inequality with new parameters p and q of the form

$$\left(\int_{[0,\infty)}(Hf)^q v d\mu\right)^{\frac{1}{q}}\leqslant C\left(\int_{[0,\infty)}f^p w d\nu\right)^{\frac{1}{p}},\tag{1.3}$$

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where

$$(Hf)(x) := (H_1 f)(x) = \int_{[0,x]} f u d\lambda \tag{1.4}$$

we may and shall restrict our studies to the inequality (1.3). All the characterizations of (1.1) can be easily reproduced from the results for (1.3).

The weighted inequality (1.3) for $f \in \mathfrak{M}\downarrow$, when $\lambda = \mu = \nu$ is the Lebesgue measure, was essentially characterized in [9] and [13] with the complement for the case $0 < q < 1 = p$ in [12] and recent contribution in [1] for the case $0 < q < p < 1$. In fact, [9], [13], [12] and [1] deal with the case $u(x) = 1$, but a weight u can be incorporated with no change in the arguments. A piece of historical remarks and the literature can be found in ([3] and [4], Chapter 6). We summarize these results in the following

THEOREM 1.1. *Let $\lambda = \mu = \nu$ be the Lebesgue measure. Then the inequality* (1.3) *holds for all $f \in \mathfrak{M}\downarrow$ if and only if:*

(a) $1 < p \leqslant q < \infty$, $\max(A_0, A_1) < \infty$, *where*

$$A_0 := \sup_{t>0} \left(\int_0^t \left(\int_0^s u\right)^q v(s)ds\right)^{\frac{1}{q}} \left(\int_0^t w\right)^{-\frac{1}{p}},$$

and

$$A_1 := \sup_{t>0} \left(\int_t^\infty v\right)^{\frac{1}{q}} \left(\int_0^t \left(\int_0^s u\right)^{p'} \left(\int_0^s w\right)^{-p'} w(s)ds\right)^{\frac{1}{p'}}$$

and $C \approx A_0 + A_1$.

(b) $0 < q < p < \infty, 1 < p < \infty$, $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$, $\max(B_0, B_1) < \infty$, *where*

$$B_0 := \left(\int_0^\infty \left(\int_0^t w\right)^{-\frac{r}{p}} \left(\int_0^t \left(\int_0^s u\right)^q v(s)ds\right)^{\frac{r}{p}} \left(\int_0^t u\right)^q v(t)\,dt\right)^{\frac{1}{r}},$$

and

$$B_1 := \left(\int_0^\infty \left(\int_t^\infty v\right)^{\frac{r}{p}} \left(\int_0^t \left(\int_0^s u\right)^{p'} \left(\int_0^s w\right)^{-p'} w(s)ds\right)^{\frac{r}{p'}} v(t)dt\right)^{\frac{1}{r}}$$

and $C \approx B_0 + B_1$.

(c) $0 < q < p \leqslant 1$. $\max(B_0, \mathscr{B}_1) < \infty$, *where*

$$\mathscr{B}_1 := \left(\int_0^\infty \left(\operatorname*{esssup}_{s\in[0,t]} \left(\int_0^s u\right)^p \left(\int_0^s w\right)^{-1}\right)^{\frac{r}{p}} \left(\int_t^\infty v\right)^{\frac{r}{p}} v(t)dt\right)^{\frac{1}{r}}$$

and $C \approx B_0 + \mathscr{B}_1$.

(d) $0 < p \leqslant q < \infty$, $0 < p \leqslant 1$, $\max(A_0, \mathscr{A}_1) < \infty$, *where*

$$\mathscr{A}_1 := \sup_{t>0} \left(\int_0^t u\right) \left(\int_t^\infty v\right)^{\frac{1}{q}} \left(\int_0^t w\right)^{-\frac{1}{p}}$$

and $C \approx A_0 + \mathscr{A}_1$.

It is important to note, that the weighted case of (1.3) for $1 < p,q < \infty$ was solved in [9] by proving *the principle of duality* which allows to reduce an inequality with a positive operator on monotone functions to an inequality with modified operator on non-negative functions. The other cases, when $p,q \notin (1,\infty)$ were studied by different methods.

Our aim is twofold. First we study the inequality (1.3) in the case $0 < p \leqslant 1$ proving a complete analog of the parts (c) and (d) of Theorem 1.1 (Section 3). In the case $0 < q < p \leqslant 1$ our method is based on the characterization of the Hardy inequality on nonnegative functions in the case $0 < q < 1 = p$, which we establish in Section 3 (Theorem 3.1). This approach is direct and different from discretization methods of [1] and [2].

Hardy inequality (1.3) on monotone functions with two different measures was recently investigated by G. Sinnamon [11]. Namely, for $1 < p < \infty$ and $0 < q < \infty$ the author established the equivalence of (1.3) with $u \equiv v \equiv w \equiv 1$ and $d\lambda = d\nu$ for $f \in \mathfrak{M}^+$ to the same inequality restricted to $f \in \mathfrak{M}\downarrow$. Moreover, such equivalence takes place also for more general operator than (1.4), that is for the operator $(Kf)(x) = \int_{[0,x]} k(x,y) f(y)\, d\lambda(y)$ with a kernel $k(x,y) \geqslant 0$, which is monotone in the variable y (see [5, Theorem 2.3]). Moreover, G. Sinnamon [11] extended the Sawyer principle of duality for measures. We apply this extension to characterize (1.3) in case $1 < p,q < \infty$ (Section 4) combining with the recent results by D.V. Prokhorov [6] for the inequality (1.3) on $f \in \mathfrak{M}^+$ with $1 < p < \infty$ and $0 < q < \infty$ extended by the same author for the Hardy operator with Oinarov kernel [7].

We use the following notations and conventions. $A \ll B$ means that $A \leqslant cB$ with c depending only on p and q, $A \approx B$ is equivalent to $A \ll B \ll A$. Uncertainties of the form $0 \cdot \infty$ are taken to be zero. We also use the notation $:=$ for introducing new quantities.

2. Preliminary remarks

Denote

$$\Lambda_f(x) := \int_{[0,x]} f\, d\lambda, \qquad \textit{and} \qquad \bar{\Lambda}_f(x) := \int_{[x,\infty)} f\, d\lambda. \tag{2.1}$$

We need the following statements.

LEMMA 2.1. ([6], Lemma 1) *If* $\gamma > 0$, *then*

$$\frac{\Lambda_f(\infty)^{\gamma+1}}{\max\{1,\gamma+1\}} \leqslant \int_{[0,\infty)} f(x)\,\Lambda_f(x)^{\gamma}\, d\lambda(x) \leqslant \frac{\Lambda_f(\infty)^{\gamma+1}}{\min\{1,\gamma+1\}} \tag{2.2}$$

holds. If $\gamma \in (-1,0)$ *and* $\Lambda_f(\infty) < +\infty$, *then* (2.2) *holds.*

LEMMA 2.2. ([6], Lemma 2) *If* $\gamma > 0$, *then*

$$\frac{\bar{\Lambda}_f(0)^{\gamma+1}}{\max\{1,\gamma+1\}} \leqslant \int_{[0,\infty)} f(x)\,\bar{\Lambda}_f(x)^{\gamma}\, d\lambda(x) \leqslant \frac{\bar{\Lambda}_f(0)^{\gamma+1}}{\min\{1,\gamma+1\}} \tag{2.3}$$

holds. If $\gamma \in (-1,0)$ *and* $\bar{\Lambda}_f(0) < +\infty$, *then* (2.3) *holds.*

The following two statements can be obtained from [[10], Lemma 1.2] (see also [[11], Proposition 1.5]).

LEMMA 2.3. *Let* $f \in \mathfrak{M}\uparrow$ *with* $f(0) = 0$ *and let* η *be a Borel measure on* $[0,\infty)$. *Then there exist* $f_0 \in \mathfrak{M}\uparrow$ *and the sequence* $\{h_n\}_{n\geqslant 1} \subset \mathfrak{M}^+$ *such that*

(1) $f_0(x) \leqslant f(x)$ *for all* $x \in [0,\infty)$.

(2) $f_0(x) = f(x)$ *for* η*-a.e.* $x \in [0,\infty)$.

(3) $f_n(x) := \int_{[0,x]} h_n d\eta \leqslant f_0(x)$ *for all* $x \in [0,\infty)$.

(4) *For all* $x \in [0,\infty)$ *the sequence* $\{f_n(x)\}_{n\geqslant 1}$ *is nondecreasing in* n *and* $f_0(x) = \lim_{n\to\infty} f_n(x)$ η*-a.e.* $x \in [0,\infty)$.

LEMMA 2.4. *Let* $f \in \mathfrak{M}\downarrow$ *with* $f(+\infty) = 0$ *and let* η *be a Borel measure on* $[0,\infty)$. *Then there exist* $f_0 \in \mathfrak{M}\downarrow$ *and the sequence* $\{h_n\}_{n\geqslant 1} \subset \mathfrak{M}^+$ *such that*

(1) $f_0(x) \leqslant f(x)$ *for all* $x \in [0,\infty)$.

(2) $f_0(x) = f(x)$ *for* η*-a.e.* $x \in [0,\infty)$.

(3) $f_n(x) := \int_{[x,\infty)} h_n d\eta \leqslant f_0(x)$ *for all* $x \in [0,\infty)$.

(4) *For all* $x \in [0,\infty)$ *the sequence* $\{f_n(x)\}_{n\geqslant 1}$ *is nondecreasing in* n *and* $f_0(x) = \lim_{n\to\infty} f_n(x)$ η*-a.e.* $x \in [0,\infty)$.

REMARK 2.5. Two similar lemmas are valid for the approximation from above.

The following statements are taken from [7] and concern the weighted $L^p_\lambda[0,\infty) - L^q_\mu[0,\infty)$ inequality with the operator of the form

$$(K_u f)(x) = \int_{[0,x]} k(x,y)\,u(y) f(y)\,d\lambda(y).$$

Here the kernel $k(x,y) \geqslant 0$ is $\mu \times \lambda$ - measurable on $[0,\infty)\times[0,\infty)$ and satisfies the following Oinarov condition. There is a constant $D \geqslant 1$ such that

$$D^{-1}k(x,y) \leqslant k(x,z) + k(z,y) \leqslant Dk(x,y), \qquad 0 \leqslant y \leqslant z \leqslant x. \tag{2.4}$$

THEOREM 2.6. *Let* $1 < p \leqslant q < \infty$. *Then the inequality*

$$\left(\int_{[0,\infty)} (K_u f)^q\, v d\mu\right)^{\frac{1}{q}} \leqslant C\left(\int_{[0,\infty)} f^p d\lambda\right)^{\frac{1}{p}} \tag{2.5}$$

holds for all $f \in \mathfrak{M}^+$ *if and only if* $\mathbb{A} := \max(\mathbb{A}_{0,1}, \mathbb{A}_{0,2}) < \infty$, *where*

$$\mathbb{A}_{0,1} := \sup_{t\in[0,\infty)} \left(\int_{[t,\infty)} v(x)k(x,t)^q\, d\mu(x)\right)^{\frac{1}{q}} \left(\int_{[0,t]} u^{p'} d\lambda\right)^{\frac{1}{p'}},$$

$$\mathbb{A}_{0,2} := \sup_{t\in[0,\infty)} \left(\int_{[t,\infty)} v d\mu\right)^{\frac{1}{q}} \left(\int_{[0,t]} k(t,y)^{p'}\, u(y)^{p'}\, d\lambda(y)\right)^{\frac{1}{p'}}.$$

If $1<q<p<\infty$ *and* $\frac{1}{r}=\frac{1}{q}-\frac{1}{p}$, *then inequality* (2.5) *holds for all* $f\in\mathfrak{M}^+$ *if and only if* $\mathbb{B}:=\max\left(\mathbb{B}_{0,1},\mathbb{B}_{0,2}\right)<\infty$, *where*

$$\mathbb{B}_{0,1}:=\left(\int_{[0,\infty)}\left(\int_{[t,\infty)}v(x)k(x,t)^q\,d\mu(x)\right)^{\frac{r}{q}}\left(\int_{[0,t]}u^{p'}d\lambda\right)^{\frac{r}{q'}}u(t)^{p'}\,d\lambda(t)\right)^{\frac{1}{r}},$$

$$\mathbb{B}_{0,2}:=\left(\int_{[0,\infty)}\left(\int_{[t,\infty)}vd\mu\right)^{\frac{r}{p}}\left(\int_{[0,t]}k(t,y)^{p'}u(y)^{p'}\,d\lambda(y)\right)^{\frac{r}{p'}}v(t)\,d\mu(t)\right)^{\frac{1}{r}}.$$

The next statement is an analog of the previous theorem for the operator K_u^* of the dual form

$$\left(K_u^*f\right)(x)=\int\limits_{[x,\infty)}k(y,x)\,u(y)f(y)\,d\lambda(y)$$

with a kernel satisfying Oinarov's condition (2.4).

THEOREM 2.7. *Let* $1<p\leqslant q<\infty$. *Then the inequality*

$$\left(\int_{[0,\infty)}\left(K_u^*f\right)^q vd\mu\right)^{\frac{1}{q}}\leqslant C\left(\int_{[0,\infty)}f^p d\lambda\right)^{\frac{1}{p}} \tag{2.6}$$

holds for all $f\in\mathfrak{M}^+$ *if and only if* $\mathbb{A}^*:=\max\left(\mathbb{A}_{0,1}^*,\mathbb{A}_{0,2}^*\right)<\infty$, *where*

$$\mathbb{A}_{0,1}^*:=\sup_{t\in[0,\infty)}\left(\int_{[0,t])}v(x)k(t,x)^q\,d\mu(x)\right)^{\frac{1}{q}}\left(\int_{[t,\infty)}u^{p'}d\lambda\right)^{\frac{1}{p'}},$$

$$\mathbb{A}_{0,2}^*:=\sup_{t\in[0,\infty)}\left(\int_{[0,t]}vd\mu\right)^{\frac{1}{q}}\left(\int_{[t,\infty)}k(y,t)^{p'}u(y)^{p'}\,d\lambda(y)\right)^{\frac{1}{p'}}.$$

If $1<q<p<\infty$ *and* $\frac{1}{r}=\frac{1}{q}-\frac{1}{p}$, *then inequality* (2.6) *holds for all* $f\in\mathfrak{M}^+$ *if and only if* $\mathbb{B}^*:=\max\left(\mathbb{B}_{0,1}^*,\mathbb{B}_{0,2}^*\right)<\infty$, *where*

$$\mathbb{B}_{0,1}^*:=\left(\int_{[0,\infty)}\left(\int_{[0,t]}v(x)k(t,x)^q\,d\mu(x)\right)^{\frac{r}{q}}\left(\int_{[t,\infty)}u^{p'}d\lambda\right)^{\frac{r}{q'}}u(t)^{p'}\,d\lambda(t)\right)^{\frac{1}{r}},$$

$$\mathbb{B}_{0,2}^*:=\left(\int_{[0,\infty)}\left(\int_{[0,t]}vd\mu\right)^{\frac{r}{p}}\left(\int_{[t,\infty)}k(y,t)^{p'}u(y)^{p'}\,d\lambda(y)\right)^{\frac{r}{p'}}v(t)\,d\mu(t)\right)^{\frac{1}{r}}.$$

In the following theorems we collect weight versions of the results obtained by G. Sinnamon in [11] for embeddings the cones of monotone functions. Put

$$W(t):=\int_{[0,t]}wd\nu,\qquad \text{and}\qquad \bar{W}(x):=\int_{[x,\infty)}wd\nu. \tag{2.7}$$

THEOREM 2.8. *If* $0 < p \leqslant q < \infty$, *then*

$$\sup_{F \in \mathfrak{M}\downarrow} \frac{\left(\int_{[0,\infty)} F^q v d\mu\right)^{\frac{1}{q}}}{\left(\int_{[0,\infty)} F^p w d\nu\right)^{\frac{1}{p}}} = \sup_{x \in [0,\infty)} \frac{\left(\int_{[0,x]} v d\mu\right)^{\frac{1}{q}}}{\left(\int_{[0,x]} w d\nu\right)^{\frac{1}{p}}}. \tag{2.8}$$

THEOREM 2.9. *If* $0 < q < p < \infty$, *and* $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ *then*

$$\sup_{F \in \mathfrak{M}\downarrow} \frac{\left(\int_{[0,\infty)} F^q v d\mu\right)^{\frac{1}{q}}}{\left(\int_{[0,\infty)} F^p w d\nu\right)^{\frac{1}{p}}} \approx \left(\int_{[0,\infty)} w(y) \left(\int_{[y,\infty)} W^{-1} v d\mu\right)^{\frac{r}{q}} d\nu(y)\right)^{\frac{1}{r}}. \tag{2.9}$$

Analogous results take place for $F \in \mathfrak{M}\uparrow$.

THEOREM 2.10. *If* $0 < p \leqslant q < \infty$, *then*

$$\sup_{F \in \mathfrak{M}\uparrow} \frac{\left(\int_{[0,\infty)} F^q v d\mu\right)^{\frac{1}{q}}}{\left(\int_{[0,\infty)} F^p w d\nu\right)^{\frac{1}{p}}} = \sup_{x \in [0,\infty)} \frac{\left(\int_{[x,\infty)} v d\mu\right)^{\frac{1}{q}}}{\left(\int_{[x,\infty)} w d\nu\right)^{\frac{1}{p}}}. \tag{2.10}$$

THEOREM 2.11. *If* $0 < q < p < \infty$ *and* $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, *then*

$$\sup_{F \in \mathfrak{M}\uparrow} \frac{\left(\int_{[0,\infty)} F^q v d\mu\right)^{\frac{1}{q}}}{\left(\int_{[0,\infty)} F^p w d\nu\right)^{\frac{1}{p}}} \approx \left(\int_{[0,\infty)} w(y) \left(\int_{[0,y]} \bar{W}^{-1} v d\mu\right)^{\frac{r}{q}} d\nu(y)\right)^{\frac{1}{r}}. \tag{2.11}$$

Note that Theorems 2.9 and 2.11 with $q = 1$ give analogs of Sawyer's principle of duality with general Borel measures.

3. The case $0 < p \leqslant 1$

We need the following extension of ([12], Theorem 3.3) from the weighted case to the case of measures.

THEOREM 3.1. *Let* $0 < q < 1$, $\nu = \nu_a + \nu_s$, *where* $d\nu_a = \frac{d\nu_a}{d\lambda} d\lambda$ *and* $\nu_s \perp \lambda$.

Then

$$\left(\int_{[0,\infty)} \left(\int_{[0,x]} f u d\lambda\right)^q v(x)\, d\mu(x)\right)^{\frac{1}{q}} \leqslant C \int_{[0,\infty)} f w d\nu \tag{3.1}$$

holds for all $f \in \mathfrak{M}^+$ *if and only if*

$$\mathscr{B} := \left(\int_{[0,\infty)} \left(\int_{[0,y]} \frac{v(z)\, d\mu(z)}{\tilde{w}_\downarrow(z)}\right)^{\frac{q}{1-q}} v(y)\, d\mu(y)\right)^{\frac{1-q}{q}} < \infty,$$

where

$$\tilde{w} := \frac{w}{u}\frac{d\nu_a}{d\lambda} \quad and \quad \tilde{w}(x)_{\downarrow} := \operatorname*{ess\,inf}_{t\in[0,x]}{}_{\lambda}\tilde{w}(t). \tag{3.2}$$

Moreover, $C \approx \mathscr{B}$.

Proof. Let us start with proving that (3.1) is equivalent to the following inequality

$$\left(\int_{[0,\infty)}\left(\int_{[0,x]} f u d\lambda\right)^q v(x)\, d\mu(x)\right)^{\frac{1}{q}} \leqslant C\int_{[0,\infty)} f w \frac{d\nu_a}{d\lambda} d\lambda. \tag{3.3}$$

Obviously, (3.3) implies (3.1). Let (3.1) hold and $f \in \mathfrak{M}^+$. If $\nu_s \perp \lambda$, then there exists $A \subset [0,\infty)$ such that $\lambda(A) = 0, \operatorname{supp}\nu_s = A$ and $\operatorname{supp}\nu_a = [0,\infty)\backslash A$. Let $\tilde{f} = f\chi_{[0,\infty)\backslash A}$. Then

$$\left(\int_{[0,\infty)}\left(\int_{[0,x]} f u d\lambda\right)^q v(x)\, d\mu(x)\right)^{\frac{1}{q}} = \left(\int_{[0,\infty)}\left(\int_{[0,x]} \tilde{f} u d\lambda\right)^q v(x)\, d\mu(x)\right)^{\frac{1}{q}}$$
$$\leqslant C\int_{[0,\infty)} \tilde{f} w d\nu = C\left(\int_{[0,\infty)} \tilde{f} w d\nu_a + \int_{[0,\infty)} \tilde{f} w d\nu_s\right) = C\int_{[0,\infty)} f w d\nu_a.$$

Now if we use (3.2), then (3.3) is equivalent to

$$\left(\int_{[0,\infty)}\left(\int_{[0,x]} f u d\lambda\right)^q v(x)\, d\mu(x)\right)^{\frac{1}{q}} \leqslant C\int_{[0,\infty)} f u \tilde{w} d\lambda. \tag{3.4}$$

Then, by [10, Theorem 3.1] and changing fu to f, we get that (3.4) is equivalent to

$$\left(\int_{[0,\infty)}\left(\int_{[0,x]} f d\lambda\right)^q v(x)\, d\mu(x)\right)^{\frac{1}{q}} \leqslant C\int_{[0,\infty)} f \tilde{w}_{\downarrow} d\lambda. \tag{3.5}$$

Now we follow the proof of [12, Theorem 3.3]. First let $\tilde{w}_{\downarrow}(x) = \int_{[x,\infty)} b d\lambda$ for λ-a.e. $x \in [0,\infty)$, $\int_{[0,\infty)} b d\lambda = \infty$ and $\int_{[x,\infty]} b d\lambda < \infty$. Then by changing order of integration the right hand side of (3.5) is equal to

$$C\int_{[0,\infty)}\left(\int_{[0,x]} f d\lambda\right) b(x) d\lambda(x)$$

and so (3.5) is eqivalent to

$$\left(\int_{[0,\infty)}\left(\int_{[0,x]} f d\lambda\right)^q v(x)\, d\mu(x)\right)^{\frac{1}{q}} \leqslant C\int_{[0,\infty)}\left(\int_{[0,x]} f d\lambda\right) b(x) d\lambda(x). \tag{3.6}$$

Since $\int_{[0,x]} f d\lambda$ is increasing we can replace it with F and so (3.6) is equivalent to

$$\left(\int_{[0,\infty)} F^q v d\mu\right)^{\frac{1}{q}} \leqslant C\int_{[0,\infty)} F b d\lambda \text{ with } F \in \mathfrak{M}\uparrow. \tag{3.7}$$

By [11, Theorem 2.5] and using Lemma 2.2 we get

$$\begin{aligned} C &\approx \left(\int_{[0,\infty)}\left(\int_{[0,x]}\frac{v(y)\,d\mu(y)}{\tilde{w}_{\downarrow}(y)}\right)^{\frac{1}{1-q}} b(x)\,d\lambda(x)\right)^{\frac{q}{1-q}} \\ &\approx \left(\int_{[0,\infty)}\int_{[0,x]}\frac{v(y)\,d\mu(y)}{\tilde{w}_{\downarrow}(y)}\left(\int_{[0,y]}\frac{v(z)\,d\mu(z)}{\tilde{w}_{\downarrow}(z)}\right)^{\frac{q}{1-q}} b(x)\,d\lambda(x)\right)^{\frac{1-q}{q}} \\ &= \left(\int_{[0,\infty)}\left(\int_{[0,y]}\frac{v(z)\,d\mu(z)}{\tilde{w}_{\downarrow}(z)}\right)^{\frac{q}{1-q}} v(y)\,d\mu(y)\right)^{\frac{1-q}{q}}. \end{aligned}$$

For a general $\tilde{w}_{\downarrow}$ we may and shall suppose that $\tilde{w}_{\downarrow}(x) < \infty$ for all $x > 0$. Let $N \in \mathbb{N}$ and

$$w_N(x) := \chi_{[0,N]}(x)\,\tilde{w}_{\downarrow}(x).$$

Then $w_N(+\infty) = 0$ and similar to Lemma 2.4 we find $w_N^{(0)} \in \mathfrak{M}\downarrow$ and $h_n \in \mathfrak{M}^+$ $(n \in \mathbb{N})$ such that

(1) $w_N(x) \leqslant w_N^{(0)}(x)$ for all $x \in [0,\infty)$.

(2) $w_N(x) = w_N^{(0)}(x)$ for λ-a.e. $x \in [0,\infty)$.

(3) $w_{N,k}(x) := \int_{[x,\infty)} h_k d\lambda \geqslant w_N^{(0)}(x)$ for all $x \in [0,\infty)$.

(4) The sequence $\{w_{N,k}(x)\}_{k\geqslant 1}$ is nonincreasing in k for all $x \in [0,\infty)$ and $w_N^{(0)}(x) = \lim_{k\to\infty} w_{N,k}(x)$ λ-a.e. $x \in [0,\infty)$. Then by the previous part of the proof for any $f \in \mathfrak{M}^+$ we have

$$\begin{aligned} &\left(\int_{[0,N]}\left(\int_{[0,x]} f\,d\lambda\right)^{q} v(x)\,d\mu(x)\right)^{\frac{1}{q}} \\ &\qquad \ll \left(\int_{[0,N]}\left(\int_{[0,x]}\frac{v(z)\,d\mu(z)}{w_{N,k}(z)}\right)^{\frac{q}{1-q}} v(x)\,d\mu(x)\right)^{\frac{1-q}{q}} \int_{[0,N]} f\,w_{N,k}d\lambda. \end{aligned}$$

By [6, Lemma 5] this is equivalent to

$$\begin{aligned} &\left(\int_{[0,N]}\left(\int_{[0,x]} \frac{f}{w_{N,k}}d\lambda\right)^{q} v(x)\,d\mu(x)\right)^{\frac{1}{q}} \\ &\qquad \ll \left(\int_{[0,N]}\left(\int_{[0,x]}\frac{v(z)\,d\mu(z)}{w_{N,k}(z)}\right)^{\frac{q}{1-q}} v(x)\,d\mu(x)\right)^{\frac{1-q}{q}} \int_{[0,N]} f\,d\lambda. \end{aligned}$$

By (3) and (1) we have $\frac{1}{w_{N,k}(z)} \leqslant \frac{1}{w_N(z)}$ and by (4), (2) and Monotone Convergence Theorem

$$\lim_{k\to\infty}\int_{[0,x]}\frac{f}{w_{N,k}}d\lambda = \int_{[0,x]}\frac{f}{w_N^{(0)}}d\lambda = \int_{[0,x]}\frac{f}{w_N}d\lambda.$$

Making the reverse change $\frac{f}{w_N} \to f$ we find

$$\left(\int_{[0,N]}\left(\int_{[0,x]} f\,d\lambda\right)^q v(x)\,d\mu(x)\right)^{\frac{1}{q}}$$
$$\ll \left(\int_{[0,N]}\left(\int_{[0,x]} \frac{v(z)\,d\mu(z)}{w_N(z)}\right)^{\frac{q}{1-q}} v(x)\,d\mu(x)\right)^{\frac{1-q}{q}} \int_{[0,N]} f\,w_N d\lambda$$
$$= \left(\int_{[0,N]}\left(\int_{[0,x]} \frac{v(z)\,d\mu(z)}{\tilde{w}_\downarrow(z)}\right)^{\frac{q}{1-q}} v(x)\,d\mu(x)\right)^{\frac{1-q}{q}} \int_{[0,N]} f\,\tilde{w}_\downarrow d\lambda$$
$$\leqslant \mathscr{B}\int_{[0,\infty)} f\,\tilde{w}_\downarrow d\lambda.$$

Letting $N \longrightarrow \infty$ we arrive at $C \ll \mathscr{B}$. To show the reverse inequality we again approximate $\tilde{w}_\downarrow$ from above by a monotone sequence of functions $w_k(x) := \int_{[x,\infty)} b_k d\lambda \downarrow \tilde{w}_\downarrow$. Then applying (3.6), (3.7) and [11, Theorem 2.5] we find

$$\left(\int_{[0,\infty)}\left(\int_{[0,y]} \frac{v(z)\,d\mu(z)}{w_k(z)}\right)^{\frac{q}{1-q}} v(y)\,d\mu(y)\right)^{\frac{1-q}{q}} \ll C$$

and since $w_k^{-1} \uparrow \tilde{w}_\downarrow^{-1}$ the result follows. □

DEFINITION 3.2. Let $w \in \mathfrak{M}\downarrow$ and be continuous on the left. It is known ([8, Chapter 12, §3]), that there exists a Borel measure, say η_w, such that $w(x) = \int_{[x,\infty)} d\eta_w + w(+\infty)$. We say that $w \in \mathscr{I}_2(0)$ if there exist a constant $C \geqslant 1$ such that

$$\frac{1}{w(x)} - \frac{1}{w(0)} \leqslant C\int_{[0,x]} \frac{d\eta_w}{w^2}, \quad x > 0.$$

COROLLARY 3.3. *Let* $0 < q < 1$, $w \in \mathfrak{M}\downarrow$ *and* $w \in \mathscr{I}_2(0)$. *Then*

$$\left(\int_{[0,\infty)}\left(\int_{[0,x]} h d\lambda\right)^q v(x)\,d\mu(x)\right)^{\frac{1}{q}} \leqslant C\int_{[0,\infty)} hw d\lambda$$

holds for all $h \in \mathfrak{M}^+$ *if and only if*

$$\mathbb{B} := \left(\int_{[0,\infty)}\left(\int_{[0,x]} \frac{v d\mu}{w}\right)^{\frac{q}{1-q}} v(x)\,d\mu(x)\right)^{\frac{1-q}{q}} < \infty.$$

Moreover, $C \approx \mathbb{B} \approx \mathbb{B}_0 + \mathbb{B}_1$, *where*

$$\mathbb{B}_0 := \left(\int_{[0,\infty)} v d\mu\right)^{\frac{1}{q}} w(0)^{-\frac{1}{p}},$$

$$\mathbb{B}_1 := \left(\int_{[0,\infty)} w(x)^{-\frac{q}{1-q}} \left(\int_{[x,\infty)} v d\mu \right)^{\frac{q}{1-q}} v(x) \, d\mu(x) \right)^{\frac{1-q}{q}}.$$

Proof. It follows from Theorem 3.1, Lemma 2.2 and [11, Theorem 2.6]. □

Denote

$$\Lambda(t) := \Lambda_u(t) = \int_{[0,t]} u d\lambda \tag{3.8}$$

and observe that by the change $f^p \to f$ in the inequality (1.3) we get the following equivalent inequality

$$\left(\int_{[0,\infty)} \left(H f^{\frac{1}{p}} \right)^q v d\mu \right)^{\frac{p}{q}} \leqslant C^p \left(\int_{[0,\infty)} f w d\nu \right), \qquad f \in \mathfrak{M}\downarrow. \tag{3.9}$$

THEOREM 3.4. *(a) Let $0 < p \leqslant q < \infty$ and $0 < p \leqslant 1$. Then (1.3) holds for all $f \in \mathfrak{M}\downarrow$ if and only if*

$$A_0 := \sup_{t \in [0,\infty)} \left(\int_{[0,t]} w d\nu \right)^{-\frac{1}{p}} \left(\int_{[0,t]} \Lambda^q v d\mu \right)^{\frac{1}{q}} < \infty,$$

$$\mathscr{A}_1 := \sup_{t \in [0,\infty)} \Lambda(t) \left(\int_{[0,t]} w d\nu \right)^{-\frac{1}{p}} \left(\int_{[t,\infty)} v d\mu \right)^{\frac{1}{q}} < \infty$$

and $C \approx A_0 + \mathscr{A}_1$.

(b) Let $0 < q < 1 = p$. Then (1.3) holds for all $f \in \mathfrak{M}\downarrow$ if and only if

$$\mathbb{B}_0 := \left(\int_{[0,\infty)} w(y) \left(\int_{[y,\infty)} W^{-1} \Lambda^q v d\mu \right)^{\frac{1}{1-q}} d\nu(y) \right)^{\frac{1-q}{q}} < \infty,$$

$$\mathbb{B}_1 := \left(\int_{[0,\infty)} \left(\int_{[0,x)} \operatorname*{ess\,sup}_{s \in [0,t]}{}_{\lambda} \frac{\Lambda(s)}{W(s)} v(t) d\mu(t) \right)^{\frac{q}{1-q}} v(x) \, d\mu(x) \right)^{\frac{1-q}{q}} < \infty$$

and $C \approx \mathbb{B}_0 + \mathbb{B}_1$.

(c) Let $0 < q < p < 1$, $\mathscr{V}_p(t) := \operatorname*{ess\,sup}_{s \in [0,t]}{}_{\lambda} \dfrac{\Lambda^p(s)}{W(s)}$. Then (1.3) holds for all $f \in \mathfrak{M}\downarrow$ if

$$\mathscr{B}_0 := \left(\int_{[0,\infty)} w(y) \left(\int_{[y,\infty)} W^{-1} \Lambda^q v d\mu \right)^{\frac{p}{p-q}} d\nu(y) \right)^{\frac{p-q}{pq}} < \infty,$$

$$\mathscr{B}_1 := \left(\int_{[0,\infty)} \left(\int_{[0,x]} \mathscr{V}_p(t)\, v(x) d\mu(x)\right)^{\frac{q}{p-q}} v(t)\, d\mu(t)\right)^{\frac{p-q}{pq}} < \infty$$

and only if $\mathscr{B}_0 + \mathscr{B}_1 < \infty$, *provided* $\mathscr{V}_p(t)$ *is continuous on* $(0,\infty)$ *and* $\frac{1}{\mathscr{V}_p(t)} \in \mathscr{I}_2(0)$. *Then* $C \approx \mathscr{B}_0 + \mathscr{B}_1$.

Proof. (a) Since $f \in \mathfrak{M}\downarrow$, then $(H_u f)(x) \geqslant f(x)\Lambda(x)$ and (1.3) implies

$$\left(\int_{[0,\infty)} f^q \Lambda^q v d\mu\right)^{\frac{1}{q}} \leqslant C \left(\int_{[0,\infty)} f^p w d\nu\right)^{\frac{1}{p}}, \qquad f \in \mathfrak{M}\downarrow.$$

It is known (see Theorem 2.8) that $C = A_0$ for $0 < p \leqslant q < \infty$.

Now, if $f_t = \chi_{[0,t]}$ in (1.3) then

$$C\left(\int_{[0,t]} w d\nu\right)^{\frac{1}{p}} \geqslant \left(\int_{[t,\infty)} (H_u f_t)^q v d\mu\right)^{\frac{1}{q}} = \Lambda(t)\left(\int_{[t,\infty)} v d\mu\right)^{\frac{1}{q}},$$

which implies that $C \geqslant \mathscr{A}_1$. Consequently, $A_0 + \mathscr{A}_1 \leqslant 2C$.

For the sufficiency we suppose first that $f \in \mathfrak{M}\downarrow$, $f(x) = \int_{[x,\infty)} hud\lambda$ for λ-a.e. $x \in [0,\infty)$, where $h \in \mathfrak{M}^+$ and $f(x) \geqslant \int_{[x,\infty)} hud\lambda$ for all $x \in [0,\infty)$. Let $0 < p < 1$. We have by Lemma 2.2

$$\begin{aligned}
&\int_{[0,x]} \left(\int_{[s,\infty)} hud\lambda\right) u(s)\, d\lambda(s) \\
&\quad\approx \int_{[0,x]} \left(\int_{[s,\infty)} \left(\int_{[y,\infty)} hud\lambda\right)^{p-1} h(y)u(y)d\lambda(y)\right)^{\frac{1}{p}} u(s)\, d\lambda(s) \\
&\quad\ll \int_{[0,x]} \left(\int_{[s,x]} \left(\int_{[y,\infty)} hud\lambda\right)^{p-1} h(y)u(y)d\lambda(y)\right)^{\frac{1}{p}} u(s)\, d\lambda(s) + \Lambda(x) f(x) \\
&\qquad \text{[by Minkowski inequality]} \qquad\qquad (3.10) \\
&\quad\leqslant \left(\int_{[0,x]} \left(\int_{[y,\infty)} hud\lambda\right)^{p-1} h(y)u(y)\Lambda(y)^p\, d\lambda(y)\right)^{\frac{1}{p}} + \Lambda(x) f(x).
\end{aligned}$$

Applying (3.10) we obtain

$$\left(\int_{[0,\infty)} (Hf)^q v d\mu\right)^{\frac{1}{q}} \ll \left(\int_{[0,\infty)} f^q \Lambda^q v d\mu\right)^{\frac{1}{q}} + J, \qquad (3.11)$$

where

$$J := \left(\int_{[0,\infty)}\left(\int_{[0,x]}\left(\int_{[y,\infty)} hud\lambda\right)^{p-1} h(y)u(y)\Lambda(y)^p\, d\lambda(y)\right)^{\frac{q}{p}} v(x)\, d\mu(x)\right)^{\frac{1}{q}}.$$

For the first term on the right hand side of (3.11) by Theorem 2.8 we have

$$\left(\int_{[0,\infty)} f^q\Lambda^q v d\mu\right)^{\frac{1}{q}} \leqslant A_0\left(\int_{[0,\infty)} f^p w d\nu\right)^{\frac{1}{p}}. \tag{3.12}$$

For the second term on the right hand side of (3.11) by Minkowski inequality with $\frac{q}{p} \geqslant 1$ and Lemma 2.2 we find

$$\begin{aligned} J &\leqslant \left(\int_{[0,\infty)}\left(\int_{[y,\infty)} hud\lambda\right)^{p-1} h(y)u(y)\Lambda(y)^p\left(\int_{[y,\infty)} vd\mu\right)^{\frac{p}{q}} d\lambda(y)\right)^{\frac{1}{p}} \\ &\leqslant \mathscr{A}_1\left(\int_{[0,\infty)}\left(\int_{[y,\infty)} hud\lambda\right)^{p-1} h(y)u(y)\left(\int_{[0,y]} wd\nu\right) d\lambda(y)\right)^{\frac{1}{p}} \\ &\approx \mathscr{A}_1\left(\int_{[0,\infty)}\left(\int_{[s,\infty)} hud\lambda\right)^{p} w(s)d\nu(s)\right)^{\frac{1}{p}} \leqslant \mathscr{A}_1\left(\int_{[0,\infty)} f^p wd\nu\right)^{\frac{1}{p}} \end{aligned}$$

and the inequality

$$\left(\int_{[0,\infty)} (Hf)^q\, vd\mu\right)^{\frac{1}{q}} \ll (A_0+\mathscr{A}_1)\left(\int_{[0,\infty)} f^p wd\nu\right)^{\frac{1}{p}} \tag{3.13}$$

in this case follows. For an arbitrary $f \in \mathfrak{M}\downarrow$ without loss of generality we may suppose that $f(+\infty) = 0$ and find by Lemma 2.4 that $f_0 \in \mathfrak{M}\downarrow$ and a sequence $\{h_n\}_{n\geqslant 1} \subset \mathfrak{M}^+$ such that

(1) $f_0(x) \leqslant f(x)$ for all $x \in [0,\infty)$.

(2) $f_0(x) = f(x)$ for λ-a.e. $x \in [0,\infty)$.

(3) $f_n(x) := \int_{[x,\infty)} h_n ud\lambda \leqslant f_0(x)$ for all $x \in [0,\infty)$.

(4) For all $x \in [0,\infty)$ the sequence $\{f_n(x)\}_{n\geqslant 1}$ is nondecreasing in n and $f_0(x) = \lim_{n\to\infty} f_n(x)$ λ-a.e. $x \in [0,\infty)$. Then by the Monotone Convergence

Theorem and (3.13), it yields that

$$\left(\int_{[0,\infty)} (Hf)^q\, vd\mu\right)^{\frac{1}{q}} \overset{(2)}{=} \left(\int_{[0,\infty)} (Hf_0)^q\, vd\mu\right)^{\frac{1}{q}}$$

$$\overset{(4)}{=} \lim_{n\to\infty} \left(\int_{[0,\infty)} (Hf_n)^q\, vd\mu\right)^{\frac{1}{q}} \overset{(3.13)}{\ll} (A_0 + \mathscr{A}_1) \lim_{n\to\infty} \left(\int_{[0,\infty)} f_n^p w d\nu\right)^{\frac{1}{p}}$$

$$\overset{(3)}{\leqslant} (A_0 + \mathscr{A}_1) \left(\int_{[0,\infty)} f_0^p w d\nu\right)^{\frac{1}{p}} \overset{(1)}{\leqslant} (A_0 + \mathscr{A}_1) \left(\int_{[0,\infty)} f^p w d\nu\right)^{\frac{1}{p}}$$

and the upper bound $C \ll A_0 + \mathscr{A}_1$ is proved. The case $p = 1$ is treated by the same method, but even simpler.

(b) Necessity. It follows from the inequality

$$\left(\int_{[0,\infty)} (Hf)^q\, vd\mu\right)^{\frac{1}{q}} \leqslant C \int_{[0,\infty)} f w d\nu, \qquad f \in \mathfrak{M}\downarrow, \tag{3.14}$$

that

$$\left(\int_{[0,\infty)} f^q \Lambda^q v d\mu\right)^{\frac{1}{q}} \leqslant C \int_{[0,\infty)} f w d\nu, \qquad f \in \mathfrak{M}\downarrow. \tag{3.15}$$

The last inequality is characterized by $\mathbb{B}_0$ (see Theorem 2.9 with $p = 1$.) Hence, $\mathbb{B}_0 \leqslant C$. Now, suppose $h \in \mathfrak{M}^+$ and $f(x) = \int_{[x,\infty)} hud\lambda$. Then $f \in \mathfrak{M}\downarrow$ and (3.14) gives

$$\left(\int_{[0,\infty)} \left(\int_{[0,x]} \left(\int_{[s,\infty)} hud\lambda\right) u(s)\, d\lambda(s)\right)^q v(x)\, d\mu(x)\right)^{\frac{1}{q}} \leqslant C \int_{[0,\infty)} \left(\int_{[s,\infty)} hud\lambda\right) w(s)\, d\nu(s).$$

This implies

$$\left(\int_{[0,\infty)} \left(\int_{[0,x]} h\Lambda ud\lambda\right)^q v(x)\, d\mu(x)\right)^{\frac{1}{q}} \leqslant C \int_{[0,\infty)} hWud\lambda.$$

Changing the variable $h\Lambda u \to h$ we obtain

$$\left(\int_{[0,\infty)} \left(\int_{[0,x]} hd\lambda\right)^q v(x)\, d\mu(x)\right)^{\frac{1}{q}} \leqslant C \int_{[0,\infty)} h\frac{W}{\Lambda} d\lambda.$$

The last inequality is characterized by Theorem 3.1. Consequently, $\mathbb{B}_1 \ll C$.

Sufficiency. Again, suppose first, that $f \in \mathfrak{M}\downarrow$, $f(x) = \int_{[x,\infty)} hud\lambda$ for λ-a.e. $x \in [0,\infty)$, where $h \in \mathfrak{M}$ and $f(x) \geqslant \int_{[x,\infty)} hud\lambda$ for all $x \in [0,\infty)$. Then we have

$$\left(\int_{[0,\infty)} (Hf)^q\, vd\mu\right)^{\frac{1}{q}} = \left(\int_{[0,\infty)} \left(\int_{[0,x]} \left(\int_{[s,\infty)} hud\lambda\right) u(s)d\lambda(s)\right)^q vd\mu\right)^{\frac{1}{q}}$$
$$\ll \left(\int_{[0,\infty)} \left(\int_{[0,x]} \left(\int_{[s,x)} hud\lambda\right) u(s)d\lambda(s)\right)^q vd\mu\right)^{\frac{1}{q}}$$
$$+\left(\int_{[0,\infty)} \left(\int_{[x,\infty)} hud\lambda\right)^q \Lambda^q(x)v(x)d\mu(x)\right)^{\frac{1}{q}}$$
$$\leqslant \left(\int_{[0,\infty)} \left(\int_{[0,x]} h\Lambda ud\lambda\right)^q v(x)d\mu(x)\right)^{\frac{1}{q}} + \left(\int_{[0,\infty)} f^q\Lambda^q vd\mu\right)^{\frac{1}{q}}$$

[applying Theorem 3.1 and Theorem 2.9]

$$\ll \mathbb{B}_1\left(\int_{[0,\infty)} \left(\int_{[x,\infty]} hud\lambda\right) w(x)d\nu(x)\right) + \mathbb{B}_0\left(\int_{[0,\infty)} fwd\nu\right)$$
$$\leqslant (\mathbb{B}_0+\mathbb{B}_1)\int_{[0,\infty)} fwd\nu.$$

For an arbitrary $f \in \mathfrak{M}\downarrow$ we use the arguments from the end of the part (a).

(c) Sufficiency. To prove (3.9) we again, suppose first that $f \in \mathfrak{M}\downarrow$, $f(x) = \int_{[x,\infty)} hud\lambda$ for λ-a.e. $x \in [0,\infty)$, where $h \in \mathfrak{M}^+$ and $f(x) \geqslant \int_{[x,\infty)} hud\lambda$ for all $x \in [0,\infty)$. Then, arguing as before and applying Minkowskii's inequality, we find

$$\left(\int_{[0,\infty)} \left(Hf^{\frac{1}{p}}\right)^q v\,d\mu\right)^{\frac{p}{q}}$$
$$= \left(\int_{[0,\infty)} \left(\int_{[0,x]} \left(\int_{[s,\infty)} hud\lambda\right)^{\frac{1}{p}} u(s)d\lambda(s)\right)^q v(x)d\mu(x)\right)^{\frac{p}{q}}$$
$$\ll \left(\int_{[0,\infty)} \left(\int_{[0,x]} \left(\int_{[s,x)} hud\lambda\right)^{\frac{1}{p}} u(s)d\lambda(s)\right)^q v(x)d\mu(x)\right)^{\frac{p}{q}}$$
$$+\left(\int_{[0,\infty)} \left(\int_{[x,\infty)} hud\lambda\right)^{\frac{q}{p}} \Lambda^q(x)v(x)d\mu(x)\right)^{\frac{p}{q}}$$
$$\leqslant \left(\int_{[0,\infty)} \left(\int_{[0,x]} h\Lambda^p ud\lambda\right)^{\frac{q}{p}} v(x)d\mu(x)\right)^{\frac{p}{q}} + \left(\int_{[0,\infty)} f^{\frac{q}{p}}\Lambda^q vd\mu\right)^{\frac{p}{q}}$$

applying Theorem 3.1 and Theorem 2.9

$$\ll \mathscr{B}_1^p\left(\int_{[0,\infty)}\left(\int_{[x,\infty]} hud\lambda\right)w(x)d\nu(x)\right)+\mathscr{B}_0^p\left(\int_{[0,\infty)} fwd\nu\right)$$

$$\leqslant\left(\mathscr{B}_0^p+\mathscr{B}_1^p\right)\int_{[0,\infty)} fwd\nu.$$

For an arbitrary $f\in\mathfrak{M}\downarrow$ we again use the arguments from the end of the part (a).

Necessity. The inequality $\mathscr{B}_0\leqslant C$ follows by using similar arguments as in the proof of $A_0\leqslant C$ and $\mathbb{B}_0\leqslant C$ in the parts (a) and (b).

For the rest it is sufficient to show that (3.9) implies the inequality $C\gg\mathscr{B}_1$.

Suppose for simplicity, that $\mathscr{V}_p(0)=0$. Let

$$g(t):=\max\left\{2^m,\ m\in\mathbb{Z}\colon 2^m\leqslant\mathscr{V}_p^{\frac{r}{p}}(t)\right\}$$

and

$$\tau_m:=\inf\left\{y\in[0,\infty):2^m\leqslant\mathscr{V}_p^{\frac{r}{p}}(y)\right\}.$$

Since $\mathscr{V}_p(t)$ is continuous, then τ_m exists for all $m\in\mathbb{Z}$, $\tau_m\uparrow$ and

$$\frac{\Lambda(\tau_m)^r}{W(\tau_m)^{\frac{r}{p}}}=2^m=\mathscr{V}_p^{\frac{r}{p}}(\tau_m)\leqslant\mathscr{V}_p^{\frac{r}{p}}(t)\leqslant 2^{m+1},\ \ t\in[\tau_m,\tau_{m+1}],$$

$$g(\tau_m)=2^m,\qquad g(s)\leqslant 2^{m-1}\text{ for all } s\in[0,\tau_m).$$

We note that

$$g(t)=\sum_{m\in\mathbb{Z}}2^m\chi_{[\tau_m,\tau_{m+1})}(t)\leqslant\mathscr{V}_p^{\frac{r}{p}}(t)\tag{3.16}$$

and define

$$f(t):=\int_{[t,\infty)}\frac{\left(\int_{[x,\infty)} vd\mu\right)^{\frac{r}{q}}}{W(x)}dg(x).$$

Then $f\in\mathfrak{M}\downarrow$ and by Lemma 2.2

$$\begin{aligned}\int_{[0,\infty)} fwd\nu &= \int_{[0,\infty)}\left(\int_{[x,\infty)} vd\mu\right)^{\frac{r}{q}}dg(x)\\ &\approx \int_{[0,\infty)} g(x)\left(\int_{[x,\infty)} vd\mu\right)^{\frac{r}{p}}v(x)\,d\mu(x)\\ &\leqslant \int_{[0,\infty)}\mathscr{V}_p^{\frac{r}{p}}(x)\left(\int_{[x,\infty)} vd\mu\right)^{\frac{r}{p}}v(x)\,d\mu(x):=\mathscr{B}_{2,1}^r.\end{aligned}$$

On the other hand

$$\left(\int_{[0,\infty)}\left(\int_{[0,x]} f^{\frac{1}{p}}(y)d\Lambda(y)\right)^q v(x)\,d\mu(x)\right)^{\frac{1}{q}}$$

$$\geqslant \left(\sum_m \int_{[\tau_m,\tau_{m+1})} v(x)\left(\int_{[0,\tau_m]}\left(\int_{[y,\tau_m]}\frac{\left(\int_{[s,\infty)} v d\mu\right)^{\frac{r}{q}}}{W(s)} dg(s)\right)^{\frac{1}{p}} d\Lambda(y)\right)^q d\mu(x)\right)^{\frac{1}{q}}$$

$$\geqslant \left(\sum_m \left(\int_{[\tau_m,\tau_{m+1})} v d\mu\right)\left(\int_{[\tau_m,\infty)} v d\mu\right)^{\frac{r}{p}}\right.$$

$$\left.\times\left(W(\tau_m)^{-\frac{1}{p}}\int_{[0,\tau_m]}\left(g(\tau_m)-g(y)\right)^{\frac{1}{p}} d\Lambda(y)\right)^q\right)^{\frac{1}{q}}$$

$$\gg \left(\sum_m \left(\int_{[\tau_m,\tau_{m+1})} v d\mu\right)\left(\int_{[\tau_m,\infty)} v d\mu\right)^{\frac{r}{p}}\left(\frac{2^{\frac{m}{p}}\Lambda(\tau_m)}{W(\tau_m)^{\frac{1}{p}}}\right)^q\right)^{\frac{1}{q}}$$

$$\geqslant \left(\sum_m 2^m \int_{[\tau_m,\tau_{m+1})}\left(\int_{[s,\infty)} v d\mu\right)^{\frac{r}{p}} v(s)\,d\mu(s)\right)^{\frac{1}{q}}$$

$$\gg \left(\int_{[0,\infty)} \mathscr{V}_p^{\frac{r}{p}}(s)\left(\int_{[s,\infty)} v d\mu\right)^{\frac{r}{p}} v(s)\,d\mu(s)\right)^{\frac{1}{q}} =: \mathscr{B}_{2,1}^{\frac{r}{q}}$$

With such $f(x)$ the inequality (3.9) implies $C^p\mathscr{B}^r_{2,1} \gg \mathscr{B}_{2,1}^{\frac{pr}{q}} \Rightarrow C \gg \mathscr{B}_{2,1}$. Now, if we put $f = \chi_{\{0\}}$ in (3.9), we find that

$$C \geqslant \left(\int_{[0,\infty)} v d\mu\right)^{\frac{1}{q}}\left(\frac{W(0)}{\Lambda^p(0)}\right)^{-\frac{1}{p}} = \left(\int_{[0,\infty)} v d\mu\right)^{\frac{1}{q}}\left(\frac{1}{\mathscr{V}_p(0)}\right)^{-\frac{1}{p}} =: \mathscr{B}_{2,0}.$$

It follows from Corollary 3.3, that $\mathscr{B}_{2,1}+\mathscr{B}_{2,0} \gg \mathscr{B}_1$. Hence, $C \gg \mathscr{B}_1$ and the proof is complete. □

In conclusion of this section we give an analog of part (a) of the previous theorem for non-decreasing functions.

THEOREM 3.5. *Let* $0 < p \leqslant q < \infty$ *and* $0 < p \leqslant 1$. *Then,* (1.3) *holds for all* $f \in \mathfrak{M}\uparrow$ *if and only if*

$$\bar{A_1} := \sup_{t\in[0,\infty)}\left(\int_{[t,\infty)} \Lambda^q(x,t)\,v(x)\,d\mu(x)\right)^{\frac{1}{q}} \bar{W}^{-\frac{1}{p}}(t) < \infty,$$

where

$$\Lambda(x,t) := \int_{[t,x]} ud\lambda,$$

and $C \approx \bar{A_1}$.

Proof. Replacing f in (1.3) by $f_t := \chi_{[t,\infty)}$ we find $\bar{A_1} \leqslant C$. For sufficiency we suppose that

$$f(x) = \int_{[0,x]} hud\lambda, \; h \in \mathfrak{M}^+$$

and let $0 < p < 1$. Then, by Minkowskii inequality and Lemma 2.1, we find

$$\begin{aligned}
&\int_{[0,x]} \left(\int_{[0,s]} hud\lambda\right) u(s)\, d\lambda(s) \\
&\quad \approx \int_{[0,x]} \left(\int_{[0,s]} \left(\int_{[0,y]} hud\lambda\right)^{p-1} h(y)\, u(y)\, d\lambda(y)\right)^{\frac{1}{p}} u(s)\, d\lambda(s) \\
&\quad \leqslant \left(\int_{[0,x]} \left(\int_{[0,y]} hud\lambda\right)^{p-1} h(y)\, u(y)\, \Lambda^p(x,y)\, d\lambda(y)\right)^{\frac{1}{p}}.
\end{aligned}$$

Thus, again by Minkowskii inequality

$$\begin{aligned}
&\left(\int_{[0,\infty)} (Hf)^q\, vd\mu\right)^{\frac{1}{q}} \\
&\leqslant \left(\int_{[0,\infty)} \left(\int_{[0,x]} \left(\int_{[0,y]} hud\lambda\right)^{p-1} h(y)\, u(y)\, \Lambda^p(x,y)\, d\lambda(y)\right)^{\frac{q}{p}} v(x)\, d\mu(x)\right)^{\frac{1}{q}} \\
&\leqslant \left(\int_{[0,\infty)} \left(\int_{[0,y]} hud\lambda\right)^{p-1} h(y)\, u(y) \left(\int_{[y,\infty)} \Lambda^q(x,y)\, v(x)\, d\mu(x)\right)^{\frac{p}{q}} d\lambda(y)\right)^{\frac{1}{p}} \\
&\leqslant \bar{A_1} \left(\int_{[0,\infty)} \left(\int_{[0,y]} hud\lambda\right)^{p-1} h(y)\, u(y) \left(\int_{[y,\infty)} wd\nu\right) d\lambda(y)\right)^{\frac{1}{p}} \\
&\approx \bar{A_1} \left(\int_{[0,\infty)} f^p wd\nu\right)^{\frac{1}{p}}.
\end{aligned}$$

□

A general case $f \in \mathfrak{M} \uparrow$ follows by Lemma 2.3 similar to the proof of Theorem 3.4.

4. The case $1 < p, q < \infty$

The result of this section is based on the following statement, which follows from Theorems 2.9 and 2.11 with $q = 1$.

COROLLARY 4.1. *Let* $(Tf)(x) = \int_{[0,\infty)} k(x,y)f(y)u(y)d\lambda(y)$, *where* $k(x,y)$ *is a defined on* $[0,\infty)\times[0,\infty)$, *non-negative,* $\mu\times\lambda$ *-measurable kernel.*

(a) *The inequality*

$$\left(\int_{[0,\infty)}(Tf)^q v d\mu\right)^{\frac{1}{q}} \leqslant C\left(\int_{[0,\infty)} f^p w d\nu\right)^{\frac{1}{p}} \tag{4.1}$$

for $f \in \mathfrak{M}\downarrow$, *holds if and only if the inequality*

$$\left(\int_{[0,\infty)} w(y)\left(\int_{[y,\infty)} W^{-1}(T^*g)ud\lambda\right)^{p'} d\nu(y)\right)^{\frac{1}{p'}} \tag{4.2}$$

$$\leqslant C\left(\int_{[0,\infty)} q^{q'} v d\mu\right)^{\frac{1}{q'}}, \qquad g\in\mathfrak{M}^+,$$

holds with $(T^*g)(z) = \int_{[0,\infty)} k(z,x)g(z)v(z)d\mu(z)$.

(b) *The inequality* (4.1) *for* $f \in \mathfrak{M}\uparrow$ *holds if and only if the following inequality holds:*

$$\left(\int_{[0,\infty)} w(y)\left(\int_{[0,y]} \bar{W}^{-1}(T^*g)ud\lambda\right)^{p'} d\nu(y)\right)^{\frac{1}{p'}} \leqslant C\left(\int_{[0,\infty)} q^{q'} v d\mu\right)^{\frac{1}{q'}}, \qquad g\in\mathfrak{M}^+.$$

Now let us present our result for the case $1 < p, q < \infty$.

THEOREM 4.2. *Let* $\mathbf{k}(x,y) = \int_{[y,x]} W^{-1}ud\lambda$ *and* $f \in \mathfrak{M}\downarrow$. *The inequality* (1.3) *holds for* $1 < p \leqslant q < \infty$ *if and only if* $\mathscr{A} = \max\{\mathscr{A}_{0,1} + \mathscr{A}_{0,2}\} < \infty$, *where*

$$\mathscr{A}_{0,1} := \sup_{t\in[0,\infty)}\left(\int_{[0,t]} w(y)\mathbf{k}(t,y)^{p'}d\nu(y)\right)^{\frac{1}{p'}}\left(\int_{[t,\infty)} v d\mu\right)^{\frac{1}{q}},$$

$$\mathscr{A}_{0,2} := \sup_{t\in[0,\infty)}\left(\int_{[0,t]} w d\nu\right)^{\frac{1}{p'}}\left(\int_{[t,\infty)} v(x)\mathbf{k}(x,t)^q d\mu(x)\right)^{\frac{1}{q}}.$$

Moreover, if C *is the best constant in* (1.3), *then* $C = \mathscr{A}$.

In the case $1 < q < p < \infty$ *the inequality* (1.3) *holds if and only if* $\mathscr{B} = \max\{\mathscr{B}_{0,1} + \mathscr{B}_{0,2}\} < \infty$, *where*

$$\mathscr{B}_{0,1} := \left(\int_{[0,\infty)}\left(\int_{[0,t]} w(y)\mathbf{k}(t,y)^{p'}d\nu(y)\right)^{\frac{r}{p'}}\left(\int_{[t,\infty)} vd\mu\right)^{\frac{r}{p}} v(t)d\mu(t)\right)^{\frac{1}{r}},$$

$$\mathscr{B}_{0,2} := \left(\int_{[0,\infty)}\left(\int_{[0,t]} wd\nu\right)^{\frac{r}{q'}}\left(\int_{[t,\infty)} v(x)\mathbf{k}(x,t)^{q}d\mu(x)\right)^{\frac{r}{q}} w(t)d\nu(t)\right)^{\frac{1}{r}}$$

and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. *Moreover, if* C *is the best constant in* (1.3)*, then* $C = \mathscr{B}$.

Proof. Because of Corollary 4.1 (a) the inequality (1.3) is equivalent to

$$\left(\int_{[0,\infty)} w(y)\left(\int_{[y,\infty)} W(x)^{-1}\left(\int_{[x,\infty)} gvd\mu\right)u(x)d\lambda(x)\right)^{p'} d\nu(y)\right)^{\frac{1}{p'}} \leqslant C\left(\int_{[0,\infty)} q^{q'}vd\mu\right)^{\frac{1}{q'}}, \qquad g\in\mathfrak{M}^{+}. \tag{4.3}$$

By changing the order of integration in the left hand side of (4.3) we obtain the Hardy inequality with Oinarov kernel of the form

$$\left(\int_{[0,\infty)} w(y)\left(\int_{[y,\infty)} g(z)\mathbf{k}(z,y)v(z)d\mu(z)\right)^{p'} d\nu(y)\right)^{\frac{1}{p'}} \leqslant C\left(\int_{[0,\infty)} q^{q'}vd\mu\right)^{\frac{1}{q'}}.$$

By substitution $f = g^{q'}v$ and according to Lemma 7 from [7] the last inequality is equivalent to

$$\left(\int_{[0,\infty)} w(y)\left(\int_{[y,\infty)} f(z)\mathbf{k}(z,y)v(z)^{1/q}d\mu(z)\right)^{p'} d\nu(y)\right)^{\frac{1}{p'}} \leqslant C\left(\int_{[0,\infty)} f^{q'}d\mu\right)^{\frac{1}{q'}}.$$

Thus the proof follows by applying Theorem 2.7. □

Similarly we can obtain the result for non-decreasing functions as follows.

THEOREM 4.3. *Let* $\bar{\mathbf{k}}(y,x) = \int_{[x,y]} \bar{W}^{-1}ud\lambda$ *and* $f \in \mathfrak{M}\uparrow$. *The inequality* (1.3) *holds for* $1 < p \leqslant q < \infty$ *if and only if* $\bar{\mathscr{A}} = \max\left\{\bar{\mathscr{A}}_{0,1} + \bar{\mathscr{A}}_{0,2}\right\} < \infty$, *where*

$$\bar{\mathscr{A}}_{0,1} := \sup_{t\in[0,\infty)}\left(\int_{[t,\infty)} w(y)\bar{\mathbf{k}}(y,t)^{p'}d\nu(y)\right)^{\frac{1}{p'}}\left(\int_{[0,t]} vd\mu\right)^{\frac{1}{q}},$$

$$\bar{\mathscr{A}}_{0,2} := \sup_{t\in[0,\infty)} \left(\int_{[t,\infty)} w d\nu\right)^{\frac{1}{p'}} \left(\int_{[0,t]} v(x)\bar{\mathbf{k}}(t,x)^q d\mu(x)\right)^{\frac{1}{q}}.$$

Moreover, if C *is the best constant in* (1.3)*, then* $C = \bar{\mathscr{A}}$*. In the case* $1 < q < p < \infty$ *the inequality* (1.3) *holds if and only if* $\bar{\mathscr{B}} = \max\left\{\bar{\mathscr{B}}_{0,1} + \bar{\mathscr{B}}_{0,2}\right\} < \infty,$ *where*

$$\bar{\mathscr{B}}_{0,1} := \left(\int_{[0,\infty)} \left(\int_{[t,\infty)} w(y)\bar{\mathbf{k}}(y,t)^{p'} d\nu(y)\right)^{\frac{r}{p'}} \left(\int_{[0,t]} v d\mu\right)^{\frac{r}{p}} v(t) d\mu(t)\right)^{\frac{1}{r}},$$

$$\bar{\mathscr{B}}_{0,2} := \left(\int_{[0,\infty)} \left(\int_{[t,\infty)} w d\nu\right)^{\frac{r}{q'}} \left(\int_{[0,t]} v(x)\bar{\mathbf{k}}(t,x)^q d\mu(x)\right)^{\frac{r}{q}} w(t) d\nu(t)\right)^{\frac{1}{r}}$$

and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$*. Moreover, if* C *is the best constant in* (1.3)*, then* $C = \mathscr{B}^*$.

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