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INEQUALITIES INVOLVING THE ARITHMETIC AND GEOMETRIC OPERATOR MEANS

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Abstract. As a continuation of our previous research, we give an estimate of the difference between the weighted arithmetic mean and the geometric one of positive invertible operators.

1. Introduction

We recall the Kubo-Ando theory of operator means [3]: A map $(A,B) \to A\ \sigma\ B$ in the cone of positive invertible operators is called an operator mean if the following conditions are satisfied:
monotonity: $A \leqslant C$ and $B \leqslant D$ imply $A\ \sigma\ B \leqslant C\ \sigma\ D$,
upper continuity: $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n\ \sigma B_n \downarrow A\ \sigma\ B$,
transformer inequality: $T^*(A\ \sigma\ B)T \leqslant (T^*AT)\ \sigma\ (T^*BT)$ for every operator T,
normalized condition: $A\ \sigma\ A = A$.

Mićić, Pečarić and Seo in [5, 6] obtained several inequalities associated with operator means. For example, they determined the bound β in the inequality

$$\Phi(A\ \sigma_1\ B) \geqslant \alpha\Phi(A)\ \sigma_2\ \Phi(B) + \beta\Phi(A),$$

where A and B are positive invertible operators on a Hilbert space H, σ_1, σ_2 are two operator means with not affine representing functions, Φ is a unital positive linear map and $\alpha > 0$ is a given real constant.

We shall observe the weighted arithmetic mean ∇_α and the weighted geometric mean $\sharp_\alpha$, for $\alpha \in [0,1]$, defined by

$$A\ \nabla_\alpha\ B := (1-\alpha)A + \alpha B \quad \text{and} \quad A\ \sharp_\alpha\ B := A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}},$$

respectively. Like the numerical case, the arithmetic-geometric mean inequality holds:

$$A\ \sharp_\alpha\ B \leqslant A\ \nabla_\alpha\ B \qquad \text{for all } \alpha \in [0,1]. \tag{1}$$

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In [4, Corollary 10] it is obtained the following converse inequality of the arithmetic-geometric mean inequality (1): Let A and B be positive invertible operators satisfying $0 < m_1 I \leqslant A \leqslant M_1 I$ and $0 < m_2 I \leqslant B \leqslant M_2 I$. Then

$$A \nabla_\alpha B - A \sharp_\alpha B \leqslant \max\{1-\alpha+\alpha m - m^\alpha, 1-\alpha+\alpha M - M^\alpha\}A,$$

where $m = \frac{m_2}{M_1}$ and $M = \frac{M_2}{m_1}$.

Tominaga [9] showed the another converse of (1) for the arithmetic mean and the geometric one: Let A and B be positive operators on a Hilbert space H satisfying $m \leqslant A, B \leqslant M$ for some scalars $0 < m < M$. Then (like the numerical case)

$$A \nabla_\alpha B - A \sharp_\alpha B \leqslant hL(m,M)\log M_h(1) \qquad \text{for all } \alpha \in [0,1],$$

where $h = \frac{M}{m}$, the logarithmic mean is defined for $0 < m \leqslant M$ as $L(m,M) = \frac{M-m}{\log M - \log m}$ if $M \neq m$ and $L(m,m) = m$ and the Specht ratio is defined for $h \geqslant 1$ as $M_h(1) = \frac{(h-1)h^{\frac{1}{h-1}}}{e\log h}$ if $h > 1$ and $M_1(1) = 1$.

In this paper, as a continuation of papers [5, 6, 4, 7, 8], we shall consider an another estimate of the difference between the arithmetic mean and the geometric one of positive invertible operators.

2. Inequalities involving the arithmetic and geometric means

In this section we give several inequalities involving the arithmetic and geometric means of two positive real numbers.

LEMMA 2.1. *Let x, y, α, β be positive numbers and α, β such that $\alpha + \beta = 1$. If $\alpha \in (0, \frac{1}{2}]$, then*

$$\frac{1}{2} y^{\beta-\alpha}(x^\alpha - y^\alpha)^2 \leqslant \alpha x + \beta y - x^\alpha y^\beta \leqslant \frac{1}{2} x^{\alpha-\beta}\left(x^\beta - y^\beta\right)^2. \tag{2}$$

If $\alpha \in [\frac{1}{2}, 1)$, then the reverse inequalities in (2) are valid.
For $\alpha = \frac{1}{2}$ we have the identity in (2).

Proof. First, we prove the right hand inequality in (2). We start with the known arithmetic-geometric inequality [2, Chapter II]:

$$\frac{w_1 a_1 + w_2 a_2}{W} \geqslant \sqrt[W]{a_1^{w_1} a_2^{w_2}} \quad \text{holds for any } a_1, a_2, w_1, w_2 \geqslant 0 \text{ and } W = w_1 + w_2, \tag{3}$$

with the equality if and only if $a_1 = a_2$.

Since $0 < \alpha \leqslant \frac{1}{2}$ and $\alpha + \beta = 1$, then $\beta - \alpha \geqslant 0$. Putting $a_1 = \left(\frac{x}{y}\right)^{\alpha-\beta}$, $a_2 = \frac{x}{y}$, $w_1 = \frac{1}{2\beta} > 0$, $w_2 = \frac{\beta-\alpha}{2\beta} \geqslant 0$ and $W = \frac{1}{2\beta} + \frac{\beta-\alpha}{2\beta} = 1$ in (3) gives:

$$\frac{1}{2\beta}\left(\frac{x}{y}\right)^{\alpha-\beta} + \frac{\beta-\alpha}{2\beta} \cdot \frac{x}{y} \geqslant \left(\frac{x}{y}\right)^{\frac{\alpha-\beta}{2\beta}} \left(\frac{x}{y}\right)^{\frac{\beta-\alpha}{2\beta}} = 1. \tag{4}$$

It follows

$$\frac{1}{2}x^{\alpha-\beta}y^{2\beta}+\frac{1-2\alpha}{2}x\geqslant \beta y,$$

i.e.

$$\frac{1}{2}x^{\alpha-\beta}y^{2\beta}+\frac{1}{2}x\geqslant \alpha x+\beta y.$$

Substracting $x^{\alpha}y^{\beta}$ on both sides gives

$$\frac{1}{2}x^{\alpha-\beta}\left(y^{\beta}-x^{\beta}\right)^{2}\geqslant \alpha x+\beta y-x^{\alpha}y^{\beta}.$$

This is the right hand inequality in (2).

Next, we prove the right hand reverse inequality in (2). If $\frac{1}{2}\leqslant\alpha<1$ and $\alpha+\beta=1$, then we start with the known reverse arithmetic-geometric inequality [2, Chapter II]:

$$\frac{w_1a_1+w_2a_2}{W}\leqslant \sqrt[W]{a_1^{w_1}a_2^{w_2}}\quad \text{holds for any } a_1,a_2,w_1\geqslant 0,\ \ w_2\leqslant 0 \ \text{ and } \ W=w_1+w_2, \tag{5}$$

with the equality if and only if $a_1=a_2$. Putting $a_1=\left(\frac{x}{y}\right)^{\alpha-\beta}$, $a_2=\frac{x}{y}$, $w_1=\frac{1}{2\beta}>0$, $w_2=\frac{\beta-\alpha}{2\beta}\leqslant 0$ and $W=\frac{1}{2\beta}+\frac{\beta-\alpha}{2\beta}=1$ in (5), then the reverse inequality in (4) is valid. The rest of the proof is the same as above. Then we obtain the right hand inequality in (2).

Finally, we prove the left hand inequality in (2). Since $0<\alpha\leqslant\frac{1}{2}$ and $\alpha+\beta=1$, it is enough to replace α by β and x by y in the right hand reverse inequality in (2) (or putting $a_1=\left(\frac{y}{x}\right)^{\beta-\alpha}$, $a_2=\frac{y}{x}$, $w_1=\frac{1}{2\alpha}>0$, $w_2=\frac{\alpha-\beta}{2\alpha}\leqslant 0$ in (5)). In the same way we obtain the left hand reverse inequality in (2). □

If we replace x by y or α by β in (2) we can observe new inequalities. Then, for example, we obtain the following lemma.

LEMMA 2.2. *Let x,y,α,β be positive numbers and α,β such that $\alpha+\beta=1$. Let t_0 (resp. t_1) be the unique solution of the equation $\alpha t+\beta=t^{\alpha}+\frac{1}{2}(t^{\beta}-1)^2$ (resp. $\beta t+\alpha=t^{\beta}+\frac{1}{2}(t^{\alpha}-1)^2$) on $(0,1)$ for $\alpha\neq\frac{1}{2}$.*

If $\alpha\in(0,\frac{1}{2})$, then

$$\alpha x+\beta y-x^{\alpha}y^{\beta}\ \geqslant\ \frac{1}{2}y^{\alpha-\beta}\left(x^{\beta}-y^{\beta}\right)^{2}\qquad \textit{in the case}\quad 0<\frac{x}{y}\leqslant t_0, \tag{6}$$

$$\alpha x+\beta y-x^{\alpha}y^{\beta}\ \leqslant\ \frac{1}{2}y^{\alpha-\beta}\left(x^{\beta}-y^{\beta}\right)^{2}\qquad \textit{in the case}\quad t_0\leqslant\frac{x}{y}. \tag{7}$$

If $\alpha\in(\frac{1}{2},1)$, then the reverse inequality in (6) and (7) are valid given same conditions.

If $\alpha\in(\frac{1}{2},1)$, then

$$\alpha x+\beta y-x^{\alpha}y^{\beta}\ \geqslant\ \frac{1}{2}x^{\beta-\alpha}\left(x^{\alpha}-y^{\alpha}\right)^{2}\qquad \textit{in the case}\quad 0<\frac{y}{x}\leqslant t_1, \tag{8}$$

$$\alpha x+\beta y-x^{\alpha}y^{\beta}\ \leqslant\ \frac{1}{2}x^{\beta-\alpha}\left(x^{\alpha}-y^{\alpha}\right)^{2}\qquad \textit{in the case}\quad t_1\leqslant\frac{y}{x}. \tag{9}$$

If $\alpha \in (0, \frac{1}{2})$ *, then the reverse inequality in* (8) *and* (9) *are valid given same conditions.*

For $\alpha = \frac{1}{2}$ *we have the identity in* (6), (7), (8) *and* (9).

Proof. Let $\alpha \in (0, \frac{1}{2})$ hold. Then for every $\beta \in (\frac{1}{2}, 1)$, we put a function $f_\beta(t)$ derived from the inequality (6) or (7) as follows:

$$f_\beta(t) := \left(\beta - \frac{1}{2}\right) + (1-\beta)t - t^{1-\beta} + t^\beta - \frac{1}{2}t^{2\beta}, \qquad t \in (0,\infty). \tag{10}$$

We consider the sign of this function. We have

$$f_\beta'(t) = \left(1 - \beta - \beta t^{2\beta-1}\right)\left(1 - t^{-\beta}\right),$$
$$f_\beta''(t) = \frac{\beta}{t^{\beta+2}}\left[(\beta-1)\left(t^{2\beta} - t\right) + (1-2\beta)t^{3\beta}\right].$$

We have the stationary points for $t = 1$ and $t = \sqrt[\beta-\alpha]{\alpha/\beta}$. (Obviously $\sqrt[\beta-\alpha]{\alpha/\beta} = \sqrt[2\beta-1]{\frac{1}{\beta} - 1} < 1$ for every $0 < \beta < 1$). Since $\beta \in (\frac{1}{2}, 1)$, then $f_\beta''(1) < 0$, $f_\beta''(\sqrt[\beta-\alpha]{\alpha/\beta}) > 0$, i.e. $f_\beta(1)$ is a local maximum, $f_\beta(\sqrt[\beta-\alpha]{\alpha/\beta})$ is a local minimum. Then there is the unique solution t_0 of the equation $f_\beta(t) = 0$ on $(0,1)$ for $\alpha \neq \frac{1}{2}$. Since $f_\beta(0) = \beta - \frac{1}{2} \geqslant 0$, $f_\beta(t_0) = 0$, $f_\beta(\sqrt[\beta-\alpha]{\alpha/\beta}) \leqslant 0$ and $f_\beta(1) = 0$, it follows that $f_\beta \leqslant 0$ on $[t_0, \infty)$ and $f_\beta \geqslant 0$ on $(0, t_0]$.

Replacing t by $\frac{x}{y}$ in (10) and next multiplying by $y > 0$ on both sides, we obtain that

$$\alpha x + \beta y - x^\alpha y^\beta \leqslant \frac{1}{2} y^{\alpha-\beta}\left(x^\beta - y^\beta\right)^2$$

holds in the case $\frac{x}{y} \in [t_0, \infty)$ and the reverse inequality holds in the case $\frac{x}{y} \in (0, t_0]$. Then we have the desired inequalities (6) and (7).

Similarly, using (10) in the case $\alpha \in (\frac{1}{2}, 1)$, we obtain that $f_\beta \geqslant 0$ on $[t_0, \infty)$ and $f_\beta \leqslant 0$ on $(0, t_0]$. Then we have the reverse inequality in (6) (resp. (7)) in the case $0 < \frac{x}{y} \leqslant t_0$ (resp. $t_0 \leqslant \frac{x}{y}$).

Now, in the case $\alpha \in (\frac{1}{2}, 1)$, the inequalities (8) and (9) follow from (6) and (7), respectively, if we replace α by β, x by y and t_0 by t_1. If $\alpha \in (0, \frac{1}{2})$, we have in the same way that the reverse inequality in (8) (resp. (9)) holds in the case $0 < \frac{y}{x} \leqslant t_1$ (resp. $t_1 \leqslant \frac{y}{x}$) □

Now, we can compare bounds given in Lemma 2.1 and Lemma 2.2. Then we obtain the following lemma. This statement is obvious. We omit the proof.

LEMMA 2.3. *Let* x, y, α, β *be positive numbers and* α, β *such that* $\alpha + \beta = 1$. *Let* t_0 *(resp.* t_1 *) be the unique solution of the equation* $\alpha t + \beta = t^\alpha + \frac{1}{2}(t^\beta - 1)^2$ *(resp.* $\beta t + \alpha = t^\beta + \frac{1}{2}(t^\alpha - 1)^2$ *) on* $(0,1)$ *for* $\alpha \neq \frac{1}{2}$.

If $\alpha \in (0, \frac{1}{2})$, *then*

$$\begin{aligned} \frac{1}{2}x^{\beta-\alpha}(x^\alpha - y^\alpha)^2 &\leqslant \frac{1}{2}y^{\beta-\alpha}(x^\alpha - y^\alpha)^2 \leqslant \alpha x + \beta y - x^\alpha y^\beta \\ &\leqslant \frac{1}{2}y^{\alpha-\beta}\left(x^\beta - y^\beta\right)^2 \leqslant \frac{1}{2}x^{\alpha-\beta}\left(x^\beta - y^\beta\right)^2 \end{aligned} \tag{11}$$

holds in the case $t_0 \leqslant \frac{x}{y} < 1$ *and*

$$\begin{aligned}\frac{1}{2}y^{\beta-\alpha}\left(x^{\alpha}-y^{\alpha}\right)^2 &\leqslant \frac{1}{2}x^{\beta-\alpha}\left(x^{\alpha}-y^{\alpha}\right)^2 \leqslant \alpha x+\beta y-x^{\alpha}y^{\beta} \\ &\leqslant \frac{1}{2}x^{\alpha-\beta}\left(x^{\beta}-y^{\beta}\right)^2 \leqslant \frac{1}{2}y^{\alpha-\beta}\left(x^{\beta}-y^{\beta}\right)^2\end{aligned} \tag{12}$$

holds in the case $t_1 \leqslant \frac{y}{x} < 1$. *Otherwise, the inequality* (2) *holds.*

If $\alpha \in (\frac{1}{2},1)$, *then the reverse inequalities in* (11) *and* (12) *are valid given same conditions.*

For $\alpha = \frac{1}{2}$ *we have identities in* (11) *and* (12).

REMARK 2.4. We can obtain similar results if we replace $\frac{1}{2}y^{\alpha-\beta}\left(x^{\beta}-y^{\beta}\right)^2$ by $\frac{1}{2}y^{\beta-\alpha}\left(x^{\beta}-y^{\beta}\right)^2$ in (6) - (7) and $\frac{1}{2}x^{\beta-\alpha}\left(x^{\alpha}-y^{\alpha}\right)^2$ by $\frac{1}{2}x^{\alpha-\beta}\left(x^{\alpha}-y^{\alpha}\right)^2$ in (8) - (9).

If we replace the factor $\frac{1}{2}$ by $\frac{1}{2}\frac{\beta}{\alpha}$ or α by β in Lemma 2.1 and Lemma 2.2 we can observe new inequalities. Then, for example, we obtain the following lemma.

LEMMA 2.5. *Let* x,y,α,β *be positive numbers and* α,β *such that* $\alpha+\beta=1$. *Suppose that either of the following conditions holds*

(i) $x<y$ *and* $\alpha\in(0,\frac{1}{2}]$, (ii) $y<x$ *and* $\alpha\in[\frac{1}{2},1)$,
(iii) $x<y$ *and* $\alpha\in[\frac{1}{2},1)$, (iv) $y<x$ *and* $\alpha\in(0,\frac{1}{2}]$.

Then

$$\frac{1}{2}\frac{\beta}{\alpha}x^{\beta-\alpha}\left(x^{\alpha}-y^{\alpha}\right)^2 \leqslant \alpha x+\beta y-x^{\alpha}y^{\beta} \leqslant \frac{1}{2}\frac{\beta}{\alpha}y^{\beta-\alpha}\left(x^{\alpha}-y^{\alpha}\right)^2, \tag{13}$$

$$\frac{1}{2}\frac{\alpha}{\beta}y^{\alpha-\beta}\left(x^{\beta}-y^{\beta}\right)^2 \leqslant \alpha x+\beta y-x^{\alpha}y^{\beta} \leqslant \frac{1}{2}\frac{\alpha}{\beta}x^{\alpha-\beta}\left(x^{\beta}-y^{\beta}\right)^2 \tag{14}$$

hold in the cases (i) *and* (ii) *and the reverse inequalities in* (13) *and* (14) *hold in the cases* (iii) *and* (iv).

For $\alpha=\frac{1}{2}$ *we have identities in* (13) *and* (14).

Proof. First, we prove the right inequality in (13), i.e. that

$$\alpha x+\beta y-x^{\alpha}y^{\beta} \leqslant \frac{1}{2}\frac{\beta}{\alpha}y^{\beta-\alpha}\left(x^{\alpha}-y^{\alpha}\right)^2 \tag{15}$$

holds in the cases (i) and (ii) and the reverse inequality in (15) holds in the cases (iii) and (iv).

For every $\alpha,\beta\in(0,1)$, $\alpha+\beta=1$, we put a function $f_{\alpha,\beta}(t)$ deriver from the inequality (15) as follows:

$$f_{\alpha,\beta}(t) := -\frac{1}{2}\frac{\beta}{\alpha}t^{2\alpha}+\frac{\beta-\alpha}{\alpha}t^{\alpha}+\alpha t+\frac{1}{2}\frac{\beta}{\alpha}(\alpha-\beta), \qquad t\in(0,\infty). \tag{16}$$

We have $f_{\alpha,\beta}(1)=0$, $f'_{\alpha,\beta}(t)=-\beta t^{2\alpha-1}+(\beta-\alpha)t^{\alpha-1}+\alpha$, $f'_{\alpha,\beta}(1)=0$ and

$$f''_{\alpha,\beta}(t)=\beta(\alpha-\beta)t^{\alpha-2}\left(1-t^{\alpha}\right). \tag{17}$$

Let be $0 < t < 1$. Observe $f''_{\alpha,\beta}(t) \geqslant 0$ if $\alpha \geqslant \beta$ and $f''_{\alpha,\beta}(t) \leqslant 0$ if $\alpha \leqslant \beta$. It follows that $f_{\alpha,\beta}(t) \geqslant 0$ if $\alpha \geqslant \frac{1}{2}$ and $f_{\alpha,\beta}(t) \leqslant 0$ if $\alpha \leqslant \frac{1}{2}$.

Replacing t by $\frac{x}{y}$ in (16) and next multiplying by $y > 0$ on both sides, we obtain that

$$\alpha x + \beta y - x^\alpha y^\beta - \frac{1}{2}\frac{\beta}{\alpha} y^{\beta-\alpha} (x^\alpha - y^\alpha)^2 \geqslant 0 \tag{18}$$

holds if $x < y$ and $\alpha \geqslant \frac{1}{2}$ and the reverse inequality in (18) holds if $x < y$ and $\alpha \leqslant \frac{1}{2}$. Hence we have the desired inequality (15) in the case (i) and the reverse inequality in (15) in the case (iii).

Let be $t > 1$ in (17). Then $f''_{\alpha,\beta}(t) \geqslant 0$ if $\alpha \leqslant \beta$ and $f''_{\alpha,\beta}(t) \leqslant 0$ if $\beta \leqslant \alpha$. It follows that $f_{\alpha,\beta}(t) \geqslant 0$ if $\alpha \leqslant \frac{1}{2}$ and $f_{\alpha,\beta}(t) \leqslant 0$ if $\alpha \geqslant \frac{1}{2}$.

Replacing t by $\frac{x}{y}$ in (16) and next multiplying by $y > 0$ on both sides, we have the desired inequality (15) in the case (ii) and the reverse inequality in (15) in the case (iv).

Next, we prove the left inequality in (13), i.e. that

$$\frac{1}{2}\frac{\beta}{\alpha} x^{\beta-\alpha} (x^\alpha - y^\alpha)^2 \leqslant \alpha x + \beta y - x^\alpha y^\beta \tag{19}$$

holds in the cases (i) and (ii) and the reverse inequality in (19) holds in the cases (iii) and (iv). We put a function $f_{\alpha,\beta}(t)$ derived from the inequality (19) as follows:

$$f_{\alpha,\beta}(t) := -\frac{1}{2}\frac{\beta}{\alpha} t^{2\alpha} + \frac{\beta}{\alpha} t^\alpha - t^\beta + \beta t + \left(\alpha - \frac{1}{2}\frac{\beta}{\alpha}\right), \qquad t \in (0,\infty). \tag{20}$$

We have $f_{\alpha,\beta}(1) = 0$, $f'_{\alpha,\beta}(t) = -\beta t^{2\alpha-1} + \beta t^{\alpha-1} - \beta t^{\beta-1} + \beta$, $f'_{\alpha,\beta}(1) = 0$ and

$$f''_{\alpha,\beta}(t) = \frac{\beta}{t^{\alpha+1}} (1 - t^\alpha) \left[(1-2\beta) t^{2\alpha} + (1-\beta) t^\alpha + 1 - \beta\right]. \tag{21}$$

It is obvious that $(1-2\beta)t^{2\alpha} + (1-\beta)t^\alpha + 1 - \beta > 0$ if $\alpha \geqslant \frac{1}{2}$ and $(1-2\beta)t^{2\alpha} + (1-\beta)t^\alpha + 1 - \beta < 0$ if $\alpha \leqslant \frac{1}{2}$.

Let be $0 < t < 1$. Then $f''_{\alpha,\beta}(t) \geqslant 0$ if $\alpha \geqslant \frac{1}{2}$ and $f''_{\alpha,\beta}(t) \leqslant 0$ if $\alpha \leqslant \frac{1}{2}$. It follows $f_{\alpha,\beta}(t) \geqslant 0$ if $\alpha \geqslant \frac{1}{2}$ and $f_{\alpha,\beta}(t) \leqslant 0$ if $\alpha \leqslant \frac{1}{2}$.

Replacing t by $\frac{y}{x}$ in (20) and next multiplying by $x > 0$ on both sides, we obtain the inequality (19) in the case (ii) and the reverse inequality in (19) in the case (iv).

Similarly, putting $t > 1$ in (21), we obtain the inequality (19) in the case (i) and the reverse inequality in (19) in the case (iii).

Next, replacing x by y and α by β in (13) in the case (i) (resp. (ii)), we obtain (14) in the case (ii) (resp. (i)). Similarly, using the reverse inequalities in (13) we obtain the reverse inequalities in (14). □

Next, in the following lemma we compare bounds given in Lemma 2.5.

LEMMA 2.6. *Let x, y, α, β be positive numbers and α, β such that $\alpha + \beta = 1$. Suppose that either of the following conditions holds*
(i) $x < y$ *and* $\alpha \in (0, \frac{1}{2}]$, (ii) $y < x$ *and* $\alpha \in [\frac{1}{2}, 1)$,

(iii) $x < y$ *and* $\alpha \in [\frac{1}{2}, 1)$, (iv) $y < x$ *and* $\alpha \in (0, \frac{1}{2}]$.
Then

$$\lambda_1 \leqslant \alpha x + \beta y - x^\alpha y^\beta \leqslant \lambda_2, \tag{22}$$

where

$$\lambda_1 = \begin{cases} \frac{1}{2}\frac{\alpha}{\beta} y^{\alpha-\beta}\left(x^\beta - y^\beta\right)^2, \\ \frac{1}{2}\frac{\beta}{\alpha} x^{\beta-\alpha}\left(x^\alpha - y^\alpha\right)^2, \end{cases} \qquad \lambda_2 = \begin{cases} \frac{1}{2}\frac{\beta}{\alpha} y^{\beta-\alpha}\left(x^\alpha - y^\alpha\right)^2, & \textit{if } \text{(i)}, \\ \frac{1}{2}\frac{\alpha}{\beta} x^{\alpha-\beta}\left(x^\beta - y^\beta\right)^2, & \textit{if } \text{(ii)}. \end{cases}$$

In the case (iii) *(resp.* (iv)*) the reverse inequality in* (22) *is halid with the same bound as in the case* (i) *(resp.* (ii)*).*

For $\alpha = \frac{1}{2}$ *we have the identity in* (22).

Proof. Using Lemma 2.5 we obtain:

$$\max\left\{\frac{1}{2}\frac{\alpha}{\beta} y^{\alpha-\beta}\left(x^\beta - y^\beta\right)^2, \frac{1}{2}\frac{\beta}{\alpha} x^{\beta-\alpha}\left(x^\alpha - y^\alpha\right)^2\right\} \tag{23}$$
$$\leqslant \alpha x + \beta y - x^\alpha y^\beta \leqslant \min\left\{\frac{1}{2}\frac{\beta}{\alpha} y^{\beta-\alpha}\left(x^\alpha - y^\alpha\right)^2, \frac{1}{2}\frac{\alpha}{\beta} x^{\alpha-\beta}\left(x^\beta - y^\beta\right)^2\right\}$$

in the cases (i) and (ii);

$$\max\left\{\frac{1}{2}\frac{\alpha}{\beta} x^{\alpha-\beta}\left(x^\beta - y^\beta\right)^2, \frac{1}{2}\frac{\beta}{\alpha} y^{\beta-\alpha}\left(x^\alpha - y^\alpha\right)^2\right\} \tag{24}$$
$$\leqslant \alpha x + \beta y - x^\alpha y^\beta \leqslant \min\left\{\frac{1}{2}\frac{\beta}{\alpha} x^{\beta-\alpha}\left(x^\alpha - y^\alpha\right)^2, \frac{1}{2}\frac{\alpha}{\beta} y^{\alpha-\beta}\left(x^\beta - y^\beta\right)^2\right\}$$

in the cases (iii) and (iv).

First, we shall prove that

$$\lambda_2 = \min\left\{\frac{1}{2}\frac{\beta}{\alpha} y^{\beta-\alpha}\left(x^\alpha - y^\alpha\right)^2, \frac{1}{2}\frac{\alpha}{\beta} x^{\alpha-\beta}\left(x^\beta - y^\beta\right)^2\right\} = \frac{1}{2}\frac{\beta}{\alpha} y^{\beta-\alpha}\left(x^\alpha - y^\alpha\right)^2$$

in the cases (i), i.e. that

$$\frac{1}{2}\frac{\beta}{\alpha} y^{\beta-\alpha}\left(x^\alpha - y^\alpha\right)^2 \leqslant \frac{1}{2}\frac{\alpha}{\beta} x^{\alpha-\beta}\left(x^\beta - y^\beta\right)^2 \qquad \text{if} \quad \text{(i)}. \tag{25}$$

For every $\alpha, \beta \in (0,1)$, $\alpha \leqslant \beta$ we consider the sign of a function

$$f_{\alpha,\beta}(t) = \frac{t^{\frac{\alpha}{2}} - t^{-\frac{\alpha}{2}}}{\alpha} - \frac{t^{\frac{\beta}{2}} - t^{-\frac{\beta}{2}}}{\beta}, \qquad t \in (0,1).$$

Since $\lim_{t\to 0+} f_{\alpha,\beta}(t) = \infty$, $f(1) = 0$ and $f'_{\alpha,\beta}(t) = \frac{(t^{\alpha/2} - t^{\beta/2})(t^{1/2}-1)}{2t^{3/2}} \leqslant 0$, it follows that $f_{\alpha,\beta}(t) \geqslant 0$, i.e. $0 \geqslant \frac{t^{\alpha/2} - t^{-\alpha/2}}{\alpha} \geqslant \frac{t^{\beta/2} - t^{-\beta/2}}{\beta}$. Then we have $\left(\frac{t^{\alpha/2} - t^{-\alpha/2}}{\alpha}\right)^2 \leqslant \left(\frac{t^{\beta/2} - t^{-\beta/2}}{\beta}\right)^2$.

Replacing t by $\frac{x}{y}$ in the last inequality and next multiplying by $\frac{1}{2}\alpha\beta x^{\alpha}y^{\beta}>0$ on both sides, we obtain (25).

Next, multiplying by $x^{\beta-\alpha}y^{\alpha-\beta}$ on both sides (25), we obtain that $\frac{1}{2}\frac{\alpha}{\beta}y^{\alpha-\beta}\left(x^{\beta}-y^{\beta}\right)^{2} \geqslant \frac{1}{2}\frac{\beta}{\alpha}x^{\beta-\alpha}\left(x^{\alpha}-y^{\alpha}\right)^{2}$, i.e. $\lambda_1=\frac{1}{2}\frac{\alpha}{\beta}y^{\alpha-\beta}\left(x^{\beta}-y^{\beta}\right)^{2}$ in the case (i).

Now, replacing α by β and x by y in (25), we obtain we obtain that $\lambda_2=\frac{1}{2}\frac{\alpha}{\beta}x^{\alpha-\beta}\left(x^{\beta}-y^{\beta}\right)^{2} \leqslant \frac{1}{2}\frac{\beta}{\alpha}y^{\beta-\alpha}\left(x^{\alpha}-y^{\alpha}\right)^{2}$ in the case (ii). In the same way we obtain λ_1 in the cases (ii). Finally, the reverse inequalities are evident. □

Now, we obtain the following lemma comparing bounds given in Lemma 2.3 and Lemma 2.6.

LEMMA 2.7. *Let x,y,α,β be positive numbers and α,β such that $\alpha+\beta=1$. Let t_0 be the unique solution of the equation $\sqrt{\beta}(t^{\alpha}-1)=\sqrt{\alpha}(t^{\beta}-1)$ on $(0,1)$ for $\alpha\neq\frac{1}{2}$.*

If $\alpha\in(0,\frac{1}{2})$, then

$$\alpha x+\beta y-x^{\alpha}y^{\beta} \leqslant \frac{1}{2}\frac{\beta}{\alpha}y^{\beta-\alpha}\left(x^{\alpha}-y^{\alpha}\right)^{2} \qquad \text{in the case } x<y; \tag{26}$$

$$\alpha x+\beta y-x^{\alpha}y^{\beta} \geqslant \begin{cases} \frac{1}{2}y^{\beta-\alpha}\left(x^{\alpha}-y^{\alpha}\right)^{2} & \text{in the case } 0<\frac{x}{y}\leqslant t_0, \\ \frac{1}{2}\frac{\alpha}{\beta}y^{\alpha-\beta}\left(x^{\beta}-y^{\beta}\right)^{2} & \text{in the case } t_0\leqslant\frac{x}{y}<1. \end{cases} \tag{27}$$

In the case $y<x$ the reverse inequality in (26) is valid. In the case $0<\frac{y}{x}\leqslant t_0$ (resp. $t_0\leqslant\frac{y}{x}<1$) the reverse inequality in the first inequality (resp. the second inequality) in (27) is valid.

If $\alpha\in(\frac{1}{2},1)$, then

$$\alpha x+\beta y-x^{\alpha}y^{\beta} \leqslant \frac{1}{2}\frac{\alpha}{\beta}x^{\alpha-\beta}\left(x^{\beta}-y^{\beta}\right)^{2} \qquad \text{in the case } y<x; \tag{28}$$

$$\alpha x+\beta y-x^{\alpha}y^{\beta} \geqslant \begin{cases} \frac{1}{2}x^{\alpha-\beta}\left(x^{\beta}-y^{\beta}\right)^{2} & \text{in the case } 0<\frac{y}{x}\leqslant t_0, \\ \frac{1}{2}\frac{\beta}{\alpha}x^{\beta-\alpha}\left(x^{\alpha}-y^{\alpha}\right)^{2} & \text{in the case } t_0\leqslant\frac{y}{x}<1. \end{cases} \tag{29}$$

In the case $x<y$ the reverse inequality in (28) is valid. In the case $0<\frac{x}{y}\leqslant t_0$ (resp. $t_0\leqslant\frac{x}{y}<1$) the reverse inequality in the first inequality (resp. the second inequality) in (29) is valid.

For $\alpha=\frac{1}{2}$ we have identities in (26), (27), (28) and (29).

Proof. In this proof we use the same cases (i)–(iv) as in Lemma 2.6. Using results given in Lemma 2.3 and Lemma 2.6, we have to decide

(a) $\min\left\{\frac{1}{2}\frac{\beta}{\alpha}y^{\beta-\alpha}\left(x^{\alpha}-y^{\alpha}\right)^{2}, \frac{1}{2}x^{\alpha-\beta}\left(x^{\beta}-y^{\beta}\right)^{2}\right\}$ and

$\max\left\{\frac{1}{2}\frac{\alpha}{\beta}y^{\alpha-\beta}\left(x^{\beta}-y^{\beta}\right)^{2}, \frac{1}{2}y^{\beta-\alpha}\left(x^{\alpha}-y^{\alpha}\right)^{2}\right\}$

in the case (i);

(b) $\min\left\{\frac{1}{2}\frac{\alpha}{\beta}x^{\alpha-\beta}\left(x^{\beta}-y^{\beta}\right)^{2}, \frac{1}{2}y^{\beta-\alpha}\left(x^{\alpha}-y^{\alpha}\right)^{2}\right\}$ and

$\max\left\{\frac{1}{2}\frac{\beta}{\alpha}x^{\beta-\alpha}\left(x^{\alpha}-y^{\alpha}\right)^{2}, \frac{1}{2}x^{\alpha-\beta}\left(x^{\beta}-y^{\beta}\right)^{2}\right\}$

in the case (ii).

In the case (iii) (resp. (iv)) we can replace max by min in (a) (resp. (b)).

Let (i) hold. Since $\frac{\alpha}{\beta} \leqslant 1$ it follows

$$\frac{1}{2}\frac{\beta}{\alpha}y^{\beta-\alpha}\left(x^{\alpha}-y^{\alpha}\right)^{2} \leqslant \frac{1}{2}\frac{\alpha}{\beta}x^{\alpha-\beta}\left(x^{\beta}-y^{\beta}\right)^{2} \leqslant \frac{1}{2}x^{\alpha-\beta}\left(x^{\beta}-y^{\beta}\right)^{2}.$$

Then we have $\min\left\{\frac{1}{2}\frac{\beta}{\alpha}y^{\beta-\alpha}\left(x^{\alpha}-y^{\alpha}\right)^{2}, \frac{1}{2}x^{\alpha-\beta}\left(x^{\beta}-y^{\beta}\right)^{2}\right\} = \frac{1}{2}\frac{\beta}{\alpha}y^{\beta-\alpha}\left(x^{\alpha}-y^{\alpha}\right)^{2}$ and (26) is valid.

Next, for every $\alpha,\beta \in (0,1)$, $\alpha \leqslant \beta$, we consider the sign of a function

$$f_{\alpha,\beta}(t) = \frac{t^{\alpha}-1}{\sqrt{\alpha}} - \frac{t^{\beta}-1}{\sqrt{\beta}}, \qquad t \in [0,1].$$

We have $f'_{\alpha,\beta}(t) = \sqrt{\alpha}t^{\alpha-1} - \sqrt{\beta}t^{\beta-1}$, $\lim_{t\to 0+} f'_{\alpha,\beta}(t) = \infty$, $f'_{\alpha,\beta}(1) = \sqrt{\alpha} - \sqrt{\beta} \leqslant 0$ and $f'_{\alpha,\beta}(t) = 0$ for $t = t_1 = \sqrt[2(1-2\alpha)]{\frac{\alpha}{1-\alpha}} \in (0,1)$. Then $f(t_1) > 0$ is the maximum value since $f''_{\alpha,\beta}(t_1) = t_1^{\beta-2}\sqrt{\beta}(\alpha-\beta) < 0$. Then there is the unique solution t_0 of the equation $f_{\alpha,\beta}(t) = 0$ (i.e. $\sqrt{\beta}(t^{\alpha}-1) = \sqrt{\alpha}(t^{\beta}-1)$) on $(0,1)$. Since $f_{\alpha,\beta}(0) = \frac{\sqrt{\alpha}-\sqrt{\beta}}{\sqrt{\alpha}\sqrt{\beta}} \leqslant 0$, $f(t_0) = 0$, $f(t_1) \geqslant 0$ and $f(1) = 0$, it follows that $f_{\alpha,\beta}(t) \leqslant 0$ for $t \in [0,t_0]$ and $f_{\alpha,\beta}(t) \geqslant 0$ for $t \in [t_0,1]$. Then $\frac{t^{\alpha}-1}{\sqrt{\alpha}} \leqslant \frac{t^{\beta}-1}{\sqrt{\beta}} \leqslant 0$ if $t \in [0,t_0]$ and $0 \geqslant \frac{t^{\alpha}-1}{\sqrt{\alpha}} \geqslant \frac{t^{\beta}-1}{\sqrt{\beta}}$ if $t \in [t_0,1]$. It follows

$$\left(\frac{t^{\alpha}-1}{\sqrt{\alpha}}\right)^{2} \geqslant \left(\frac{t^{\beta}-1}{\sqrt{\beta}}\right)^{2} \qquad \text{if } t \in [0,t_0]$$

and the reverse inequality if $t \in [t_0,1]$.

Replacing t by $\frac{x}{y}$ in the above inequality and next multiplying by $\frac{1}{2}\alpha y > 0$ on both sides, we obtain that $\max\left\{\frac{1}{2}\frac{\alpha}{\beta}y^{\alpha-\beta}\left(x^{\beta}-y^{\beta}\right)^{2}, \frac{1}{2}y^{\beta-\alpha}\left(x^{\alpha}-y^{\alpha}\right)^{2}\right\} = \frac{1}{2}y^{\beta-\alpha}\left(x^{\alpha}-y^{\alpha}\right)^{2}$ if $t \in [0,t_0]$ and $\max\left\{\frac{1}{2}\frac{\alpha}{\beta}y^{\alpha-\beta}\left(x^{\beta}-y^{\beta}\right)^{2}, \frac{1}{2}y^{\beta-\alpha}\left(x^{\alpha}-y^{\alpha}\right)^{2}\right\} = \frac{1}{2}\frac{\alpha}{\beta}y^{\alpha-\beta}\left(x^{\beta}-y^{\beta}\right)^{2}$ if $t \in [t_0,1]$. Then (27) is valid.

Next, in the case (ii) it is enough to replace α by β and x by y in (26) and (27). Finally, the reverse inequalities are evident. $\square$

REMARK 2.8. i) If we replace α by $\frac{1}{p}$, β by $\frac{1}{q}$, x by u^p ($u > 0$), y by $w^p > 0$ ($w > 0$), in Lemma 2.1 and denote $P(u,v) = \frac{u^p}{p} + \frac{v^q}{q} - uv$, $\Phi(t) = t^{p-1}$, we obtain that

$$\frac{1}{2}w^{p-2}\left(u-w\right)^2 < P(u,\Phi(w)) < \frac{1}{2}u^{2-p}\left(\Phi(u)-\Phi(w)\right)^2 \tag{30}$$

holds if $\frac{1}{p} < \frac{1}{2}$ and $u^p \neq w^{(p-1)q}$, i.e. $p > 2$ and $u \neq w$ and the reverse inequalities in (30) are valid if $p < 2$ and $u \neq w$.

The right hand inequality in (30) is just [1, Inequality (2.11)]. Moreover, in Lemma 2.1 we give one proof more of [1, Inequality (2.11)].

Notice that $P(u,v)$ is the difference between the arithmetic mean and the geometric one of u^p and v^q. Then we have directly that $P(u,v) \geqslant 0$ given in [1, Inequality (2.10)].

ii) Using Lemma 2.6 and doing equal substitute as in i), we obtain that

$$\frac{1}{2(p-1)}w^{2-p}\left(\Phi(u)-\Phi(w)\right)^2 < P(u,\Phi(w)) < \frac{p-1}{2}w^{p-2}\left(u-w\right)^2 \tag{31}$$

holds if $p > 2$ and $0 < u < w$,

$$\frac{p-1}{2}u^{p-2}\left(u-w\right)^2 < P(u,\Phi(w)) < \frac{1}{2(p-1)}u^{2-p}\left(\Phi(u)-\Phi(w)\right)^2 \tag{32}$$

holds if $1 < p < 2$ and $u > w > 0$.

If $1 < p < 2$ and $0 < u < w$ (resp. $p > 2$ and $u > w > 0$) the reverse inequalities in (31) (resp. (32)) are valid.

If we select the right hand inequality in (32) and the left hand reverse inequality in (32), we have that

$$P(u,\Phi(w)) < \frac{1}{2(p-1)}u^{2-p}\left(\Phi(u)-\Phi(w)\right)^2 \tag{33}$$

holds if $1 < p < 2$ and the reverse inequality in (33) holds if $p > 2$, where $u > w > 0$.

Now, we define $P(u,v) = \frac{|u|^p}{p} + \frac{|v|^q}{q} - uv$, $\Phi(t) = |t|^{p-2}t$ for all $u,v \in \mathbb{R}$. Then $\Phi(-t) = -\Phi(t)$, $P(u,v) = P(-u,-v)$ in the case $uv > 0$. Using (33) we can obtain that

$$P(u,\Phi(w)) < \frac{1}{2(p-1)}|u|^{2-p}\left(\Phi(u)-\Phi(w)\right)^2 \tag{34}$$

holds if $1 < p < 2$ and the reverse inequality holds if $p > 2$, where u,w such that $|u| > |w|$, $uw > 0$. Then (34) and its reverse are just [1, Inequality (2.12)].

iii) We obtained new bounds for $P(u,\Phi(w))$, which are not given in [1, Inequality (2.11)] and [1, Inequality (2.12)].

Let $1 < p < 2$, $uw > 0$ and t_0 be the unique solution of the equation $\sqrt{p-1}(\sqrt[p]{t}-1) = (\sqrt[p-1]{t^p}-1)$ on $(0,1)$. Using Lemma 2.7 we obtain

$$P(u,\Phi(w)) \geqslant \begin{cases} \frac{1}{2}|u|^{2-p}\left(\Phi(u)-\Phi(w)\right)^2, & \text{if } 0 < \frac{|w|}{|u|} \leqslant \sqrt[p]{t_0}, \\ \frac{p-1}{2}|u|^{p-2}\left(u-w\right)^2, & \text{if } \sqrt[p]{t_0} \leqslant \frac{|w|}{|u|} < 1. \end{cases} \tag{35}$$

If $0 < \frac{|u|}{|w|} \leqslant \sqrt[p]{t_0}$, then the reverse inequality in the first inequality in (35) is valid. If $\sqrt[p]{t_0} \leqslant \frac{|u|}{|w|} < 1$ then the reverse inequality in the second inequality in (35) is valid.

Similarly, let $p > 2$. Then

$$P(u,\Phi(w)) \geqslant \begin{cases} \dfrac{p-1}{2}|w|^{2-p}(u-w)^2, & \text{if } 0 < \dfrac{|u|}{|w|} \leqslant \sqrt[p]{t_0}, \\ \dfrac{1}{2}w^{p-2}(\Phi(u)-\Phi(w))^2, & \text{if } \sqrt[p]{t_0} \leqslant \dfrac{|u|}{|w|} < 1. \end{cases} \tag{36}$$

If $0 < \frac{|w|}{|u|} \leqslant \sqrt[p]{t_0}$, then the reverse inequality in the first inequality in (36) is valid. If $\sqrt[p]{t_0} \leqslant \frac{|w|}{|u|} < 1$ then the reverse inequality in the second inequality in (36) is valid.

REMARK 2.9. We can compare our results with the one given in [9].

i) We remark that our results from Lemma 2.1 are better than the results from [9, Theorem 2.3] for some α, x and y, but not all.

ii) We can not compare the results from Lemma 2.6 and [9, Theorem 2.3], since their hypotheses are not the same.

3. Inequalities involving the arithmetic and geometric operator means

In this section we give several inequalities involving the arithmetic and geometric means of two positive invertible operators using results obtained in Section 2. In the next result we observe the generalized geometric mean defined for every positive invertible operators A and B as follows:

$$A \natural_\alpha B := A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}, \qquad \alpha \in \mathbb{R}.$$

For $\alpha \in (-\infty,0)\cup(1,\infty)$, it cannot be Kubo-Ando mean and is often denoted by $\natural_\alpha$.

In the next proofs we shall use the known property of monotonity for operator functions: If X is bounded selfadjoint operator on a Hilbert space H with a spectra $\mathsf{Sp}(X)$, then

$$f(t) \geqslant g(t) \quad (t \in \mathsf{Sp}(X)) \qquad \Longrightarrow \qquad f(X) \geqslant g(X). \tag{$\clubsuit$}$$

Using Lemma 2.1 we obtain the following theorem.

THEOREM 3.1. *Let A and B be positive invertible operators on a Hilbert space H. If $\alpha \in (0,\frac{1}{2}]$, then*

$$\frac{1}{2}(A \natural_{2\alpha} B + A) \leqslant A \nabla_\alpha B \leqslant \frac{1}{2}(A \natural_{2\alpha-1} B + B). \tag{37}$$

If $\alpha \in [\frac{1}{2},1)$, then the reverse inequalities in (37) are valid.

For $\alpha = \frac{1}{2}$ we have the identity in (37).

Proof. We only prove the case $\alpha \in (0, \frac{1}{2}]$. Putting $y = 1$ in Lemma 2.1 for this α, we have the following inequality:

$$\frac{1}{2}\left(x^{\alpha}-1\right)^{2} \leqslant \alpha x+\beta-x^{\alpha} \leqslant \frac{1}{2} x^{\alpha-\beta}\left(x^{\beta}-1\right)^{2},$$

i.e.

$$\frac{1}{2}\left(x^{2\alpha}+1\right) \leqslant \alpha x+(1-\alpha) \leqslant \frac{1}{2}\left(x^{2\alpha-1}+x\right).$$

Using (♣) we can replace x by $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in the above inequalities. Then we have

$$\begin{aligned}\frac{1}{2}\left((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{2\alpha}+1_H\right) &\leqslant \alpha A^{-\frac{1}{2}}BA^{-\frac{1}{2}}+(1-\alpha)1_H\\ &\leqslant \frac{1}{2}\left((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{2\alpha-1}+A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right).\end{aligned}$$

It follows

$$\frac{1}{2}\left(A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{2\alpha}A^{\frac{1}{2}}+A\right) \leqslant \alpha B+(1-\alpha)A \leqslant \frac{1}{2}\left(A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{2\alpha-1}A^{\frac{1}{2}}+B\right).$$

From here the desired inequalities follow (37). □

Using Lemma 2.3 we obtain better results than the one from Theorem 3.1 given some conditions.

THEOREM 3.2. *Let A and B be positive invertible operators on a Hilbert space H. Let t_0 (resp. t_1) be the unique solution of the equation $\alpha t+1-\alpha = t^{\alpha}+\frac{1}{2}(t^{1-\alpha}-1)^2$ (resp. $(1-\alpha)t+\alpha = t^{1-\alpha}+\frac{1}{2}(t^{\alpha}-1)^2$) on $(0,1)$ for $\alpha \in (0,1)$, $\alpha \neq \frac{1}{2}$.*

Let $\alpha \in (0, \frac{1}{2})$. If $t_0 A \leqslant B < A$, then

$$\begin{aligned}\frac{1}{2}(A \sharp_{2\alpha} B-2A \sharp_{\alpha} B+A) &\leqslant A \nabla_{\alpha} B-A \sharp_{\alpha} B\\ &\leqslant \frac{1}{2}(A \sharp_{2-2\alpha} B-2A \sharp_{1-\alpha} B+A);\end{aligned} \tag{38}$$

if $A < B \leqslant \frac{1}{t_1} A$, then

$$\begin{aligned}\frac{1}{2}(A \sharp_{1-2\alpha} B-2A \sharp_{1-\alpha} B+B) &\leqslant A \nabla_{\alpha} B-A \sharp_{\alpha} B\\ &\leqslant \frac{1}{2}(A \sharp_{2\alpha-1} B-2A \sharp_{\alpha} B+B).\end{aligned} \tag{39}$$

In the case $\alpha \in (\frac{1}{2}, 1)$ the reverse inequalities in (38) and (39) are valid given same conditions.

For $\alpha = \frac{1}{2}$ we have identities in (38) and (39).

Proof. We only prove the case $\alpha \in (0, \frac{1}{2})$.

(a) Putting $y = 1$ in Lemma 2.3 for $\alpha \in (0, \frac{1}{2})$, we have:

$$\frac{1}{2}\left(x^{2\alpha}-2x^{\alpha}+1\right) \leqslant \alpha x+(1-\alpha)-x^{\alpha} \leqslant \frac{1}{2}\left(x^{2-2\alpha}-2x^{1-\alpha}+1\right), \quad \text{if } x \in [t_0,1),$$

$$\frac{1}{2}\left(x-2x^{1-\alpha}+x^{1-2\alpha}\right) \leqslant \alpha x+(1-\alpha)-x^{\alpha} \leqslant \frac{1}{2}\left(x-2x^{\alpha}+x^{2\alpha-1}\right), \text{ if } x \in (1,1/t_1].$$

Using (♣) we can replace x by $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. We obtain the following: if $\mathsf{Sp}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \subseteq [t_0, 1]$, then

$$\begin{aligned}\frac{1}{2}\left((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{2\alpha} - 2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha} + 1_H\right) \\ \leqslant \alpha A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + (1-\alpha)1_H - (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha} \\ \leqslant \frac{1}{2}\left((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{2-2\alpha} - 2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-\alpha} + 1_H\right);\end{aligned}$$

but if $\mathsf{Sp}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \subseteq (1, 1/t_1]$, then

$$\begin{aligned}\frac{1}{2}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - 2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-\alpha} + (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-2\alpha}\right) \\ \leqslant \alpha A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + (1-\alpha)1_H - (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha} \\ \leqslant \frac{1}{2}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - 2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha} + (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{2\alpha-1}\right).\end{aligned}$$

Consequently, if $\mathsf{Sp}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \subseteq [t_0, 1]$, then

$$\begin{aligned}\frac{1}{2}\left(A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{2\alpha}A^{\frac{1}{2}} - 2A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}} + A\right) \\ \leqslant \alpha B + (1-\alpha)A - A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}} \\ \leqslant \frac{1}{2}\left(A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{2-2\alpha}A^{\frac{1}{2}} - 2A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-\alpha}A^{\frac{1}{2}} + A\right);\end{aligned} \tag{40}$$

but if $\mathsf{Sp}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \subseteq [1, 1/t_1]$, then

$$\begin{aligned}\frac{1}{2}\left(B - 2A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-\alpha}A^{\frac{1}{2}} + A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-2\alpha}A^{\frac{1}{2}}\right) \\ \leqslant \alpha B + (1-\alpha)A - A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}} \\ \leqslant \frac{1}{2}\left(B - 2A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}} + A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{2\alpha-1}A^{\frac{1}{2}}\right).\end{aligned} \tag{41}$$

(b) Now, if $t_0\ A \leqslant B < A$, then $t_0\ 1_H \leqslant A^{-\frac{1}{2}}BA^{-\frac{1}{2}} < 1_H$. It follows, $\mathsf{Sp}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \subseteq [t_0, 1]$. Then (40) holds, i.e. if $t_0\, A \leqslant B < A$, then

$$\frac{1}{2}(A \sharp_{2\alpha} B - 2A \sharp_{\alpha} B + A) \leqslant A \nabla_{\alpha} B - A \sharp_{\alpha} B \leqslant \frac{1}{2}(A \sharp_{2-2\alpha} B - 2A \sharp_{1-\alpha} B + A).$$

This is the desired inequality (38).

Similarly, if $A < B \leqslant \frac{1}{t_1} A$, then $\mathsf{Sp}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \subseteq [1, \frac{1}{t_1}]$. Then (41) holds and we have the desired inequality (39). □

REMARK 3.3. We can obtain similar results as in Theorem 3.2 if we observe inequalities as in Remark 2.4.

Using Lemma 2.5 we have the following theorem.

THEOREM 3.4. *Let A and B be positive invertible operators on a Hilbert space H. Suppose that either of the following conditions holds*

(I) $B < A$ *and* $\alpha \in (0, \frac{1}{2}]$, (II) $A < B$ *and* $\alpha \in [\frac{1}{2}, 1)$,

(III) $B < A$ *and* $\alpha \in [\frac{1}{2}, 1)$, (IV) $A < B$ *and* $\alpha \in (0, \frac{1}{2}]$.

Then

$$\begin{aligned} \frac{1-\alpha}{2\alpha}(A \sharp_{1-2\alpha} B - 2A \sharp_{1-\alpha} B + B) &\leqslant A \nabla_\alpha B - A \sharp_\alpha B \\ &\leqslant \frac{1-\alpha}{2\alpha}(A \sharp_{2\alpha} B - 2A \sharp_\alpha B + A), \end{aligned} \tag{42}$$

$$\begin{aligned} \frac{\alpha}{2(1-\alpha)}(A \sharp_{2-2\alpha} B - 2A \sharp_{1-\alpha} B + A) &\leqslant A \nabla_\alpha B - A \sharp_\alpha B \\ &\leqslant \frac{\alpha}{2(1-\alpha)}(A \sharp_{2\alpha-1} B - 2A \sharp_\alpha B + B) \end{aligned} \tag{43}$$

hold in the cases (I) *and* (II) *and the reverse inequalities in* (42) *and* (43) *hold in the cases* (III) *and* (IV).

For $\alpha = \frac{1}{2}$ *we have identities in* (42) *and* (43).

Proof. The proof is quite similar to the proof of Theorem 3.2. We give the quick proof.

Putting $y = 1$ in Lemma 2.5, we obtain that

$$\frac{1}{2}\frac{\beta}{\alpha}\left(x^{1-2\alpha} - 2x^{1-\alpha} + x\right) \leqslant \alpha x + (1-\alpha) - x^\alpha \leqslant \frac{1}{2}\frac{\beta}{\alpha}\left(x^{2\alpha} - 2x^\alpha + 1\right),$$
$$\frac{1}{2}\frac{\alpha}{\beta}\left(x^{2-2\alpha} - 2x^{1-\alpha} + 1\right) \leqslant \alpha x + (1-\alpha) - x^\alpha \leqslant \frac{1}{2}\frac{\alpha}{\beta}\left(x^{2\alpha-1} - 2x^\alpha + x\right)$$

hold in the cases (i) $x \leqslant 1$, $\alpha \in (0, \frac{1}{2}]$ and (ii) $1 \leqslant x$, $\alpha \in [\frac{1}{2}, 1)$ and that their reverse inequalities hold in the cases (iii) $x \leqslant 1$, $\alpha \in [\frac{1}{2}, 1)$ and (iv) $1 \leqslant x$, $\alpha \in (0, \frac{1}{2}]$. It follows:

$$\begin{aligned} \frac{1}{2}\frac{\beta}{\alpha}(A \sharp_{1-2\alpha} B - 2A \sharp_{1-\alpha} B + B) &\leqslant A \nabla_\alpha B - A \sharp_\alpha B \\ &\leqslant \frac{1}{2}\frac{\beta}{\alpha}(A \sharp_{2\alpha} B - 2A \sharp_\alpha B + A), \\ \frac{1}{2}\frac{\alpha}{\beta}(A \sharp_{2-2\alpha} B - 2A \sharp_{1-\alpha} B + A) &\leqslant A \nabla_\alpha B - A \sharp_\alpha B \\ &\leqslant \frac{1}{2}\frac{\alpha}{\beta}(A \sharp_{2\alpha-1} B - 2A \sharp_\alpha B + B) \end{aligned}$$

hold in the cases (I_1) $\mathrm{Sp}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \subset (0, 1]$, $\alpha \in (0, \frac{1}{2}]$ and (II_1) $\mathrm{Sp}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \subset [1, \infty)$, $\alpha \in [\frac{1}{2}, 1)$ and that their reverse inequalities hold in the cases (III_1) $\mathrm{Sp}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \subset (0, 1]$, $\alpha \in [\frac{1}{2}, 1)$ and (IV_1) $\mathrm{Sp}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \subset [1, \infty)$, $\alpha \in (0, \frac{1}{2}]$.

Now, if $B < A$, then $A^{-\frac{1}{2}}BA^{-\frac{1}{2}} < 1_H$ and it follows $\mathrm{Sp}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \subset (0, 1]$. But, if $A < B$, then $\mathrm{Sp}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \subset [1, \infty)$. Then (I) implies ($\mathrm{I}_1$), (II) implies

(II_1), (III) implies (III_1) and (IV) implies (IV_1). It follows the desired inequalities (42) and (43) in the cases (I) and (II) and their reverse inequalities in the cases (III) and (IV). □

Finally, using Lemma 2.7 we obtain the following theorem where we compare bounds given in Theorem 3.1 and Theorem 3.4.

THEOREM 3.5. *Let A and B be positive invertible operators on a Hilbert space H. Let t_0 be the unique solution of the equation $\sqrt{1-\alpha}(t^{\alpha}-1)=\sqrt{\alpha}(t^{1-\alpha}-1)$ on $(0,1)$ for $\alpha\in(0,1)$, $\alpha\neq\frac{1}{2}$.*

Let $\alpha\in(0,\frac{1}{2})$ hold. Then

$$A\nabla_{\alpha}B-A\sharp_{\alpha}B \leqslant \frac{1-\alpha}{2\alpha}(A\sharp_{2\alpha}B-2A\sharp_{\alpha}B+A), \quad \text{if} \quad B<A, \tag{44}$$

$$A\nabla_{\alpha}B-A\sharp_{\alpha}B \geqslant \frac{1}{2}(A\sharp_{2\alpha}B-2A\sharp_{\alpha}B+A), \quad \text{if} \quad B\leqslant t_0A, \tag{45}$$

$$A\nabla_{\alpha}B-A\sharp_{\alpha}B \geqslant \frac{\alpha}{2(1-\alpha)}(A\sharp_{2-2\alpha}B-2A\sharp_{1-\alpha}B+A), \quad \text{if} \quad t_0A\leqslant B<A. \tag{46}$$

If $A<B$ the reverse inequality in (44) is valid. If $A\leqslant t_0B$ the reverse inequality in (45) is valid, but if $t_0B\leqslant A<B$ the reverse inequality in (46) is valid.

Let $\alpha\in(\frac{1}{2},1)$ hold. Then

$$A\nabla_{\alpha}B-A\sharp_{\alpha}B \leqslant \frac{\alpha}{2(1-\alpha)}(A\sharp_{2\alpha-1}B-2A\sharp_{\alpha}B+B), \quad \text{if} \quad A<B, \tag{47}$$

$$A\nabla_{\alpha}B-A\sharp_{\alpha}B \geqslant \frac{1}{2}(A\sharp_{2\alpha-1}B-2A\sharp_{\alpha}B+B), \quad \text{if} \quad A\leqslant t_0B, \tag{48}$$

$$A\nabla_{\alpha}B-A\sharp_{\alpha}B \geqslant \frac{1-\alpha}{2\alpha}(A\sharp_{1-2\alpha}B-2A\sharp_{1-\alpha}B+B), \quad \text{if} \quad t_0B\leqslant A<B. \tag{49}$$

If $B<A$ the reverse inequality in (47) is valid. If $B\leqslant t_0A$ the reverse inequality in (48) is valid, but if $t_0A\leqslant B<A$ the reverse inequality in (49) is valid.

For $\alpha=\frac{1}{2}$ we have the identity in all the above inequalities.

Proof. Putting $y=1$ in Lemma 2.7, we obtain the following:

Let $\alpha\in(0,\frac{1}{2})$ hold. Then

$$\alpha x+\beta-x^{\alpha} \leqslant \frac{1}{2}\frac{\beta}{\alpha}\left(x^{2\alpha}-2x^{\alpha}+1\right), \quad \text{if} \quad x<1, \tag{50}$$

$$\alpha x+\beta-x^{\alpha} \geqslant \frac{1}{2}\left(x^{2\alpha}-2x^{\alpha}+1\right), \quad \text{if} \quad x\leqslant t_0, \tag{51}$$

$$\alpha x+\beta-x^{\alpha} \geqslant \frac{1}{2}\frac{\alpha}{\beta}\left(x^{2-2\alpha}-2x^{1-\alpha}+1\right), \quad \text{if} \quad t_0\leqslant x<1. \tag{52}$$

If $1<x$, then the reverse inequality in (50) is valid. If $x\geqslant\frac{1}{t_0}$, then the reverse inequality in (51) is valid, but if $1<x\leqslant\frac{1}{t_0}$, then the reverse inequality in (52) is valid.

Let $\alpha \in (\frac{1}{2}, 1)$ hold. Then

$$\alpha x + \beta - x^{\alpha} \leqslant \frac{1}{2}\frac{\alpha}{\beta}\left(x^{2\alpha-1} - 2x^{\alpha} + x\right), \qquad \text{if} \quad 1 < x, \tag{53}$$

$$\alpha x + \beta - x^{\alpha} \geqslant \frac{1}{2}\left(x^{2\alpha-1} - 2x^{\alpha} + x\right), \qquad \text{if} \quad x \geqslant \frac{1}{t_0}, \tag{54}$$

$$\alpha x + \beta - x^{\alpha} \geqslant \frac{1}{2}\frac{\beta}{\alpha}\left(x^{1-2\alpha} - 2x^{1-\alpha} + x\right), \qquad \text{if} \quad 1 < x \leqslant \frac{1}{t_0}. \tag{55}$$

If $x < 1$, then the reverse inequality in (53) is valid. If $x \leqslant t_0$, then the reverse inequality in (54) is valid, but if $t_0 \leqslant x < 1$, then the reverse inequality in (55) is valid.

The remainder of the proof is the same as the proof of Theorem 3.2. We omit the details. □

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