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SOME NEW INEQUALITIES FOR MEANS IN TWO VARIABLES

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Abstract. In this paper, some new bounds for $L^p(x,y)$ and $I^p(x,y)$ in terms of $A^p(x,y)$ and $G^p(x,y)$ are established.

1. Introduction

Assume x and y to be two different positive numbers, let $A(x,y) = \frac{x+y}{2}$, $G(x,y) = \sqrt{xy}$, $L(x,y) = \frac{x-y}{\log x - \log y}$, and $I(x,y) = \frac{1}{e}(\frac{x^x}{y^y})^{\frac{1}{x-y}}$ be arithmetic, geometric, logarithmic, and identric mean, respectively. Then it is well-known that (see [1-4])

$$G(x,y) < L(x,y) < I(x,y) < A(x,y). \tag{1}$$

Carlson [5] gives bounds for $L(x,y)$ in terms of $G(x,y)$ and $A(x,y)$ as follows

$$L(x,y) < \frac{1}{3}A(x,y) + \frac{2}{3}G(x,y). \tag{2}$$

The further results are obtained by [6] and [7] (or see [8]) as follows

$$\alpha_1 A(x,y) + (1-\alpha_1)G(x,y) < L(x,y) < \beta_1 A(x,y) + (1-\beta_1)G(x,y) \tag{3}$$

holds if and only if $\alpha_1 \leqslant 0$ and $\beta_1 \geqslant 1/3$.

On the other hand, Sandor [9] proves that

$$\frac{2}{3}A(x,y) + \frac{1}{3}G(x,y) < I(x,y). \tag{4}$$

Alzer and Qiu [6] obtains the following interesting result:

$$\lambda_1 A(x,y) + (1-\lambda_1)G(x,y) < I(x,y) < \mu_1 A(x,y) + (1-\mu_1)G(x,y) \tag{5}$$

holds if and only if $\lambda_1 \leqslant 2/3$ and $\mu_1 \geqslant 2/e$.

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Another counterpart of (4) has been obtained by Sandor and Trif [10]. In [10], they prove that

$$I^2(x,y) < \frac{2}{3}A^2(x,y) + \frac{1}{3}G^2(x,y). \tag{6}$$

Recently, Trif [11] obtains

$$\lambda_p A^p(x,y) + (1-\lambda_p)G^p(x,y) < I^p(x,y) < \mu_p A^p(x,y) + (1-\mu_p)G^p(x,y) \tag{7}$$

holds with $p \geqslant 2$ if and only if $\lambda_p \leqslant (\frac{2}{e})^p$ and $\mu_p \geqslant \frac{2}{3}$.

In this paper, we give the following generalizations of (3) and (5), and in the process a simpler of (7).

THEOREM 1. *Let* $p \geqslant 1$*, then*

$$\alpha_p A^p(x,y) + (1-\alpha_p)G^p(x,y) < L^p(x,y) < \beta_p A^p(x,y) + (1-\beta_p)G^p(x,y) \tag{8}$$

holds if and only if $\alpha_p \leqslant 0$ *and* $\beta_p \geqslant 1/3$.

THEOREM 2. *Let* $p > 0$*, then:*

(1) *if* $0 < p \leqslant 6/5$*, the double inequality*

$$\lambda_p A^p(x,y) + (1-\lambda_p)G^p(x,y) < I^p(x,y) < \mu_p A^p(x,y) + (1-\mu_p)G^p(x,y) \tag{9}$$

holds if and only if $\lambda_p \leqslant \frac{2}{3}$ *and* $\mu_p \geqslant (\frac{2}{e})^p$*;*

(2) *if* $p \geqslant 2$*, the double inequality*

$$\lambda_p A^p(x,y) + (1-\lambda_p)G^p(x,y) < I^p(x,y) < \mu_p A^p(x,y) + (1-\mu_p)G^p(x,y) \tag{10}$$

holds if and only if $\lambda_p \leqslant (\frac{2}{e})^p$ *and* $\mu_p \geqslant \frac{2}{3}$.

2. A Lemma

LEMMA 1. ([12-13]) *Let* $f, g : [a,b] \to \mathbb{R}$ *be two continuous functions which are differentiable on* (a,b)*. Further, let* $g' \neq 0$ *on* (a,b)*. If* f'/g' *is increasing (or decreasing) on* (a,b)*, then the functions*

$$\frac{f(x) - f(b)}{g(x) - g(b)}$$

and

$$\frac{f(x) - f(a)}{g(x) - g(a)}$$

are also increasing (or decreasing) on (a,b).

3. A Short Proof of Theorem 1

Theorem 1 is equivalent to the following inequalities

$$0<\frac{L^p(x,y)-G^p(x,y)}{A^p(x,y)-G^p(x,y)}<\frac{1}{3}. \tag{11}$$

Without loss of generality, we set $0<x<y$. Let $u=\sqrt{y/x}$, then $u>1$ and

$$\frac{L^p(x,y)-G^p(x,y)}{A^p(x,y)-G^p(x,y)}=\frac{\left(\frac{u^2-1}{2\log u}\right)^p-u^p}{\left(\frac{u^2+1}{2}\right)^p-u^p}.$$

Let $\log u=t$ or $u=e^t$, then $t>0$ and

$$\frac{L^p(x,y)-G^p(x,y)}{A^p(x,y)-G^p(x,y)}=\frac{\left(\frac{\sinh t}{t}\right)^p-1}{(\cosh t)^p-1}\equiv H(t).$$

Using power series expansions we have that $\lim\limits_{t\to 0^+}H(t)=\frac{1}{3}$. Rewriting $H(t)$ as $\frac{\left(\frac{\tanh t}{t}\right)^p-\left(\frac{1}{\cosh t}\right)^p}{1-\left(\frac{1}{\cosh t}\right)^p}$ we see that $\lim\limits_{t\to+\infty}H(t)=0$. Hence Theorem 1 will follow if we prove that the function $H(t)$ is decreasing on $(0,+\infty)$.

Now, let $H(t)=\frac{f_1(t)-f_1(0^+)}{g_1(t)-g_1(0^+)}$, where $f_1(t)=\left(\frac{\sinh t}{t}\right)^p$, and $g_1(t)=(\cosh t)^p$. Then

$$\frac{f_1'(t)}{g_1'(t)}=\left(\frac{f_2(t)}{g_2(t)}\right)^{p-1}\frac{f_3(t)}{g_3(t)}, \tag{12}$$

where $f_2(t)=\sinh t$, $g_2(t)=t\cosh t$, $f_3(t)=t\cosh t-\sinh t$, and $g_3(t)=t^2\sinh t$.

We find that $\frac{f_2'(t)}{g_2'(t)}=\frac{1}{1+t\tanh t}$ is decreasing on $(0,+\infty)$, so $\frac{\sinh t}{t\cosh t}=\frac{f_2(t)-f_2(0)}{g_2(t)-g_2(0)}$ is decreasing on $(0,+\infty)$ by Lemma 1. At the same time, $\frac{f_3'(t)}{g_3'(t)}=\frac{1}{2+(t\cosh t)/\sinh t}$ is also decreasing on $(0,+\infty)$. So $\frac{f_3(t)}{g_3(t)}=\frac{f_3(t)-f_3(0)}{g_3(t)-g_3(0)}$ is decreasing on $(0,+\infty)$ by Lemma 1. Since for $p\geqslant 1$, we have that $\frac{f_1'(t)}{g_1'(t)}$ is decreasing on $(0,+\infty)$ from (12). This leads to that $H(t)=\frac{f_1(t)-f_1(0^+)}{g_1(t)-g_1(0^+)}$ is decreasing on $(0,+\infty)$ by Lemma 1.

4. A Concise Proof of Theorem 2

By the same way, we can compute and obtain

$$\frac{I^p(x,y)-G^p(x,y)}{A^p(x,y)-G^p(x,y)}=\frac{e^{(t\coth t-1)p}-1}{(\cosh t)^p-1}\equiv J(t),$$

where $t>0$.

Let $J(t)=\frac{q_1(t)-q_1(0^+)}{r_1(t)-r_1(0^+)}$, where $q_1(t)=e^{(t\coth t-1)p}$, and $r_1(t)=(\cosh t)^p$. Then

$$\frac{q_1'(t)}{r_1'(t)}=\frac{1}{2}e^{(t\coth t-1)p}\frac{\sinh 2t-2t}{(\cosh t)^{p-1}(\sinh t)^3}\equiv k(t). \tag{13}$$

we compute

$$\begin{aligned}
&\frac{k'(t)}{e^{(t\coth t-1)p}}\\
&=\frac{p(\sinh 2t-2t)^2+2(\cosh 2t-1)^2-2(p-1)(\sinh 2t-2t)(\sinh t)^2\tanh t-3\sinh 2t(\sinh 2t-2t)}{4(\cosh t)^{p-1}(\sinh t)^5}\\
&=\frac{p(\sinh 2t-2t)^2+2(\cosh 2t-1)^2-(p-1)(\sinh 2t-2t)\frac{(\cosh 2t-1)\sinh 2t}{\cosh 2t+1}-3\sinh 2t(\sinh 2t-2t)}{4(\cosh t)^{p-1}(\sinh t)^5}\\
&=\frac{u(t)}{4(\cosh t)^{p-1}(\sinh t)^5(\cosh 2t+1)},
\end{aligned}$$

where

$$\begin{aligned}
u(t)=&\,p(\sinh 2t-2t)^2(\cosh 2t+1)+2(\cosh 2t-1)^2(\cosh 2t+1)\\
&-2(p-1)(\sinh 2t-2t)(\cosh 2t-1)\sinh 2t-3\sinh 2t(\sinh 2t-2t)(\cosh 2t+1).
\end{aligned}$$

Let $w=2t$, then $w>0$ and

$$\begin{aligned}
v(w)=u(t)&=p(\sinh w-w)^2(\cosh w+1)+2(\cosh w-1)^2(\cosh w+1)\\
&\quad-2(p-1)(\sinh w-w)(\cosh w-1)\sinh w-3\sinh w(\sinh w-w)(\cosh w+1)\\
&=\frac{1}{2}(2-p)w\sinh 2w+pw^2\cosh w+(p-3)(\cosh 2w-1)+(4-3p)w\sinh w+pw^2\\
&=\frac{2-p}{2}w\sum_{k=0}^{\infty}\frac{(2w)^{2k+1}}{(2k+1)!}+pw^2\sum_{k=0}^{\infty}\frac{(w)^{2k}}{(2k)!}+(p-3)\left[\sum_{k=0}^{\infty}\frac{(2w)^{2k}}{(2k)!}-1\right]\\
&\quad+(4-3p)w\sum_{k=0}^{\infty}\frac{(w)^{2k+1}}{(2k+1)!}+pw^2\\
&=\sum_{k=3}^{\infty}a_{2k}w^{2k},
\end{aligned}$$

where

$$\begin{aligned}
a_{2k}&=\frac{((k-3)2^{2k}+8k)-((\frac{k}{2}-1)2^{2k}+8k-4k^2)p}{(2k)!}\\
&=\frac{(\frac{k}{2}-1)2^{2k}+8k-4k^2}{(2k)!}\left(\frac{(k-3)2^{2k}+8k}{(\frac{k}{2}-1)2^{2k}+8k-4k^2}-p\right).
\end{aligned}$$

Since that the function $I(x)\equiv((x-3)2^{2x}+8x)/((\frac{x}{2}-1)2^{2x}+8x-4x^2)$ is strictly increasing from $[3,+\infty)$ onto $[6/5,2)$, we obtain results in two cases:

(i) If $0 < p \leqslant 6/5$, we have $a_{2k} \geqslant 0$ (equality holds if and only if $k = 3$) and $v(w) \equiv u(t) > 0$. So $q_1'(t)/r_1'(t) = k(t)$ is increasing on $(0, +\infty)$. This leads to that $J(t)$ is increasing on $(0, +\infty)$ by Lemma 1. At the same time, following the same method as that we had used in determining the limits of function $H(t)$, we have $\lim_{t\to 0^+} J(t) = 2/3$ and $\lim_{t\to +\infty} J(t) = (2/e)^p$. So the proof of (1) in Theorem 2 is complete;

(ii) If $p \geqslant 2$, we obtain (2) in Theorem 2 by the same way.

REMARK 1. In Section 3, we obtain essentially that the inequality

$$\left(\frac{\sinh t}{t}\right)^p < (1-\beta_p) + \beta_p(\cosh t)^p \tag{14}$$

holds for $p \geqslant 1$ and $t > 0$ if and only if $\beta_p \geqslant 1/3$.

In particular if $p = 1$ we get from (14) that

$$\frac{\sinh t}{t} < (1-\beta_1) + \beta_1 \cosh t \tag{15}$$

holds for $t > 0$ if and only if $\beta_1 \geqslant 1/3$.

Taking $\beta_1 = 1$ in (15) leads to the well-known inequality (see [14], [15, p.9])

$$\frac{\sinh t}{t} < \cosh t, \quad t > 0. \tag{16}$$

REMARK 2. In Section 4, we obtain two results as follows:

(a) If $0 < p \leqslant 6/5$, the double inequality

$$(1-\lambda_p) + \lambda_p(\cosh t)^p < e^{(t\coth t - 1)p} < (1-\mu_p) + \mu_p(\cosh t)^p \tag{17}$$

holds for $t > 0$ if and only if $\lambda_p \leqslant \frac{2}{3}$ and $\mu_p \geqslant (\frac{2}{e})^p$.

(b) If $p \geqslant 2$, the double inequality

$$(1-\lambda_p) + \lambda_p(\cosh t)^p < e^{(t\coth t - 1)p} < (1-\mu_p) + \mu_p(\cosh t)^p \tag{18}$$

holds for $t > 0$ if and only if $\lambda_p \leqslant (\frac{2}{e})^p$ and $\mu_p \geqslant \frac{2}{3}$.

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