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REFINEMENTS, EXTENSIONS AND GENERALIZATIONS OF THE SECOND KERSHAW'S DOUBLE INEQUALITY

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Abstract. In the paper, the second Kershaw's double inequality concerning the ratio of two gamma functions is refined, extended and generalized elegantly.

1. Introduction

It is well known that the classical Euler's gamma function $\Gamma(x)$ is defined for $x > 0$ as

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, \mathrm{d}t. \tag{1}$$

The logarithmic derivative of $\Gamma(x)$, denoted by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \tag{2}$$

is called the psi or digamma function, and $\psi^{(i)}(x)$ for $i \in \mathbb{N}$ are known as the polygamma or multigamma functions. These functions play central roles in the theory of special functions and have lots of extensive applications in many branches, for example, statistics, physics, engineering, and other mathematical sciences.

The generalized logarithmic mean $L_p(a,b)$ of order $p \in \mathbb{R}$ for positive numbers a and b with $a \neq b$ is defined in [4, p. 385] by

$$L_p(a,b) = \begin{cases} \left[\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & p \neq -1, 0; \\ \dfrac{b-a}{\ln b - \ln a}, & p = -1; \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & p = 0. \end{cases} \tag{3}$$

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It is well known that

$$L_{-2}(a,b)=\sqrt{ab}=G(a,b), \quad L_{-1}(a,b)=L(a,b), \tag{4}$$

$$L_0(a,b)=I(a,b) \quad \text{and} \quad L_1(a,b)=\frac{a+b}{2}=A(a,b) \tag{5}$$

are called respectively the geometric mean, the logarithmic mean, the identric or exponential mean and the arithmetic mean. It is also known [4, pp. 386–387, Theorem 3] that the generalized logarithmic mean $L_p(a,b)$ of order p is increasing in p for $a\ne b$. Therefore, inequalities

$$G(a,b)<L(a,b)<I(a,b)<A(a,b) \tag{6}$$

are valid for $a>0$ and $b>0$ with $a\ne b$. See also [26, 27].

In [13], the following two double inequalities were established for $0<s<1$ and $x\geqslant 1$:

$$\left(x+\frac{s}{2}\right)^{1-s}<\frac{\Gamma(x+1)}{\Gamma(x+s)}<\left(x-\frac12+\sqrt{s+\frac14}\right)^{1-s}, \tag{7}$$

$$\exp\left[(1-s)\psi\left(x+\sqrt{s}\right)\right]<\frac{\Gamma(x+1)}{\Gamma(x+s)}<\exp\left[(1-s)\psi\left(x+\frac{s+1}{2}\right)\right]. \tag{8}$$

They are called the first and the second Kershaw's double inequality respectively. There have been a lot of literature, such as [5, 7, 9, 10, 11, 12, 14, 19, 20, 21, 22, 23, 30, 32, 34, 35, 36, 37, 39, 40, 50] and the references therein, about these two double inequalities and their history, background, refinements, extensions, generalizations and applications.

In [2, Theorem 2.7], the double inequality (8) was generalized to

$$-\left|\psi^{(n+1)}(L_{-(n+2)}(x,y))\right|<\frac{\left|\psi^{(n)}(x)\right|-\left|\psi^{(n)}(y)\right|}{x-y}<-\left|\psi^{(n+1)}(A(x,y))\right|, \tag{9}$$

where x and y are positive numbers, n is a positive integer.

In [23], the following generalization, extension and refinement of the second Kershaw's double inequality (8) were obtained: For $s,t\in\mathbb{R}$ with $s\ne t$, the function

$$\left[\frac{\Gamma(x+s)}{\Gamma(x+t)}\right]^{1/(s-t)}\frac{1}{e^{\psi(L(s,t;x))}} \tag{10}$$

is decreasing in $x>-\min\{s,t\}$. In [23, 40], the function

$$\left[\frac{\Gamma(x+s)}{\Gamma(x+t)}\right]^{1/(t-s)}e^{\psi(A(s,t;x))} \tag{11}$$

is proved to be logarithmically completely monotonic in $x>-\min\{s,t\}$. Consequently, for $s,t\in\mathbb{R}$ and $x>-\min\{s,t\}$ with $s\ne t$,

$$e^{\psi(L(s,t;x))}<\left[\frac{\Gamma(x+s)}{\Gamma(x+t)}\right]^{1/(s-t)}<e^{\psi(A(s,t;x))}, \tag{12}$$

where

$$L(s,t;x) = L(x+s,x+t) \quad \text{and} \quad A(s,t;x) = A(x+s,x+t) \tag{13}$$

for $s,t \in \mathbb{R}$ and $x > -\min\{s,t\}$ with $s \neq t$.

In [41, 42], the right-hand side inequalities in (9) and (12) were refined as follows: For $s,t \in \mathbb{R}$ with $s \neq t$ and $x > -\min\{s,t\}$, inequalities

$$\left[\frac{\Gamma(x+s)}{\Gamma(x+t)}\right]^{1/(s-t)} \leqslant e^{\psi(I(s,t;x))} \tag{14}$$

and

$$\frac{(-1)^n\left[\psi^{(n-1)}(x+s) - \psi^{(n-1)}(x+t)\right]}{s-t} \leqslant (-1)^n\psi^{(n)}(I(s,t;x)), \tag{15}$$

hold, where

$$I(s,t;x) = I(x+s,x+t) \tag{16}$$

and $n \in \mathbb{N}$. These inequalities (14) and (15) refine, extend and generalize the right-hand side inequality in the second Kershaw's double inequality (8).

The aim of this paper is to generalize, extend and refine the right-hand side inequalities in (8), (9) and (12). Meanwhile, the left-hand side inequalities in (9) and (12) and inequalities (14) and (15) are recovered.

The main results of this paper are the following theorems.

THEOREM 1. *For real numbers $s > 0$ and $t > 0$ with $s \neq t$ and an integer $i \geqslant 0$, the inequality*

$$(-1)^i\psi^{(i)}(L_p(s,t)) \leqslant \frac{(-1)^i}{t-s}\int_s^t \psi^{(i)}(u)\,\mathrm{d}u \leqslant (-1)^i\psi^{(i)}(L_q(s,t)) \tag{17}$$

holds if $p \leqslant -i-1$ and $q \geqslant -i$.

THEOREM 2. *The inequality*

$$e^{\psi(L_p(s,t;x))} < \left[\frac{\Gamma(x+s)}{\Gamma(x+t)}\right]^{1/(s-t)} < e^{\psi(L_q(s,t;x))} \tag{18}$$

for $s,t \in \mathbb{R}$ with $s \neq t$ and $x > -\min\{s,t\}$ or, equivalently,

$$e^{\psi(L_p(a,b))} < \left[\frac{\Gamma(a)}{\Gamma(b)}\right]^{1/(a-b)} < e^{\psi(L_q(a,b))} \tag{19}$$

for $a > 0$ and $b > 0$, holds if $p \leqslant -1$ and $q \geqslant 0$.

THEOREM 3. *For $i \geqslant 0$ being an integer and $s,t \in \mathbb{R}$ with $s \neq t$ and $x > -\min\{s,t\}$, the function*

$$(-1)^i\left[\psi^{(i)}(L_p(s,t;x)) - \frac{1}{t-s}\int_s^t \psi^{(i)}(x+u)\,\mathrm{d}u\right] \tag{20}$$

is increasing in x if either $p \leqslant -(i+2)$ or $p = -(i+1)$ and decreasing in x if $p \geqslant 1$.

REMARK 1. It is conjectured that the function (20) is decreasing (even completely monotonic) in x if $p \geqslant -i$ and that its negative is decreasing (even completely monotonic) in x if $p \leqslant -(i+1)$.

2. Proofs of theorems

Proof of Theorem 1. It is apparent that the function $f(x) = (-1)^i \psi^{(i)}(x)$ for $i \geqslant 0$ is strictly increasing, the function $g(x) = x^p$ for $p \neq 0$ is monotonic in $(0,\infty)$, and the inverse function of g is $g^{-1}(x) = x^{1/p}$. Straightforward computation gives

$$g^{-1}\left(\frac{1}{t-s}\int_s^t g(u)\,\mathrm{d}u\right) = L_p(s,t), \tag{21}$$

$$h(x) \triangleq f \circ g^{-1}(x) = (-1)^i \psi^{(i)}\big(x^{1/p}\big) \tag{22}$$

and

$$\begin{aligned} h''(x) &= (-1)^i \frac{x^{1/p-2}}{p^2}\left[x^{1/p}\psi^{(i+2)}\big(x^{1/p}\big) - (p-1)\psi^{(i+1)}\big(x^{1/p}\big)\right] \\ &= (-1)^i \frac{x^{1/p-2}}{p^2}\left[u\psi^{(i+2)}(u) - (p-1)\psi^{(i+1)}(u)\right] \\ &= \frac{x^{1/p-2}}{p^2}\Big[(1-p)\big|\psi^{(i+1)}(u)\big| - u\big|\psi^{(i+2)}(u)\big|\Big], \end{aligned}$$

where $u = x^{1/p}$. When $p \geqslant 1$, we have $h''(x) \leqslant 0$. It was proved in [43] that the function

$$x\big|\psi^{(i+1)}(x)\big| - \alpha\big|\psi^{(i)}(x)\big|$$

is completely monotonic in $(0,\infty)$ if and only if $0 \leqslant \alpha \leqslant i \in \mathbb{N}$ and that the function

$$\alpha\big|\psi^{(i)}(x)\big| - x\big|\psi^{(i+1)}(x)\big|$$

is completely monotonic in $(0,\infty)$ if and only if $\alpha \geqslant i+1$. A function f is called completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^k f^{(k)}(x) \geqslant 0$$

for all $k \geqslant 0$ on I, see [3, 25, 51]. This means that if $1-p \leqslant i+1$ for $i \geqslant 0$ then $h''(x) \leqslant 0$ and that if $1-p \geqslant i+2$ for $i \geqslant 0$ then $h''(x) \geqslant 0$. In conclusion, for $i \geqslant 0$, if $p \geqslant -i$ then $h''(x) \leqslant 0$, if $p \leqslant -i-1$ then $h''(x) \geqslant 0$. It was obtained in [6] (see also [4, p. 274, Lemma 2]) that if g is strictly monotonic, f is strictly increasing and $f \circ g^{-1}$ is convex (or concave, respectively) on an interval I, then

$$g^{-1}\left(\frac{1}{t-s}\int_s^t g(u)\,\mathrm{d}u\right) \leqslant f^{-1}\left(\frac{1}{t-s}\int_s^t f(u)\,\mathrm{d}u\right) \tag{23}$$

holds (or reverses, respectively) for $s,t \in I$. Therefore, when $p \leqslant -i-1$ for $i \geqslant 0$, inequality

$$(-1)^i \psi^{(i)}(L_p(s,t)) \leqslant \frac{(-1)^i}{t-s}\int_s^t \psi^{(i)}(u)\,\mathrm{d}u \tag{24}$$

holds for positive numbers s and t; when $p \geqslant -i$ for $i \geqslant 0$, inequality (24) reverses. The proof of Theorem 1 is complete. □

Proof of Theorem 2. Taking logarithm on all sides of (18) yields

$$\psi(L_p(s,t;x)) < \frac{\ln\Gamma(x+s)-\ln\Gamma(x+t)}{s-t} = \frac{1}{s-t}\int_t^s \psi(x+u)\,\mathrm{d}u < \psi(L_q(s,t;x))$$

which is the same as inequality (17) for the case of $i=0$. The proof is complete. □

Proof of Theorem 3. Easy calculation gives

$$\frac{\partial L_p(s,t;x)}{\partial x} = \left[\frac{L_{p-1}(s,t;x)}{L_p(s,t;x)}\right]^{p-1}. \tag{25}$$

Since the generalized logarithmic mean $L_p(a,b)$ is strictly increasing in p, hence $\frac{\partial L_p(s,t;x)}{\partial x} \gtreqless 1$ if $p \lesseqgtr 1$. It is clear that the derivative of the function defined by (20) equals

$$\begin{aligned}
Q_{p,i,s,t}(x) &= (-1)^i\left[\psi^{(i+1)}(L_p(s,t;x))\frac{\partial L_p(s,t;x)}{\partial x} - \frac{1}{t-s}\int_s^t \psi^{(i+1)}(x+u)\,\mathrm{d}u\right]\\
&= \left|\psi^{(i+1)}(L_p(s,t;x))\right|\frac{\partial L_p(s,t;x)}{\partial x} - \frac{1}{t-s}\int_s^t \left|\psi^{(i+1)}(x+u)\right|\mathrm{d}u\\
&\gtreqless \left|\psi^{(i+1)}(L_p(s,t;x))\right| - \frac{1}{t-s}\int_s^t \left|\psi^{(i+1)}(x+u)\right|\mathrm{d}u
\end{aligned}$$

if $p \lesseqgtr 1$. Combining this with Theorem 1 yields that if $p \leqslant 1$ and $p \leqslant -i-2$ the derivative of (20) is non-negative and that if $p \geqslant 1$ and $p \geqslant -i-1$ the derivative of (20) is non-positive. Consequently, when $p \leqslant -i-2$ the function (20) is increasing, when $p \geqslant 1$ the function (20) is decreasing in $x > -\min\{s,t\}$.

In [1, p. 260, 6.4.10], the following formula is listed: For $z \neq 0,-1,-2,\dots$ and $n \in \mathbb{N}$,

$$\psi^{(n)}(z) = (-1)^{n+1}n!\sum_{k=0}^{\infty}\frac{1}{(z+k)^{n+1}}. \tag{26}$$

Further considering (25) gives

$$\begin{aligned}
\frac{Q_{p,i,s,t}(x)}{(i+1)!} &= \sum_{k=0}^{\infty}\left\{\left[\frac{L_{p-1}(s,t;x)}{L_p(s,t;x)}\right]^{p-1}\frac{1}{[L_p(s,t;x)+k]^{i+2}} - \frac{1}{t-s}\int_s^t \frac{1}{(x+u+k)^{i+2}}\,\mathrm{d}u\right\}\\
&= \sum_{k=0}^{\infty}\left\{\left[\frac{L_{p-1}(s,t;x)}{L_p(s,t;x)}\right]^{p-1}\frac{1}{[L_p(s,t;x)+k]^{i+2}} - \frac{1}{[L_{-(i+2)}(s,t;x+k)]^{i+2}}\right\}\\
&= \frac{1}{[L_p(s,t;x)+k]^{i+2}}\sum_{k=0}^{\infty}\left\{\left[\frac{L_{p-1}(s,t;x)}{L_p(s,t;x)}\right]^{p-1} - \left[\frac{L_p(s,t;x)+k}{L_{-(i+2)}(s,t;x+k)}\right]^{i+2}\right\}.
\end{aligned}$$

Inequality (25) implies that the function $L_p(s,t;x+k)-k$ is increasing in k for $p<1$. Thus, inequality

$$L_p(s,t;x) \leqslant L_p(s,t;x+k)-k < A(s,t;x) \tag{27}$$

holds for $p<1$ and $k \geqslant 0$. This means that

$$\frac{L_p(s,t;x)+k}{L_{-(i+2)}(s,t;x+k)} \leqslant \frac{L_p(s,t;x+k)}{L_{-(i+2)}(s,t;x+k)} \tag{28}$$

and

$$\begin{aligned}
\left[\frac{L_{p-1}(s,t;x)}{L_p(s,t;x)}\right]^{p-1} &- \left[\frac{L_p(s,t;x)+k}{L_{-(i+2)}(s,t;x+k)}\right]^{i+2} \\
&\geqslant \left[\frac{L_{p-1}(s,t;x)}{L_p(s,t;x)}\right]^{p-1} - \left[\frac{L_p(s,t;x+k)}{L_{-(i+2)}(s,t;x+k)}\right]^{i+2} \\
&= \left[\frac{L_{p-1}(s,t;x)}{L_p(s,t;x)}\right]^{p-1} - \left[\frac{L_{-(i+2)}(s,t;x+k)}{L_p(s,t;x+k)}\right]^{-(i+2)} \\
&\geqslant \left[\frac{L_{p-1}(s,t;x)}{L_p(s,t;x)}\right]^{p-1} - \left[\frac{L_{-(i+2)}(s,t;x+k)}{L_{-(i+1)}(s,t;x+k)}\right]^{-(i+2)} \\
&\geqslant \left[\frac{L_{p-1}(s,t;x)}{L_p(s,t;x)}\right]^{p-1} - \left[\frac{L_{-(i+2)}(s,t;x)}{L_{-(i+1)}(s,t;x)}\right]^{-(i+2)}
\end{aligned} \tag{29}$$

for $-i-2<p<1$, where the following fact is used in the final line above:

$$\begin{aligned}
\frac{\partial}{\partial x}\left[\frac{L_q(s,t;x)}{L_p(s,t;x)}\right] &= \frac{L_q(s,t;x)}{L_p(s,t;x)}\left[\frac{1}{E(p,p+1;x+s,x+t)}\right. \\
&\qquad \left. + \frac{1}{E(q,q+1;x+s,x+t)}\right] > 0,
\end{aligned} \tag{30}$$

where $E(p,q;a,b)$ is defined for $p,q \in \mathbb{R}$ and $a,b>0$ by

$$\begin{aligned}
E(p,q;a,b) &= \left[\frac{p}{q}\cdot\frac{b^q-a^q}{b^p-a^p}\right]^{1/(q-p)}, && pq(p-q)(a-b)\neq 0; \\
E(p,0;a,b) &= \left[\frac{1}{p}\cdot\frac{b^p-a^p}{\ln b-\ln a}\right]^{1/p}, && p(a-b)\neq 0; \\
E(p,p;a,b) &= \frac{1}{e^{1/p}}\left[\frac{a^{a^p}}{b^{b^p}}\right]^{1/(a^p-b^p)}, && p(a-b)\neq 0; \\
E(0,0;a,b) &= \sqrt{ab}, && a\neq b; \\
E(p,q;a,a) &= a, && a=b.
\end{aligned}$$

It is remarked that the monotonicity, Schur-convexity, logarithmic convexity, comparison, generalizations, applications and history of the extended mean values $E(p,q;a,b)$ have been investigated in many articles such as [8, 15, 16, 17, 18, 24, 26, 27, 28, 29, 31, 33, 38, 44, 45, 46, 47, 48, 49, 52, 53, 54] and the references listed in [4, pp. 393–399].

As a result, the function $Q_{-(i+1),i,s,t}(x)$ is non-negative, and then the function (20) for $p = -(i+1)$ is increasing in x. The proof of Theorem 3 is complete. □

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