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INTERPOLATION OF COMPACT OPERATORS IN SPACES OF MEASURABLE FUNCTIONS

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Abstract. It is shown that one-sided interpolation of compactness property is possible by any method on the class of *pseudo-lattices*, i.e., spaces of measurable functions, where the operators $P_D f = f\chi_D$ are uniformly bounded. Analogous results are shown for sequence spaces, ordered quasinormed Abelian groups and even extended to abstract Banach couples, satisfying some *weak approximation hypothesis*.

1. Introduction

Interpolation theory of linear operators is intended for obtaining properties of such operators on intermediate spaces of some scale, when these properties are known at the endpoint spaces. In fact, the only property of operators, which is really interpolated, is their boundedness with corresponding estimates of norms (cf. inequalities (3.1) and (3.2) below). The solution of any other problem consists thereafter in constructing some operators whose boundedness and estimates of norm imply the needed result. For example, this is the way for studying spectral properties of operators, convergence of approximations, Fourier transforms and coefficients, Sobolev embeddings and so on.

The problem of interpolation of compactness property stands in the same line. Already the first theorem of such a kind from [10] used approximation by special (averaging) finite rank operators, a priori bounded in all considered spaces; the same operators were used in many subsequent papers. A more general result was obtained in [14], where the existence of needed approximating operators (the approximation hypothesis) was only postulated, without specifying their nature. The modern investigations, as a rule, may be divided into two main parts: study of special interpolation methods on arbitrary spaces or construction of large classes of spaces where interpolation of the compact operators is possible by any method.

Papers of the first type consider the compact operators in real and complex interpolation, in some orbit methods and methods with certain maximal and minimal properties. It should be noticed that the problem of interpolation of compactness property by the real method may be now regarded as solved almost completely (see, e.g., [5], [7] and their references). Interpolation of compactness by the complex method is still far from

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complete solution; various partial results can be found in [9], [15] and many others. Other special methods are considered, e.g., in [4], [6], [8].

The classes of spaces suitable for interpolation of compactness by any method were constructed in papers of the second type by applying special approximation hypotheses. While the paper [14] considered as examples only general L_p spaces, the paper [12] contains results for arbitrary rearrangement-invariant spaces (in fact, this is the largest class of spaces with desired properties, presented in the literature till now). The same spaces appeared many years before in the papers [11] and [16] as interpolation ones in the couples (L_p, L_q). The main achievement of the last mentioned papers is obtaining necessary and sufficient conditions for interpolation of compactness in considered spaces, which may be used as necessary conditions for the general real method as well. Just this fact confirms completeness of the results from [5].

In the present paper we propose an essentially larger class of spaces of measurable functions, which contains arbitrary function and sequence lattices as a partial case. Moreover, for the case of ordered couple, we can include also quasinormed Abelian groups studied in [13]. The only required property is the boundedness of the multiplication operators $P_D f = f\chi_D$, where χ_D is a characteristic function of arbitrary measurable set D (in the sequence spaces we consider only $D = [n, n+1, \ldots)$ with arbitrary natural n). Such spaces will be called *pseudo-lattices*. The compactness of sets in these spaces can be split into two properties that can be verified separately: compactness in measure and absolute equicontinuity of norms. When interpolating, the first property is inherited from the largest spaces and the second property results from interpolation of the corresponding multiplication operators with estimates of their norms.

Throughout the paper we suppose the reader is familiar with the basics of interpolation theory of linear operators; for more details and definitions, we refer to the monograph [2]. For simplicity, we only give detailed proofs for Banach spaces of functions, indicating in the last section all needed modifications for passing to the spaces of sequences and Abelian groups. For the same reason, we consider only bounded sets, since an extension to the general case can be done by standard methods for any σ-finite measure. At last, we formulate some new approximation hypothesis, generalizing properties of pseudo-lattices to abstract normed spaces.

2. Compactness in pseudo-lattices of functions

We start with consideration of measurable finite a.e. functions $f : \Omega \to \mathbb{R}$, where Ω is an arbitrary given set with positive nonatomic measure μ such that $\mu\Omega < \infty$. Let X be a Banach space of such functions f, which is intermediate in the couple (L_0, L_∞), where L_0 is the space from [13] with the "norm" $\|f\|_{L_0} = \mu\{\operatorname{supp} f\}$. In particular, this means that the topology in X is stronger than that in the space $S = L_0 + L_\infty$. The space S is a quasinormed Abelian group with "1-norm"

$$\|f\|_S = \inf_{a>0}(a + \mu\{t : |f(t)| > a\});$$

the convergence in this "norm" is the usual convergence in measure.

DEFINITION 2.1. *A space X is said to be a pseudo-lattice if $\|f\chi_D\|_X \leqslant C_X\|f\|_X$ for any measurable $D \subset \Omega$ with a constant C_X independent of f and D.*

A pseudo-lattice is not necessarily a lattice in the standard sense. For instance, given a non-positive linear operator $T: L_p \to L_q$ with $p, q > 1$, we can define a space X as completion of L_p in the norm

$$\|f\|_X = \|f\|_{L_1} + \sup_D \|T(f\chi_D)\|_{L_q},$$

where the supremum is taken over all measurable subsets $D \subset \Omega$. In result the space X with this norm is a pseudo-lattice but not necessarily a lattice.

We can illustrate the last assertion by the following elementary example. Let $\Omega = (0,4)$, $v = \chi_{(0,2]} - \chi_{(2,4)}$ and let $Tf(t) = u(t)\int_0^4 v(s)f(s)\,ds$ for some function $u \in L_q$. Now we take $f = v$ and $g = \chi_{(0,1]} - \chi_{(1,2]} + \chi_{(2,3]} - \chi_{(3,4)}$, so that $|f(s)| = |g(s)|$ for all $s \in \Omega$. At the same time, it is easy to calculate that $\|f\|_X = 4 + 4\|u\|_{L_q}$, while $\|g\|_X = 4 + 2\|u\|_{L_q}$.

DEFINITION 2.2. *We say that a function $f \in X$ has absolutely continuous norm if, for any $\epsilon > 0$, there exists $\delta = \delta(f)$ such that $\|f\chi_D\|_X < \epsilon$ whenever $\mu D < \delta$. If all functions from X are of such a kind, we say that X itself "has absolutely continuous norms".*

This property is well known for separable Banach function lattices with Lebesgue measure. In the general case we can use the following statement.

PROPOSITION 2.3. *A pseudo-lattice X has absolutely continuous norms if and only if the space L_∞ is dense in X and, for any $\epsilon > 0$, there exists δ such that $\|\chi_D\|_X < \epsilon$ whenever $\mu D < \delta$ (that is, one only needs to check the absolute continuity of the norm for one function $f(t) \equiv 1$).*

Proof. In order to prove the necessity part we consider truncations of functions: $f_n(t) = \min\{|f(t)|, n\}\ \mathrm{sign}\, f(t)$, $n = 1, 2, \ldots$ Since all functions f are finite a.e., the measure of the set of t where $|f(t)| > n$ tends to 0 as $n \to \infty$. Due to absolutely continuous norm of any $f \in X$ this implies that $\|f - f_n\|_X \to 0$, i.e., any f is the limit of a sequence of bounded functions.

On the other hand, let the density of L_∞ in X be given, i.e., for any function $f \in X$ and any $\epsilon > 0$, there exists a bounded function g such that $\|f - g\|_X < \epsilon$. Let now $h = c_1\chi_{\Omega_1} + \ldots + c_m\chi_{\Omega_m}$ be a simple function such that $\|g - h\|_{L_\infty} < \epsilon$. Without loss of generality, we may suppose that the embedding constant of L_∞ into X is no greater than 1, and thus $\|f - h\|_X < 2\epsilon$.

Now we observe that the function h has an absolutely continuous norm in X, that is, for sufficiently small μD,

$$\|h\chi_D\|_X \leqslant \sum_{i=1}^{m} |c_i|\, \|\chi_{\Omega_i \cap D}\|_X < \epsilon, \tag{2.1}$$

since all characteristic functions have absolutely continuous norms by assumption. In result, using the pseudo-lattice property of X, we get

$$\|f\chi_D\|_X \leqslant \|h\chi_D\|_X + \|(f - h)\chi_D\|_X < (1 + 2C_X)\epsilon,$$

as desired. □

As usual, a set $M \subset X$ is said to be compact in measure if any sequence from M contains a subsequence convergent in measure. The compactness in measure is equivalent to compactness with respect to convergence a.e. Indeed, any sequence convergent a.e. converges in measure; any sequence convergent in measure contains a subsequence convergent a.e. Any set M, which is compact in some X as above, is compact in measure. The second necessary component of compactness in X is the AEN property. Moreover, these two components together are not only necessary but also sufficient for compactness, as proved in the following assertion.

PROPOSITION 2.4. *A set M is compact in a pseudo-lattice X with absolutely continuous norms if and only if it is compact in measure and has absolutely equicontinuous norms (AEN), that is, for any $\epsilon > 0$, there exists common δ such that $\|f\chi_D\|_X < \epsilon$ for all $f \in M$ whenever $\mu D < \delta$.*

Proof. Let a set M be compact in X and let us show that it has AEN property. For a given $\epsilon > 0$, let $f_1, \dots, f_m$ be an $\epsilon/(2C_X)$-net of this set and let δ be such that $\|f_i\chi_D\|_X < \epsilon/2,\ i = 1, \dots, m$ for any set D with $\mu D < \delta$. Now, for arbitrary given function $f \in M$, we take a function f_k from the net such that $\|f - f_k\|_X < \epsilon/(2C_X)$, and then, with the same D as above, we obtain that $\|f\chi_D\|_X \leqslant \|f_k\chi_D\|_X + \|(f - f_k)\chi_D\|_X < \epsilon$.

Conversely, let a set $M \subset X$ be compact in measure and have the AEN property in X. To prove compactness of this set in X it is enough to show that every sequence $\{f_n\}$, converging in measure to 0, converges to 0 in X. For any given $\epsilon > 0$, we have δ such that the inequality $\mu D < \delta$ implies $\|f\chi_D\|_X < \epsilon/2$ for all $f \in M$. At the same time, the convergence $f_n \to 0$ in measure means existence of a number N such that, for all $n > N$, the measure of the set $D_n = \{t : |f_n(t)| > \epsilon/2\}$ is less than δ. Due to the choice of δ this implies that $\|f\chi_{D_n}\|_X < \epsilon/2$ for all $f \in M$. On the other hand, $|f_n(t)| \leqslant \epsilon/2$ on the remaining set $\Omega \setminus D_n$ and thus $\|f_n\chi_{\Omega\setminus D_n}\|_{L_\infty} \leqslant \epsilon/2$. Since L_∞ is normally embedded into X, this means that $\|f_n\|_X < \epsilon$, and we are done. □

REMARK 1. It is easy to see that, in fact, we have used absolute continuity of norm only for functions from the set M and may not require this property of other $f \in X$.

3. Interpolation theorem

We recall that a rule $\mathscr{F}$, which to any Banach couple $\vec{X} = (X_0, X_1)$ assigns some intermediate space $X = \mathscr{F}(\vec{X})$, is called an *interpolation functor* if, for any other couple $\vec{Y} = (Y_0, Y_1)$ and any linear operator $T \in \mathscr{L}(\vec{Y}, \vec{X})$, this operator is bounded from $Y = \mathscr{F}(\vec{Y})$ to X (the case $\vec{X} = \vec{Y}$ is also permitted). Using the closed graph theorem, one can show existence of a constant C such that

$$\|T\|_{Y\to X} \leqslant C \max\{\|T\|_{Y_0\to X_0},\ \|T\|_{Y_1\to X_1}) \tag{3.1}$$

for any T as above. A functor $\mathscr{F}$ is called *bounded* if there exists a universal C, suitable for all couples $\vec{X}, \vec{Y}$.

Even for very simple and standard functors and couples, the norm estimate (3.1) is insufficient for interpolation of compactness property. As shown in [11] for rearrangement-invariant spaces and in [3] for the cases when one of the couples $\vec{X}, \vec{Y}$ is reduced to a single space, the norm $\|T\|_{Y \to X}$ should necessarily tend to zero together with that norm $\|T\|_{Y_i \to X_i}$, $i = 0$ or 1, where the compactness of operators is given. In our main theorem below we also include an analogous condition. Moreover, we do not specify our proofs for situations when $X_0 = X_1$ or $Y_0 = Y_1$, since these cases are investigated in [3] as fully as possible.

Notice that an analogous extreme situation occurs in the cases when at least one of the couples $\vec{X}, \vec{Y}$ is trivial. Recall that a couple $\vec{X}$ is called *trivial* if the space $X_0 \cap X_1$ is closed in both spaces X_0 and X_1. As shown in [1] (see also [2, page 133]), any trivial couple has no interpolation spaces different from the endpoint spaces of the couple, their sum and intersection, so any interpolation functor reduces to at most four individual spaces and can be investigated similarly to the cases $X_0 = X_1$ or $Y_0 = Y_1$. That is why the trivial couples also will not be considered below.

THEOREM 3.1. *Let $\vec{X} = (X_0, X_1)$ be a nontrivial couple of pseudo-lattices such that X_0 has absolutely continuous norms. Let $\vec{Y} = (Y_0, Y_1)$ be an arbitrary nontrivial Banach couple and let an interpolation functor $\mathscr{F}$ possess the property*

$$\|T\|_{\mathscr{F}(\vec{Y}) \to \mathscr{F}(\vec{X})} \leqslant \Phi(\|T\|_{Y_0 \to X_0}, \|T\|_{Y_1 \to X_1}) \tag{3.2}$$

for all linear operators $T : \vec{Y} \to \vec{X}$, where the function $\Phi(\alpha, \beta)$ is non-decreasing and $\lim_{\alpha \to 0} \Phi(\alpha, \beta) = 0$ for any fixed β. Then if $T \in \mathscr{L}(\vec{Y}, \vec{X})$ is compact as an operator from Y_0 to X_0, it is also compact as an operator from $\mathscr{F}(\vec{Y})$ to $\mathscr{F}(\vec{X})$.

Proof. Let $X = \mathscr{F}(\vec{X})$, $Y = \mathscr{F}(\vec{Y})$ and let a linear operator T be bounded from Y_1 to X_1 and compact from Y_0 to X_0. Then it is bounded, acting from Y_1, and compact, acting from Y_0 into one and the same space $X_0 + X_1$. Applying Theorem 3.9 from [3], we obtain that either T is compact, when acting from Y to $X_0 + X_1$, or $Y_1^\circ \hookrightarrow Y$, where, as usual, Y_1° means the closure of $Y_0 \cap Y_1$ in Y_1. Let us show that the second possibility can only happen if $Y_1^\circ \hookrightarrow Y_0$.

Note, first of all, that the embedding $Y_1^\circ \hookrightarrow Y_0$ is equivalent to the embedding $Y_1^\circ \hookrightarrow Y_0^\circ$. The Banach couple Y_0°, Y_1° is regular and thus the conjugate spaces $(Y_0^\circ)^*, (Y_1^\circ)^*$ also form a Banach couple. Suppose that there exists a constant c such that

$$\|g\|_{(Y_0^\circ)^*} \geqslant c\,\|g\|_{(Y_1^\circ)^*} \qquad \text{for all } g \in (Y_0^\circ)^* \cap (Y_1^\circ)^*. \tag{3.3}$$

This inequality is possible only in two cases: either $(Y_0^\circ)^* \cap (Y_1^\circ)^* = (Y_0^\circ)^*$ or the space $(Y_0^\circ)^* \cap (Y_1^\circ)^*$ is closed in $(Y_0^\circ)^*$. In the first case we have that $(Y_0^\circ)^* \hookrightarrow (Y_1^\circ)^*$, that is, $Y_1^\circ \hookrightarrow Y_0^\circ$ and we are done. In the second case we may use the theorem of Aronszajn and Gagliardo from [1] (see also [2, page 131]), getting that in the couple $(Y_0^\circ)^*, (Y_1^\circ)^*$ there are no interpolation spaces, lying strictly between $(Y_0^\circ)^* \cap (Y_1^\circ)^* = (Y_0^\circ + Y_1^\circ)^*$ and $(Y_0^\circ)^*$. But this conclusion contradicts the properties of functors in real interpolation that gives infinitely many different interpolation spaces of the couple Y_0°, Y_1° in which $Y_0^\circ \cap Y_1^\circ$ is dense and which lie between the spaces Y_0° and $Y_0^\circ + Y_1^\circ$. Passing to the dual functors, we obtain that all these spaces have different conjugate spaces which all are interpolation in the conjugate couple $(Y_0^\circ)^*, (Y_1^\circ)^*$ and lie just in the "forbidden interval". This contradiction shows that the second case cannot be realized.

Now let us check the possibility of a situation when the inequality (3.3) does not hold. Such a situation implies existence of linear functionals $g_n \in (Y_0^\circ)^* \cap (Y_1^\circ)^*$, $n = 1,2,\ldots$, such that $\|g_n\|_{(Y_1^\circ)^*} = 1$ for every n while $\|g_n\|_{(Y_0^\circ)^*} \to 0$. Moreover, the Hahn-Banach theorem allows us to extend these functionals (separately) to the whole Y_0 and to the whole Y_1 with the same norms. Taking some function $f \in X_0 \cap X_1$, we define one-dimensional operators $T_n y = g_n(y)f$, acting from Y_i to X_i, $i = 0,1$, with the norms

$$\|T_n\|_{Y_1 \to X_1} = \|f\|_{X_1}, \qquad \|T_n\|_{Y_0 \to X_0} = \|g_n\|_{Y_0^*} \|f\|_{X_0} \to 0.$$

Using the inequality (3.2), we obtain that also

$$\|T_n\|_{Y \to X} = \|g_n\|_{Y^*} \|f\|_X \to 0,$$

which contradicts the embedding $Y_1^\circ \hookrightarrow Y$ and thus disproves the considered possibility for the inequality (3.3).

The proven embedding $Y_1^\circ \hookrightarrow Y_0$ admits only two possibilities for the space Y: either $Y = Y_1 \not\hookrightarrow Y_0$ or $Y \hookrightarrow Y_0$, where the first relation is incompatible with the inequality (3.2) for any nontrivial couples. Thus we remain with the embedding $Y \hookrightarrow Y_0$ which again implies compactness of T as an operator from Y to $X_0 + X_1$. Denoting by B_Y the unit ball of Y, we obtain that the set of functions TB_Y is relatively compact in $X_0 + X_1$ and thus in measure.

For arbitrary $D \subset \Omega$, let P_D denote the operator of multiplication by χ_D. Recall that any such operator is bounded in pseudo-lattices with norm estimates independent of D. By conditions of the theorem, the set of functions TB_{Y_0} is relatively compact in X_0 and by Proposition 2.4 it has AEN property there. This means that $\|P_D Ty\|_{X_0} \to 0$ uniformly on B_{Y_0} when $\mu D \to 0$, i.e., for any given $\alpha > 0$, we can find δ such that $\|P_D T\|_{Y_0 \to X_0} < \alpha$ whenever $\mu D < \delta$. Since on the second endpoint of interpolation we have $\|P_D T\|_{Y_1 \to X_1} < C_{X_1} \|T\|_{Y_1 \to X_1}$, we obtain that, for all D with sufficiently small measure,

$$\|P_D T\|_{Y \to X} \leqslant \Phi(\alpha, \beta), \qquad \beta = C_{X_1} \|T\|_{Y_1 \to X_1}.$$

But $\Phi(\alpha,\beta) \to 0$ when $\alpha \to 0$. Therefore, for any $\epsilon > 0$, there exists α such that $\Phi(\alpha,\beta) < \epsilon$. Taking δ suitable for this α, we obtain that $\|P_D T\|_{Y \to X} < \epsilon$ for any set D with $\mu D < \delta$, that is, the set of functions TB_Y has absolutely equicontinuous norms. By Proposition 2.4 this entails the desired compactness of T as an operator from Y to X. □

REMARK 2. The properties of the function $\Phi(\alpha,\beta)$ are required in Theorem 3.1 only for the considered couples $\vec{X}, \vec{Y}$, so that the functor $\mathscr{F}$ may be even unbounded on the whole. Moreover, this theorem can be reformulated for two given triples (X_0, X_1, X) and (Y_0, Y_1, Y) with needed properties even without any mention of functors.

At the first sight, the conditions of our theorem look similar to the approximation hypothesis from [14] (see also [12]), and it is interesting to compare them. Recall the components of this hypothesis as they were formulated in [12]:

For each compact set $K \subset X_0$, there exists a constant C such that, for each $\epsilon > 0$, there exists an operator $P \in \mathscr{L}(\vec{X}, \vec{X})$ with the norms $\|P\|_{X_i \to X_i} \leqslant C$ $(i = 0, 1)$ and such that

a) $P(X_i) \subset X_0 \cap X_1, \quad i = 0, 1$;

b) $\|Pf - f\|_{X_0} < \epsilon \quad$ *for all* $f \in K$.

The only possible substitutes for operators P in our constructions are the operators P_D with arbitrary measurable sets $D \subset \Omega$, which indeed satisfy the condition b) after the special choice of D. At the same time, the condition a) could be realized only in very special spaces, where the singularities of all functions are concentrated on some set with zero measure. For instance, given a Banach function space E, we may take the weight spaces $X_i = E(\omega_i)$, $i = 0, 1$ with arbitrary monotone decreasing weights ω_i.

The situation changes if the spaces X_0, X_1 are rearrangement-invariant, since in such spaces the AEN property of an arbitrary set M is equivalent to the AEN property of the set of non-increasing rearrangements f^* for all functions from M. And for the non-increasing functions, the common set D such that the corresponding operator P_D satisfies the condition a) always exists — one may take $D = \Omega \setminus (0, \delta)$ with an appropriate δ. Recall that the class of rearrangement-invariant spaces was precisely the largest class of spaces of measurable functions, considered in [12] by the use of the approximation hypothesis.

4. Abelian groups, sequence spaces and other generalizations

It is easy to see that the proof of Theorem 3.1 is based only on Proposition 2.4 independently of a particular form of the measure μ and of the topology in the considered spaces. This gives a possibility to extend this theorem to other kinds of spaces, where an appropriate analog of Proposition 2.4 can be proved. For example, we may consider quasinormed Abelian groups instead of Banach spaces.

Interpolation of bounded homomorphisms acting in Abelian groups of measurable functions was defined and studied in [13]. These spaces are intermediate in the same couple (L_0, L_∞). Their topology can be described, using a special kind of quasinorm which is not homogeneous, i.e., a constant factor cannot be taken out of this quasinorm. Extending our definitions and proofs to Abelian groups, we should be only careful with constant factors and the triangle inequality.

From this point of view, we find immediately that both Definitions 2.1 and 2.2 may be applied to Abelian groups with no changes. This is not true for Proposition 2.3 where we cannot consider now the function $f(t) \equiv 1$ alone and should require that all constant functions have absolutely continuous quasinorms. Thereafter, in the proof, we can replace the inequality (2.1) by

$$\|h\chi_D\|_X \leqslant C \sum_{i=1}^{m} \|c_i \chi_{\Omega_i \cap D}\|_X < \epsilon$$

and finish as before. At last, the formulation and the proof of Proposition 2.4 can be accepted without any changes.

Passing to Theorem 3.1, we observe that the unit balls of spaces are now not sufficient representatives for stating properties of operators and should be replaced by arbitrary bounded sets. It is easy to ascertain that this fact has no influence on the given proof. More serious is the use of Theorem 3.9 from [3], presented there only in a context of Banach spaces. It seems very likely that this theorem could be extended to the abstract Abelian groups, but in the existent situation we may not use it. This obstacle can be avoided if we consider only *ordered* couples $\vec{Y}$, so that $Y \subset Y_0$. Thus any bounded set $B \subset Y$ will be bounded in the space Y_0 too and the set TB will be relatively compact in $X_0 + X_1$ as desired.

Let now X be a Banach space of sequences $a = \{a(n)\}$ with a topology stronger than the componentwise one (this means that $\lim_{k\to\infty} \|a_k - a\|_X = 0$ implies $a_k(n) \to a(n)$ for any fixed n). We say that X is a *pseudo-lattice* if $\|a\chi_{[n,\infty)}\|_X \leqslant C_X\|a\|_X$ for any natural n with a constant C_X independent of a and n. As a consequence of this definition, we obtain immediately that any such X contains standard basic sequences $e_m = \{e_m(n)\} = \{\delta_{mn}\}$ with any $m = 1, 2, \ldots$ such that $a(m) \neq 0$ at least for one $a \in X$. We say that X has *absolutely continuous norms* if, for any ϵ and each sequence $a \in X$, there exists $N = N(a)$ such that $\|a\chi_{[n,\infty)}\|_X < \epsilon$ whenever $n > N$. Let us show that after such definitions of the needed concepts one obtains a complete analog of Proposition 2.4 for the sequence spaces.

PROPOSITION 4.1. *A set of sequences M is compact in a pseudo-lattice X with absolutely continuous norms if and only if it is compact with respect to the componentwise convergence and has absolutely equicontinuous norms (AEN), that is, for any $\epsilon > 0$, there exists a common N such that $\|a\chi_{[n,\infty)}\|_X < \epsilon$ for all $a \in M$ whenever $n > N$.*

Proof. Let a set of sequences M be compact in X; we show that it has the AEN property. For a given $\epsilon > 0$, let $a_1, \ldots, a_m$ be an $\epsilon/(2C_X)$-net of this set and let N be such that $\|a_i\chi_{[n,\infty)}\|_X < \epsilon/2$, $i = 1, \ldots, m$, for any $n > N$. Now, for arbitrary given element $a \in X$, we take a corresponding element a_k from the net such that $\|a - a_k\|_X < \epsilon/(2C_X)$, and then, with the same N as above, we obtain for any $n > N$ that $\|a\chi_{[n,\infty)}\|_X \leqslant \|a_k\chi_{[n,\infty)}\|_X + \|(a - a_k)\chi_{[n,\infty)}\|_X < \epsilon$.

Conversely, let a set $M \subset X$ be compact with respect to the componentwise convergence and have the AEN-property in X. To show compactness of M in X it is enough to show that any sequence (of sequences) $\{a_k\}$ with components convergent to 0 tends to 0 in the norm of X. Given $\epsilon > 0$, define the corresponding n such that $\|a_k\chi_{[n+1,\infty)}\|_X < \epsilon/2$ for all k. Fixing this n, since all norms in final dimensional spaces are equivalent, there exists a number k_0 such that, for all $k > k_0$, one has $\|a_k\chi_{[1,n]}\|_X < \epsilon/2$. Consequently,

$$\|a_k\|_X \leqslant \|a_k\chi_{[1,n]}\|_X + \|a_k\chi_{[n+1,\infty)}\|_X < \epsilon \qquad \text{for all } k > k_0,$$

as desired. □

We leave it to the reader to verify that, with Proposition 4.1 instead of 2.2, the proof of Theorem 3.1 can be repeated for sequence spaces almost word by word.

Let us attempt now to formulate some new approximation hypothesis which extends the cases considered above to abstract normed spaces (of course, some requirements of this hypothesis should be weaker than those from [14] so as to correspond to pseudo-lattices). Let X_0, X_1 form a couple of such spaces and, for arbitrary relatively compact set $K \subset X_0 + X_1$ and $\alpha > 0$, let $\mathscr{P}_{K,\alpha}$ denote the set of all linear operators $P \in \mathscr{L}(\vec{X}, \vec{X})$ such that $\|Px - x\|_{X_0} < \alpha$ for all $x \in K \cap X_0$ (the set $\mathscr{P}_{K,\alpha}$ is never empty, since it contains, at least, the identity operator I). We say that the couple $\vec{X}$ satisfies the *weak approximation hypothesis* if, for each relatively compact set K and for any sequence $x_n \in K$ convergent in $X_0 + X_1$, there exists a constant b such that, for any α, there exist an operator $P \in \mathscr{P}_{K,\alpha}$ and a number N such that $\|P(x_n - x_m)\|_{X_0 \cap X_1} < \alpha$ for all $m, n > N$ while $\|P\|_{X_1 \to X_1} \leqslant b$.

By Propositions 2.3, 2.4 and 4.1, it is easy to ascertain that in pseudo-lattices having absolutely continuous norms the operators $P = I - P_D$ with appropriate sets D satisfy all requirements of the weak approximation hypothesis. Let us show that this hypothesis entails interpolation of compactness in the abstract case as well.

THEOREM 4.2. *Let a Banach couple $\vec{X} = (X_0, X_1)$ satisfy the weak approximation hypothesis and let $T \in \mathscr{L}(\vec{Y}, \vec{X})$ for some Banach couple $\vec{Y} = (Y_0, Y_1)$. If T is compact as an operator from Y_0 to X_0 and if an interpolation functor $\mathscr{F}$ satisfies conditions of Theorem 3.1, then T is compact as an operator from $Y = \mathscr{F}(\vec{Y})$ to $X = \mathscr{F}(\vec{X})$.*

Proof. Exactly as in Theorem 3.1, we can show that T is compact as an operator from Y to $X_0 + X_1$, so that the weak approximation hypothesis may be applied to the set $K = TB_Y \cup TB_{Y_0}$. Any sequence of elements from TB_Y contains a subsequence $\{Tx_n\}$ convergent in $X_0 + X_1$, hence it is enough to show that Tx_n is convergent in X as well. Taking arbitrary $\alpha > 0$, we obtain an operator $P \in \mathscr{P}_{K,\alpha}$ and an integer N such that

$$\|P(Tx_n - Tx_m)\|_X \leqslant C\|P(Tx_n - Tx_m)\|_{X_0 \cap X_1} < C\alpha \qquad \text{for all } m, n > N,$$

where C is the embedding constant of $X_0 \cap X_1$ into X. At the same time, we obtain that the inequality $\|Px - x\|_{X_0} < \alpha$ holds for all $x \in TB_{Y_0} \subset K \cap X_0$, which means that $\|(P - I)T\|_{Y_0 \to X_0} < \alpha$. The operator $(P - I)T$ is also bounded from Y_1 to X_1 where its norm can be estimated by some β independent of α. Applying (3.2), we get that $\|(P - I)T\|_{Y \to X} \leqslant \Phi(\alpha, \beta)$. In result,

$$\|Tx_n - Tx_m\|_X \leqslant \|P(Tx_n - Tx_m)\|_X + \|(P - I)T(x_n - x_m)\|_X < C\alpha + 2\Phi(\alpha, \beta),$$

and the last term can be made less than any given ϵ by a suitable choice of α. □

REMARK 3. Theorem 4.2 is more general than Theorem 3.1 and may be applied to abstract Banach spaces. Unfortunately, at the moment, the author is not able to give any interesting example of such an application and only conjectures that spaces like BMO or spaces of functions with prescribed properties of Fourier coefficients should satisfy the needed conditions.

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