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A SHARPENED VERSION OF THE FUNDAMENTAL TRIANGLE INEQUALITY

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Abstract. In this note, we show a sharpened version of the classical fundamental triangle inequality, as follows

$$2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr}\cos\phi \leqslant s^2 \leqslant 2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}\cos\phi,$$

where $\phi = \min\limits_{1\leqslant i<j\leqslant 3} |A_i - A_j|$.

1. Introduction

In what follows, we denote by A_1, A_2, A_3 the angles of the triangle $A_1A_2A_3$, and let R, r, s and F denote respectively the circumradius, the inradius, the semi-perimeter and the area of the triangle. In addition, we will customarily use the symbols $\prod$ to signify cyclic product, such as $\prod f(A_1) = f(A_1)f(A_2)f(A_3)$.

Since Euler established the celebrated inequality $R \geqslant 2r$ in 1765, the geometric inequalities relating elements R, r and s has attracted remarkable interest of many mathematicians and has been studied widely.

In 1851, as an answer to Ramus's question, Rouché [1] proved the following result:

$$r\sqrt{2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr}} \\ \leqslant F \leqslant r\sqrt{2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}}. \tag{1}$$

Obviously, inequality (1) is equivalent to the inequality:

$$2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr} \\ \leqslant s^2 \leqslant 2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}. \tag{2}$$

It is well-known that inequality (2) is a necessary and sufficient condition for the existence of a triangle with elements R, r and s, and it is called the fundamental triangle inequality (see [2]). The fundamental triangle inequality is one of the most

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investigated classical geometric inequality, a large number of results involving its proofs, generalizations and applications have been presented in the literatures (see e.g. [2–10] and references therein).

The aim of this paper is to establish a new sharp version of the fundamental triangle inequality by introducing a parameter. Our result is given in the following theorem.

THEOREM. *Let* $\phi = \min_{1 \leqslant i<j \leqslant 3} |A_i - A_j|$. *Then for any triangle* $A_1A_2A_3$, *we have*

$$2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R^2-2Rr}\cos\phi \leqslant s^2 \leqslant 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R^2-2Rr}\cos\phi. \quad (3)$$

Both equalities hold in (3) if and only if the triangle is equilateral.

2. Lemma

We start with the following lemma.

LEMMA. *For any triangle* $A_1A_2A_3$, *if* $A_1 \geqslant A_2 \geqslant A_3$, *then we have the following inequalities*

$$2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R^2-2Rr}\,\cos(A_2-A_3) \leqslant s^2 \leqslant 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R^2-2Rr}\,\cos(A_1-A_2), \quad (4)$$

where the left equality holds if and only if $A_2 = A_3$, *the right equality holds if and only if* $A_1 = A_2$.

Proof. Firstly, we prove the left-hand side of inequality (4), that is,

$$s^2 \geqslant 2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R^2-2Rr}\cos(A_2-A_3). \quad (5)$$

Using the well-known identities for a triangle:

$$s = 4R\prod\cos\frac{A_1}{2}\ ,\ r = 4R\prod\sin\frac{A_1}{2}, \quad (6)$$

and substituting them into (5), we find

$$(5) \iff -8\prod\cos^2\frac{A_1}{2} + 1 + 20\prod\sin\frac{A_1}{2} - 8\prod\sin^2\frac{A_1}{2} - \cos(A_2-A_3)\left(1-8\prod\sin\frac{A_1}{2}\right)^{\frac{3}{2}} \leqslant 0. \quad (7)$$

Direct calculation gives

$$-8\prod\cos^2\frac{A_1}{2}+1+20\prod\sin\frac{A_1}{2}-8\prod\sin^2\frac{A_1}{2}-\cos(A_2-A_3)\left(1-8\prod\sin\frac{A_1}{2}\right)^{\frac{3}{2}}$$
$$= -2\cos^2\frac{A_2-A_3}{2}+1+6\cos\frac{A_2-A_3}{2}\sin\frac{A_1}{2}-12\sin^2\frac{A_1}{2}+8\cos\frac{A_2-A_3}{2}\sin^3\frac{A_1}{2}$$

$$-\cos(A_2-A_3)\left[\left(2\cos\frac{A_2-A_3}{2}\sin\frac{A_1}{2}-1\right)^2+4\sin^2\frac{A_2-A_3}{2}\sin^2\frac{A_1}{2}\right]^{\frac{3}{2}}. \tag{8}$$

On the other hand, from $A_1 \geqslant A_2 \geqslant A_3$ we conclude that $0 \leqslant A_2 - A_3 < \frac{\pi}{2}$, $\frac{\pi}{3} \leqslant A_1 < \pi$.

Let

$$\cos\frac{A_2-A_3}{2} = t \ \left(\frac{\sqrt{2}}{2} < t \leqslant 1\right) \quad \text{and} \quad \sin\frac{A_1}{2} = x \ \left(\frac{1}{2} \leqslant x < 1\right).$$

From (8) and the above assumption, we have

$$\begin{aligned}
&-8\prod\cos^2\frac{A_1}{2}+1+20\prod\sin\frac{A_1}{2}-8\prod\sin^2\frac{A_1}{2}-\cos(A_2-A_3)\left(1-8\prod\sin\frac{A_1}{2}\right)^{\frac{3}{2}}\\
&\leqslant -2\cos^2\frac{A_2-A_3}{2}+1+6\cos\frac{A_2-A_3}{2}\sin\frac{A_1}{2}-12\sin^2\frac{A_1}{2}+8\cos\frac{A_2-A_3}{2}\sin^3\frac{A_1}{2}\\
&\quad -\cos(A_2-A_3)\left(2\cos\frac{A_2-A_3}{2}\sin\frac{A_1}{2}-1\right)^3\\
&= -2t^2+1+6tx-12x^2+8tx^3-(2t^2-1)(2tx-1)^3\\
&= -2t^2+2+(8tx^3-12x^2)(1-t^2)-(2t^2-2)(2tx-1)^3\\
&= 4x(1-t^2)[2t(2t^2+1)x^2-3(2t^2+1)x+3t]\\
&= 4x(1-t^2)\left[-2(2t^2+1)(x-\frac{1}{2})(1-x)+(2t^2+1)(2t-2)x^2-2t^2+3t-1\right]\\
&= 4x(1-t^2)\left[-2(2t^2+1)(x-\frac{1}{2})(1-x)-(1-t)((4t^2+2)x^2-2t+1\right)]\\
&= -4x(1-t^2)\left[2(2t^2+1)(x-\frac{1}{2})(1-x)+(1-t)((4t^2+2)(x^2-\frac{1}{4})+(t-1)^2+\frac{1}{2})\right]\\
&\leqslant 0\,.
\end{aligned}$$

Thus inequality (7) holds, which leads to the inequality (5).

Next, we prove the right-hand side of inequality (4), that is,

$$s^2 \leqslant 2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}\cos(A_1-A_2). \tag{9}$$

Applying the identities $s = 4R\prod\cos\frac{A_1}{2}$ and $r = 4R\prod\sin\frac{A_1}{2}$ to (9), we find

$$\begin{aligned}
(9) \iff\ &-8\prod\cos^2\frac{A_1}{2}+1+20\prod\sin\frac{A_1}{2}-8\prod\sin^2\frac{A_1}{2}\\
&+\cos(A_1-A_2)\left(1-8\prod\sin\frac{A_1}{2}\right)^{\frac{3}{2}} \geqslant 0.
\end{aligned} \tag{10}$$

A calculation shows that

$$-8\prod\cos^2\frac{A_1}{2}+1+20\prod\sin\frac{A_1}{2}-8\prod\sin^2\frac{A_1}{2}+\cos(A_1-A_2)\left(1-8\prod\sin\frac{A_1}{2}\right)^{\frac{3}{2}}$$

$$= -2\cos^2\frac{A_1-A_2}{2}+1+6\cos\frac{A_1-A_2}{2}\sin\frac{A_3}{2}-12\sin^2\frac{A_3}{2}+8\cos\frac{A_1-A_2}{2}\sin^3\frac{A_3}{2}$$
$$+\cos(A_1-A_2)\left[\left(1-2\cos\frac{A_1-A_2}{2}\sin\frac{A_3}{2}\right)^2+4\sin^2\frac{A_1-A_2}{2}\sin^2\frac{A_3}{2}\right]^{\frac{3}{2}}. \tag{11}$$

By $\cos(A_1-A_2)=\cos(A_2-A_1)$, it is easy to see that (10) is symmetric with respect to variable A_1 and A_2. Without loss of generality we can assume that $A_1 \geqslant A_2$.

Case (I). When $0 \leqslant A_1-A_2 \leqslant \frac{\pi}{2}$, it implies $0 < A_3 \leqslant \frac{\pi}{3}$ and $\cos(A_1-A_2) \geqslant 0$. Let

$$\cos\frac{A_1-A_2}{2}=t\ \left(\frac{\sqrt{2}}{2}\leqslant t\leqslant 1\right),\quad \sin\frac{A_3}{2}=x\ \ \left(0<x\leqslant\frac{1}{2}\right).$$

From (11) and the above assumption, we have

$$-8\prod\cos^2\frac{A_1}{2}+1+20\prod\sin\frac{A_1}{2}-8\prod\sin^2\frac{A_1}{2}+\cos(A_1-A_2)\left(1-8\prod\sin\frac{A_1}{2}\right)^{\frac{3}{2}}$$
$$\geqslant -2\cos^2\frac{A_1-A_2}{2}+1+6\cos\frac{A_1-A_2}{2}\sin\frac{A_3}{2}-12\sin^2\frac{A_3}{2}$$
$$+8\cos\frac{A_1-A_2}{2}\sin^3\frac{A_3}{2}+\cos(A_1-A_2)\left(1-2\cos\frac{A_1-A_2}{2}\sin\frac{A_3}{2}\right)^3$$
$$=-2t^2+1+6tx-12x^2+8tx^3+(2t^2-1)(1-2tx)^3$$
$$=-2t^2+2+(8tx^3-12x^2)(1-t^2)+(2t^2-2)(1-2tx)^3$$
$$=4x(1-t^2)[\,2t(2t^2+1)x^2-3(2t^2+1)x+3t\,].$$

On the other hand, from the assumption in Lemma $A_1 \geqslant A_2 \geqslant A_3$, we find

$$\cos\frac{A_1-A_2}{2}=\sin\frac{A_3}{2}+2\sin\frac{A_1}{2}\sin\frac{A_2}{2}\geqslant\sin\frac{A_3}{2}+\sin\frac{A_2}{2}\geqslant 2\sin\frac{A_3}{2},$$

which implies

$$0<x\leqslant\frac{1}{2}t.$$

Now, in view of the fact that the function $f(x)=2t(2t^2+1)x^2-3(2t^2+1)x+3t$ is decreasing on $(0,\frac{3}{4t})$, together with $0<x\leqslant\frac{1}{2}t<\frac{3}{4t}$, we have

$$\begin{aligned}2t(2t^2+1)x^2-3(2t^2+1)x+3t &\geqslant 2t(2t^2+1)\left(\frac{1}{2}t\right)^2-3(2t^2+1)\left(\frac{1}{2}t\right)+3t\\ &= t(t^2-1)\left(t^2-\frac{3}{2}\right)\\ &\geqslant 0.\end{aligned}$$

Hence we conclude that the inequality (10) is valid, the inequality (9) follows immediately.

Case (II). When $A_1-A_2>\frac{\pi}{2}$, we have $\cos(A_1-A_2)<0$.

Since

$$\left[\left(1-2\cos\frac{A_1-A_2}{2}\sin\frac{A_3}{2}\right)^2+4\sin^2\frac{A_1-A_2}{2}\sin^2\frac{A_3}{2}\right]^{\frac{3}{2}}=\left(1-8\prod\sin\frac{A_1}{2}\right)^{\frac{3}{2}}<1,$$

it follows from (11) that

$$\begin{aligned}
&-8\prod\cos^2\frac{A_1}{2}+1+20\prod\sin\frac{A_1}{2}-8\prod\sin^2\frac{A_1}{2}+\cos(A_1-A_2)\left(1-8\prod\sin\frac{A_1}{2}\right)^{\frac{3}{2}}\\
&\quad>-2\cos^2\frac{A_1-A_2}{2}+1+6\cos\frac{A_1-A_2}{2}\sin\frac{A_3}{2}-12\sin^2\frac{A_3}{2}\\
&\qquad+8\cos\frac{A_1-A_2}{2}\sin^3\frac{A_3}{2}+\cos(A_1-A_2)\\
&\quad=6\sin\frac{A_3}{2}\left(\cos\frac{A_1-A_2}{2}-2\sin\frac{A_3}{2}\right)+8\cos\frac{A_1-A_2}{2}\sin^3\frac{A_3}{2}. \qquad (12)
\end{aligned}$$

By the assumption of the Lemma $A_1 \geqslant A_2 \geqslant A_3$, we find that

$$\cos\frac{A_1-A_2}{2}-2\sin\frac{A_3}{2}=2\sin\frac{A_1}{2}\sin\frac{A_2}{2}-\sin\frac{A_3}{2}\geqslant\sin\frac{A_2}{2}-\sin\frac{A_3}{2}\geqslant 0.$$

Combining (12) and the above inequality yields the inequality (10). Consequently, the inequality (9) is proved.

In addition, the condition of equality in (4) can be obtained from the process of the proof of Lemma. The proof of Lemma is complete. □

3. Proof of Theorem

Note that the inequality (3) is symmetrical with respect to variables A_1, A_2 and A_3. Without loss of generality, we assume that $A_1 \geqslant A_2 \geqslant A_3$.

Case (I). When $A_1-A_2 \geqslant A_2-A_3$, it implies $A_1-A_3 \geqslant A_1-A_2 \geqslant A_2-A_3$.

We need now to prove that

$$\begin{aligned}
&2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr}\cos(A_2-A_3)\\
&\quad\leqslant s^2\leqslant 2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}\cos(A_2-A_3),
\end{aligned}$$

which can be obtained from Lemma with $\cos(A_1-A_2)\leqslant\cos(A_2-A_3)$.

Case (II). When $A_1-A_2<A_2-A_3$, it implies $A_1-A_3 \geqslant A_2-A_3 > A_1-A_2$.

To prove the inequality (3), it is enough to prove that

$$\begin{aligned}
&2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr}\cos(A_1-A_2)\\
&\quad\leqslant s^2\leqslant 2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}\cos(A_1-A_2),
\end{aligned}$$

which follows from Lemma with $-\cos(A_2-A_3) > -\cos(A_1-A_2)$.

The condition of equality in (3) can be deduced from Lemma immediately. This completes the proof of Theorem.

REMARK. Euler's inequality $R-2r \geqslant 0$ and the inequality $\cos\phi \leqslant 1$ show that the inequality (3) is a sharpened version of the fundamental triangle inequality.

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