Mathematical
Inequalities
& Applications
Volume 11, Number 3 (2008), 483–493

TRUNCATION INEQUALITIES FOR PROBABILITY MEASURES ON LOCALLY COMPACT ABELIAN GROUPS

HERBERT HEYER AND GYULA PAP

(*communicated by N. Elezović*)

Abstract. The so-called truncation inequality is generalized in several ways for probability measures on locally compact Abelian groups. Some applications are presented as well.

1. Introduction

The aim of the present paper is to generalize the classical inequality

$$\mu\left(\left\{x \in \mathbf{R} : |x| > \frac{1}{a}\right\}\right) \leqslant \frac{c}{a} \int_0^a \left(1 - \operatorname{Re} \widehat{\mu}(t)\right) \mathrm{d}t \tag{1.1}$$

valid for any probability measure μ on the real line and for arbitrary $a > 0$, where $\widehat{\mu}$ denotes the Fourier transform (i.e., characteristic function) of μ and $c > 0$ is an absolute constant (i.e., not depending on μ or a). The constant c can be chosen $c = 1/(1 - \sin 1)$, or simply $c = 7$. (See, for example, Loève [6, 12.4 B$'$], Shiryayev [9, III. §3, Lemma 3], Dudley [1, 9.8.1] or Fristedt and Gray [3, 14.6, Lemma 11].)

Inequality (1.1) is sometimes called a *truncation inequality* (see, e.g., Loève [6] or Dudley [1]). Inequality (1.1) estimates the 'tails' of a measure in terms of the behavior of its Fourier transform in a neighborhood of zero (Shiryayev [9]), or, in other words, (1.1) shows that the smoothness of the characteristic function at 0 is related to the decay of the measure at ∞.

A slightly modified version of (1.1) is

$$\mu\left(\left\{x \in \mathbf{R} : |x| > \frac{1}{a}\right\}\right) \leqslant \frac{1}{2a} \int_{-2a}^{2a} \left(1 - \widehat{\mu}(t)\right) \mathrm{d}t, \tag{1.2}$$

see, e.g., Durett [2] or Jacod and Protter [5]. Inequality (1.2) can be easily generalized

---

Mathematics subject classification (2000): 60B15.

Key words and phrases: Truncation inequality, locally compact Abelian group, tail behavior, Fourier transform, character group, Plancherel formula, almost periodic function, invariant mean.

This research has been supported by the Deutsche Forschungsgemeinschaft and the Hungarian Academy of Sciences. The second author has been partially supported by the Hungarian Scientific Research Fund under Grant No. OTKA-T048544/2005.

Paper MIA-11-38

to $\mathbf{R}^d$,

$$\mu\left(\left\{x \in \mathbf{R}^d : \max_{1\leqslant j\leqslant d} |x_j| > \frac{1}{a}\right\}\right) \leqslant \frac{2}{(4a)^d} \int_{\max_{1\leqslant j\leqslant d} |t_j|\leqslant 2a} \big(1 - \widehat{\mu}(t)\big)\,\mathrm{d}t, \tag{1.3}$$

see Parthasarathy [8, Proposition 53.15] for a similar result.

2. A truncation inequality for a class of locally compact Abelian groups

The generalization of the inequalities quoted in the Introduction will be established for locally compact Abelian group G with identity element e_G. To G there is attached its character group X with identity e_X and a Haar measure θ. The main tool in proving our results is as in the classical case the notion of Fourier transform which in the general situation will be defined as a mapping $\mu \mapsto \widehat{\mu}$ from the space of bounded measures on G into the space of bounded continuous functions on X by

$$\widehat{\mu}(\chi) := \int_G \overline{\chi(x)}\,\mu(\mathrm{d}x)$$

for all $\chi \in X$. The Fourier transform is injective, and its restriction to L^2-functions provides a Plancherel formula, where the Plancherel measure can be chosen to be θ.

The following general truncation inequality is valid for locally compact Abelian groups G whose character group X is compactly generated. We phrase the hypothesis in more concrete way which indicates explicitely for which special types of groups the proof can be carried out.

THEOREM 2.1. *Let G be a group of the form $\mathbf{R}^d \times D \times K$, where d is a nonnegative integer, D is a discrete Abelian group and K is a second countable compact Abelian group.*

(i) *For each Borel neighborhood B of the identity e_G and for each $\delta > 0$ there is a Borel subset H of X with $0 < \theta(H) < \infty$ such that*

$$\mu(G \setminus B) \leqslant \frac{1+\delta}{\theta(H)} \int_H \big(1 - \operatorname{Re}\widehat{\mu}(\chi)\big)\,\mathrm{d}\theta(\chi) \tag{2.2}$$

holds for each probability measure μ on G.

(ii) *There is a sequence $\{H_n\}_{n=1}^{\infty}$ of Borel subsets of X such that $0 < \theta(H_n) < \infty$ for all n, and such that for each Borel neighborhood B of the identity e_G and for each $\delta > 0$ there is a positive integer n_0 such that* (2.2) *holds with $H = H_n$ for all $n \geqslant n_0$.*

(iii) *The sequence $\{H_n\}_{n=1}^{\infty}$ in* (ii) *can be choosen as an increasing sequence of open symmetric neighborhoods of the identity e_X such that the closure H_n^- is compact for each n and such that $\cup_{n=1}^{\infty} H_n = G$.*

3. Preparatory results

PROPOSITION 3.1. *Let G be an arbitrary locally compact Abelian group. Suppose that there is a sequence $\{H_n\}_{n=1}^{\infty}$ of Borel subsets of X such that $0 < \theta(H_n) < \infty$ for all n, and such that for each Borel neighborhood B of the identity e_G we have*

$$\frac{1}{\theta(H_n)} \int_{H_n} \chi(x)\, \mathrm{d}\theta(\chi) \to 0 \qquad \text{as } n \to \infty \text{ uniformly in } x \in G \setminus B. \tag{3.2}$$

Then for each Borel neighborhood B of the identity e_G and for each $\varepsilon > 0$ there is a positive integer n_0 such that (2.2) *holds with $H = H_n$ for all $n \geqslant n_0$.*

Proof. For each n, we have

$$\begin{aligned} I_n &:= \frac{1}{\theta(H_n)} \int_{H_n} \big(1 - \operatorname{Re} \widehat{\mu}(\chi)\big)\, \mathrm{d}\theta(\chi) \\ &= \frac{1}{\theta(H_n)} \int_{H_n} \left(1 - \operatorname{Re} \int_G \overline{\chi(x)}\, \mathrm{d}\mu(x)\right) \mathrm{d}\theta(\chi) \\ &= \frac{1}{\theta(H_n)} \int_{H_n} \left(\int_G \big(1 - \operatorname{Re} \chi(x)\big)\, \mathrm{d}\mu(x)\right) \mathrm{d}\theta(\chi). \end{aligned}$$

We can use Fubini's theorem since $0 \leqslant 1 - \operatorname{Re} \chi(x) \leqslant 2$ for all $x \in G$ and $\chi \in X$, $\mu(G) = 1$ and $\theta(H_n) < \infty$. Hence

$$\begin{aligned} I_n &= \frac{1}{\theta(H_n)} \int_G \left(\int_{H_n} \big(1 - \operatorname{Re} \chi(x)\big)\, \mathrm{d}\theta(\chi)\right) \mathrm{d}\mu(x) \\ &= \int_G \left(1 - \frac{1}{\theta(H_n)} \operatorname{Re} \int_{H_n} \chi(x)\, \mathrm{d}\theta(\chi)\right) \mathrm{d}\mu(x) \\ &\geqslant \int_G \left(1 - \frac{1}{\theta(H_n)} \left|\int_{H_n} \chi(x)\, \mathrm{d}\theta(\chi)\right|\right) \mathrm{d}\mu(x) \\ &\geqslant \int_{G \setminus B} \left(1 - \frac{1}{\theta(H_n)} \left|\int_{H_n} \chi(x)\, \mathrm{d}\theta(\chi)\right|\right) \mathrm{d}\mu(x) \\ &\geqslant \mu(G \setminus B) \inf_{x \in G \setminus B} \left(1 - \frac{1}{\theta(H_n)} \left|\int_{H_n} \chi(x)\, \mathrm{d}\theta(\chi)\right|\right). \end{aligned}$$

Thus (2.2) holds for $H = H_n$ once

$$\inf_{x \in G \setminus B} \left(1 - \frac{1}{\theta(H_n)} \left|\int_{H_n} \chi(x)\, \mathrm{d}\theta(\chi)\right|\right) \geqslant \frac{1}{1+\delta}.$$

But this is equivalent to

$$\sup_{x \in G \setminus B} \frac{1}{\theta(H_n)} \left|\int_{H_n} \chi(x)\, \mathrm{d}\theta(\chi)\right| \leqslant \frac{\delta}{1+\delta},$$

hence the hypothesis (3.2) implies the assertion. □

PROPOSITION 3.3. *Let G_1 and G_2 be locally compact Abelian groups. Let X_1 and X_2 denote the character groups of G_1 and G_2 with Haar measure θ_1 and θ_2 respectively. Suppose that for each $j=1,2$, there is a sequence $\{H_n^{(j)}\}_{n=1}^{\infty}$ of Borel subsets of X_j such that $0<\theta_j(H_n^{(j)})<\infty$ for all n, and such that for each Borel neighborhood B_j of the identity e_{G_j} we have*

$$\frac{1}{\theta_j(H_n^{(j)})}\int_{H_n^{(j)}}\chi_j(x_j)\,\mathrm{d}\theta_j(\chi_j)\to 0 \qquad \text{as } n\to\infty \text{ uniformly in } x_j\in G_j\setminus B_j.$$

Consider the group $G:=G_1\times G_2$. Then the character group X of G can be identified with $X_1\times X_2$, and $\theta:=\theta_1\otimes\theta_2$ is a Haar measure on X. Moreover for each Borel neighborhood B of the identity $e_{G_1\times G_2}$ we have (3.2) *with $H_n:=H_n^{(1)}\times H_n^{(2)}$.*

Proof. For all Borel neighborhood U of the identity $e_{G_1\times G_2}$ there are Borel neighborhoods B_1 and B_2 of the identities e_{G_1} and e_{G_2} respectively, such that $B_1\times B_2\subset U$. Clearly it suffices to show (3.2) for $B=B_1\times B_2$. Then we have

$$(G_1\times G_2)\setminus(B_1\times B_2)\subset\big[(G_1\setminus B_1)\times G_2\big]\cup\big[G_1\times(G_2\setminus B_2)\big],$$

hence it suffices to show

$$\lim_{n\to\infty}\sup_{x\in(G_1\setminus B_1)\times G_2}\frac{1}{\theta(H_n)}\left|\int_{H_n}\chi(x)\,\mathrm{d}\theta(\chi)\right|=0,$$
$$\lim_{n\to\infty}\sup_{x\in G_1\times(G_2\setminus B_2)}\frac{1}{\theta(H_n)}\left|\int_{H_n}\chi(x)\,\mathrm{d}\theta(\chi)\right|=0.$$

Any character $\chi\in X$ can be written in the form $\chi(x)=\chi_1(x_1)\chi_2(x_2)$ for $x=(x_1,x_2)\in G_1\times G_2$, where $\chi_1\in X_1$ and $\chi_2\in X_2$. Thus Fubini's theorem yield

$$\frac{1}{\theta(H_n)}\int_{H_n}\chi(x)\,\mathrm{d}\theta(\chi)=\frac{1}{\theta_1(H_n^{(1)})}\int_{H_n^{(1)}}\chi_1(x_1)\,\mathrm{d}\theta_1(\chi_1)\,\frac{1}{\theta_2(H_n^{(2)})}\int_{H_n^{(2)}}\chi_2(x_j)\,\mathrm{d}\theta_2(\chi_2).$$

Moreover

$$\left|\frac{1}{\theta_j(H_n^{(j)})}\int_{H_n^{(j)}}\chi_j(x_j)\,\mathrm{d}\theta_j(\chi_j)\right|\leqslant 1$$

for all n and for $j=1,2$, hence the assertion. □

4. Proof of Theorem 2.1

Obviously (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i), hence it remains to prove (iii).

By Propositions 3.1 and 3.3, it suffices to show that for each of the groups $G=\mathbf{R}$, $G=D$ and $G=K$ there is an increasing sequence of open symmetric neighborhoods of the identity e_X of the character group X of G such that $0<\theta(H_n)<\infty$, the closure H_n^- is compact for each n, $\cup_{n=1}^{\infty}H_n=G$, and such that for each Borel neighborhood B of the identity e_G condition (3.2) holds.

In case of $G = \mathbf{R}$, the character group X of $\mathbf{R}$ can be identified with $\mathbf{R}$. For $n = 1,2,\ldots,$ put $H_n := (-n,n)$. For each Borel neighborhood U of the identity $e_{\mathbf{R}} = 0$, there is $\delta > 0$ such that $(-\delta,\delta) \subset U$. Hence it suffices to show that condition (3.2) holds for $B_\delta := (-\delta,\delta)$ for all $\delta > 0$. The characters have the form $\chi(x) = \mathrm{e}^{\mathrm{i}\chi x}$ for $x \in \mathbf{R}$ and $\chi \in X \cong \mathbf{R}$, where $\mathrm{i} := \sqrt{-1}$. Hence

$$\frac{1}{\theta(H_n)} \int_{H_n} \chi(x)\,\mathrm{d}\theta(\chi) = \frac{1}{2n}\int_{-n}^{n} \mathrm{e}^{\mathrm{i}\chi x}\mathrm{d}\chi = \frac{\sin(nx)}{nx}.$$

Consequently, for all $\delta > 0$,

$$\sup_{x\in\mathbf{R}\setminus B_\delta} \frac{1}{\theta(H_n)}\left|\int_{H_n} \chi(x)\,\mathrm{d}\theta(\chi)\right| = \sup_{|x|\geqslant\delta}\left|\frac{\sin(nx)}{nx}\right| = \sup_{|y|\geqslant n\delta}\left|\frac{\sin y}{y}\right| \to 0$$

as $n \to \infty$.

Consider the case $G = D$. Then the character group X of D is compact. For $n = 1,2,\ldots,$ put $H_n := X$. For each Borel neighborhood U of the identity e_D we have $\{e_D\} \subset U$, hence it suffices to show that condition (3.2) holds for $B = \{e_D\}$. For each $x \in D \setminus \{e_D\}$ we have

$$\frac{1}{\theta(H_n)} \int_{H_n} \chi(x)\,\mathrm{d}\theta(\chi) = \frac{1}{\theta(X)}\int_X \chi(x)\,\mathrm{d}\theta(\chi) = 0,$$

hence condition (3.2) holds trivially.

In case of $G = K$, the character group X of K is discrete and countable, hence σ–compact. By Theorem 18.14 of Hewitt and Ross [4], there is an increasing sequence $\{H_n\}_{n=1}^{\infty}$ of open symmetric neighborhoods of the identity e_X in X such that $\theta(H_n) > 0$, the closure H_n^- is compact for each n, $\cup_{n=1}^{\infty} H_n = X$ and

$$\lim_{n\to\infty} \frac{1}{\theta(H_n)}\int_{H_n} f(\chi)\,\mathrm{d}\theta(\chi) = M(f) \tag{4.1}$$

for all continuous almost periodic function $f : X \to \mathbf{C}$, where $M(f)$ denotes the invariant mean of f. (The possibility of the symmetry of H_n is not stated in the cited theorem, but it can be assured easily.) Consider the functions $g_n : K \to \mathbf{C}$ defined by

$$g_n(x) := \frac{1}{\theta(H_n)}\int_{H_n} \chi(x)\,\mathrm{d}\theta(\chi), \qquad n = 1,2,\ldots, \quad x \in G.$$

For each Borel neighborhood U of the identity e_K, there is an open neighborhood B of the identity e_K such that $B \subset U$, hence it suffices to show that condition (3.2) holds for each open neighborhood B of the identity e_K. Then the set $K \setminus B$ is closed hence compact, thus in order to check condition (3.2), it suffices to show that

(a) $g_n(x) \to 0$ as $n \to \infty$ for each $x \in K \setminus B$;

(b) the functions $\{g_n\}_{n=1}^{\infty}$ are uniformly bounded on $K \setminus B$;

(c) the functions $\{g_n\}_{n=1}^{\infty}$ are uniformly equicontinuous on $K \setminus B$.

For (a), let $x \in K \setminus B$. Consider the function $f^{(x)} : X \to \mathbf{C}$ defined by $f^{(x)}(\chi) := \chi(x)$. Then $f^{(x)}$ is in fact a character of the group X, hence continuous and almost periodic. By (4.1), we obtain $\lim_{n\to\infty} g_n(x) = M(f^{(x)})$. By the invariance of the mean M, we have $M(f^{(x)}) = M(f^{(x)}_{\chi_1})$ for each $\chi_1 \in X$, where

$$f^{(x)}_{\chi_1}(\chi) = f^{(x)}(\chi\chi_1) = (\chi\chi_1)(x) = \chi(x)\chi_1(x),$$

hence

$$M(f^{(x)}) = \chi_1(x)M(f^{(x)}) \qquad \text{for each } \chi_1 \in X.$$

Since $x \in K \setminus B$ implies $x \neq e_K$, there is a character $\chi_1 \in X$ such that $\chi_1(x) \neq 1$. Consequently, $M(f^{(x)}) = 0$, thus we have proved (a).

For each n and $x \in K$ we have $|g_n(x)| \leqslant 1$, hence (b) follows trivially.

In order to check (c), we have to show that for each $\delta > 0$ there is an open neighborhood V of e_K such that for each n and for each $x, y \in V$, $|g_n(x)-g_n(y)| \leqslant \delta$ holds. We have

$$\begin{aligned} |g_n(x) - g_n(y)| &= \frac{1}{\theta(H_n)} \left| \int_{H_n} (\chi(x) - \chi(y))\, \mathrm{d}\theta(\chi) \right| \\ &\leqslant \frac{1}{\theta(H_n)} \int_{H_n} |\chi(x) - \chi(y)|\, \mathrm{d}\theta(\chi). \end{aligned}$$

Consequently $|g_n(x) - g_n(y)| \leqslant \delta$ holds if $|\chi(x) - \chi(y)| \leqslant \delta$. We have $|\chi(x) - \chi(y)| = |\chi(xy^{-1}) - 1| \leqslant \delta$. Indeed,

$$|\chi(x)-\chi(y)| = |\chi(xy^{-1}y)-\chi(y)| = |\chi(xy^{-1})\chi(y)-\chi(y)| = |(\chi(xy^{-1})-1)\chi(y)| = |\chi(xy^{-1})-1|$$

The uniform continuity of χ assures that there is an open neighborhood V of e_K such that for each $x, y \in V$, $|\chi(x) - \chi(y)| = |\chi(xy^{-1}) - 1| \leqslant \delta$ holds. $\square$

5. Comments on Theorem 2.1

Analysing the proof of Proposition 3.1, one obtains the following statement.

PROPOSITION 5.1. *Let G be an arbitrary locally compact Abelian group. Then for each Borel subset B of G and for each Borel subset H of X with $0 < \theta(H) < \infty$ we have*

$$\mu\,(G \setminus B) \leqslant c_1(B,H) \int_H \left(1 - \operatorname{Re} \widehat{\mu}(\chi)\right) \mathrm{d}\theta(\chi),$$

where

$$c_1(B,H) := \frac{1}{\inf\limits_{x\in G\setminus B} \int_H \left(1 - \operatorname{Re} \chi(x)\right) \mathrm{d}\theta(\chi)} \in \left[\frac{1}{2\theta(H)}, \infty\right].$$

(Here $\frac{1}{0} := \infty$.)

A lower estimate (motivated by Fristedt and Gray [3, 14.6, Lemma 11]) is the following.

PROPOSITION 5.2. *Let G be an arbitrary locally compact Abelian group. Then for each Borel subset B of G and for each Borel subset H of X with $0 < \theta(H) < \infty$ we have*

$$\mu(G \setminus B) \geqslant \frac{1}{2\theta(H)} \int_H \big(1 - \operatorname{Re}\widehat{\mu}(\chi)\big)\,\mathrm{d}\theta(\chi) - \frac{c_2(B,H)}{2\theta(H)},$$

where

$$c_2(B,H) := \int_H \sup_{x\in B} \big(1 - \operatorname{Re}\chi(x)\big)\,\mathrm{d}\theta(\chi) \in [0, 2\theta(H)].$$

Proof. Clearly

$$\begin{aligned} 1 - \operatorname{Re}\widehat{\mu}(\chi) &= \int_G \big(1 - \operatorname{Re}\chi(x)\big)\,\mathrm{d}\mu(x) \\ &= \int_{G\setminus B} \big(1 - \operatorname{Re}\chi(x)\big)\,\mathrm{d}\mu(x) + \int_B \big(1 - \operatorname{Re}\chi(x)\big)\,\mathrm{d}\mu(x) \\ &\leqslant 2\mu(G\setminus B) + \sup_{x\in B}\big(1 - \operatorname{Re}\chi(x)\big) \end{aligned}$$

implies the statement. □

6. A general truncation inequality

Another type of truncation inequality for $\mathbf{R}^d$ can be found in Vakhania, Tarieladze and Chobanyan [10, IV.2.2, Proposition 2.5],

$$\mu\left(\left\{x \in \mathbf{R}^d : \|x\| > \frac{1}{a}\right\}\right) \leqslant 3 \int_{\mathbf{R}^d} \big(1 - \widehat{\mu}(at)\big)\,\mathrm{d}\gamma(t), \tag{6.1}$$

where γ denotes the standard Gauss measure on $\mathbf{R}^d$, and $\|x\|$ denotes the Euclidean norm of $x \in \mathbf{R}^d$. Inequality (6.1) can also be written in the form

$$\mu\left(\left\{x \in \mathbf{R}^d : \|x\| > \frac{1}{a}\right\}\right) \leqslant \frac{3}{\big(\sqrt{2\pi}a\big)^d} \int_{\mathbf{R}^d} \big(1 - \widehat{\mu}(t)\big)\mathrm{e}^{-a^2\|t\|^2/2}\,\mathrm{d}t. \tag{6.2}$$

Fristedt and Gray [3, 14.6, Lemma 11] provide a lower estimates as well,

$$\mu\left(\left\{x \in \mathbf{R} : |x| > \frac{1}{a}\right\}\right) \geqslant \frac{1}{2a^2} \int_0^{a^2} \big(1 - \operatorname{Re}\widehat{\mu}(t)\big)\,\mathrm{d}t - \frac{a^2}{4}. \tag{6.3}$$

Vakhania, Tarieladze and Chobanyan [10, IV.2.2, Proposition 2.5] proved the following inequality for normed spaces, but the proof works for locally compact Abelian groups as well.

PROPOSITION 6.4. *Let G be a locally compact Abelian group. Let μ be a probability measure on G and let ν be a symmetric probability measure on X. Then for each $\delta \in (0,1)$ we have*

$$\mu\left(\{x \in G : \widehat{\nu}(x) \leqslant \delta\}\right) \leqslant \frac{1}{1-\delta} \int_X \big(1 - \widehat{\mu}(\chi)\big)\,\mathrm{d}\nu(\chi).$$

Proof. The Plancherel theorem implies the following equality

$$\int_X \widehat{\mu}(\chi)\,\mathrm{d}\nu(\chi) = \int_G \widehat{\nu}(x)\,\mathrm{d}\mu(x).$$

The second integral is real–valued since the Fourier transform $\widehat{\nu}$ is real–valued in view of the symmetry of ν. Now, for each $\delta \in (0,1)$,

$$\begin{aligned} &\mu\left(\{x \in G : \widehat{\nu}(x) \leqslant \delta\}\right) = \mu\left(\{x \in G : 1 - \widehat{\nu}(x) \geqslant 1 - \delta\}\right) \\ &\leqslant \frac{1}{1-\delta}\int_G \big(1 - \widehat{\nu}(x)\big)\,\mathrm{d}\mu(x) = \frac{1}{1-\delta}\int_X \big(1 - \widehat{\mu}(\chi)\big)\,\mathrm{d}\nu(\chi). \end{aligned}$$

□

COROLLARY 6.5. *Let G be a locally compact Abelian group. Then for each probability measure μ on G, for each symmetric Borel subset H of X with $0 < \theta(H) < \infty$ and for each $\delta \in (0,1)$ we have*

$$\mu\left(\left\{x \in G : \frac{1}{\theta(H)}\int_H \chi(x)\,\mathrm{d}\theta(\chi) \leqslant \delta\right\}\right) \leqslant \frac{1}{(1-\delta)\theta(H)}\int_H \big(1 - \widehat{\mu}(\chi)\big)\,\mathrm{d}\theta(\chi). \tag{6.6}$$

Moreover, the set

$$\left\{x \in G : \frac{1}{\theta(H)}\int_H \chi(x)\,\mathrm{d}\theta(\chi) \geqslant \delta\right\}$$

is compact.

Proof. Let $\nu := \frac{1}{\theta(H)}\theta\big|_H$. Then ν is a symmetric probability measure on X with

$$\widehat{\nu}(x) = \frac{1}{\theta(H)}\int_H \chi(x)\,\mathrm{d}\theta(\chi), \qquad x \in G$$

and

$$\int_X \big(1 - \widehat{\mu}(\chi)\big)\,\mathrm{d}\nu(\chi) = \frac{1}{\theta(H)}\int_H \big(1 - \widehat{\mu}(\chi)\big)\,\mathrm{d}\nu(\chi).$$

Clearly the measure ν is absolutely continuous with respect to the Haar measure θ on X and admits a θ-integrable Radon-Nikodym derivative

$$\frac{\mathrm{d}\nu}{\mathrm{d}\theta} = \frac{1}{\theta(H)}\mathbb{1}_H.$$

By the Riemann-Lebesgue Lemma (see, for example, Hewitt and Ross [4, C.26]) the function $\widehat{\nu}$ vanishes at infinity, that is, for all $\varepsilon > 0$ there exists a compact subset $F \subset G$ such that $|\widehat{\nu}(x)| < \varepsilon$ for all $x \notin F$. This implies that the set

$$\left\{x \in G : \frac{1}{\theta(H)}\int_H \chi(x)\,\mathrm{d}\theta(\chi) \geqslant \delta\right\} = \{x \in G : \widehat{\nu}(x) \geqslant \delta\}$$

is compact. □

7. Applications

Inequality (6.6) can be used for deriving the tightness of a sequence of probability measures.

PROPOSITION 7.1. *Let* $(\mu_n)_{n\in\mathbf{N}}$ *be a sequence of probability measures on a locally compact Abelian group* G *such that for each* $\chi \in X$, *the limit* $\lim_{n\to\infty} \widehat{\mu}_n(\chi) =: \varphi(\chi)$ *exists and the limiting function* φ *is continuous at* e_X. *Then the sequence* $(\mu_n)_{n\in\mathbf{N}}$ *is uniformly tight, i.e., for each* $\varepsilon > 0$ *there exists a compact subset* $K \subset G$ *such that* $\mu_n(G \setminus K) < \varepsilon$ *for all* $n \in \mathbf{N}$.

Proof. By Corollary 6.5, for each $n \in \mathbf{N}$ and for each symmetric Borel subset H of X with $0 < \theta(H) < \infty$ we have

$$\mu_n\left(\left\{x \in G : \frac{1}{\theta(H)} \int_H \chi(x)\, \mathrm{d}\theta(\chi) \leqslant \frac{1}{2}\right\}\right) \leqslant \frac{2}{\theta(H)} \int_H \big(1 - \widehat{\mu}_n(\chi)\big)\, \mathrm{d}\theta(\chi).$$

By the dominated convergence theorem

$$\int_H \big(1 - \widehat{\mu}_n(\chi)\big)\, \mathrm{d}\theta(\chi) \to \int_H \big(1 - \varphi(\chi)\big)\, \mathrm{d}\theta(\chi) \qquad \text{as } n \to \infty. \tag{7.2}$$

Since φ is continuous at e_X and $\varphi(e_X) = 1$, for each $\varepsilon > 0$ there exists a neighborhood H_0 of e_X such that

$$\frac{2}{\theta(H_0)} \int_{H_0} \big(1 - \widehat{\mu}_n(\chi)\big)\, \mathrm{d}\theta(\chi) < \frac{\varepsilon}{2}.$$

By (7.2) there exists $n_0 \in \mathbf{N}$ such that

$$\frac{2}{\theta(H_0)} \int_{H_0} \big(1 - \widehat{\mu}_n(\chi)\big)\, \mathrm{d}\theta(\chi) < \frac{2}{\theta(H_0)} \int_{H_0} \big(1 - \varphi(\chi)\big)\, \mathrm{d}\theta(\chi) + \frac{\varepsilon}{2} < \varepsilon$$

for all $n \geqslant n_0$. For each $n \in \mathbf{N}$ with $n < n_0$ the continuity of $\widehat{\mu}_n$ implies the existence of a symmetric Borel subset H_n of X with $0 < \theta(H_n) < \infty$ such that

$$\frac{2}{\theta(H_n)} \int_{H_n} \big(1 - \widehat{\mu}_n(\chi)\big)\, \mathrm{d}\theta(\chi) < \varepsilon.$$

Consequently,

$$\mu_n(G \setminus K) < \varepsilon$$

for all $n \in \mathbf{N}$ with the set

$$K := \bigcup_{k=0}^{n_0} \left\{x \in G : \frac{1}{\theta(H)} \int_H \chi(x)\, \mathrm{d}\theta(\chi) \leqslant \frac{1}{2}\right\}.$$

As in the proof of Corollary 6.5, the set K is compact, hence the assertion. □

Proposition 7.1 together with Prokhorov's theorem implies the crucial part of Lévy's continuity theorem for probability measures on locally compact Abelian groups which with a different proof is contained in Parthasarathy [7].

THEOREM 7.3. *Let $(\mu_n)_{n\in\mathbf{N}}$ be a sequence of probability measures on a locally compact Abelian group G such that for each $\chi \in X$, the limit $\lim_{n\to\infty} \widehat{\mu}_n(\chi) =: \varphi(\chi)$ exists and the limiting function φ is continuous at e_X. Then there exists a probability measure μ on G such that $\varphi = \widehat{\mu}$ and $\mu_n \to \mu$ weakly as $n \to \infty$.*

As a generalization of Theorem 14.6 of Fristedt and Gray [3], we derive another useful result.

THEOREM 7.4. *Let $\mathscr{Q}$ be a family of probability measures on a locally compact Abelian group G. Let $\widehat{\mathscr{Q}}$ be the set of the Fourier transforms of the measures in $\mathscr{Q}$. Then the following statements are equivalent.*

(i) *$\mathscr{Q}$ is relatively compact, i.e., every net $(\mu_\alpha)_{\alpha\in A}$ of members of $\mathscr{Q}$ has a convergent subnet;*

(ii) *$\mathscr{Q}$ is uniformly tight, i.e., for each $\varepsilon > 0$ there exists a compact subset $K \subset G$ such that $\mu(G \setminus K) < \varepsilon$ for all $\mu \in \mathscr{Q}$;*

(iii) *$\widehat{\mathscr{Q}}$ is equicontinuous at e_X, i.e., for each $\varepsilon > 0$ there exists a neighborhood H of e_X such that $|1 - \widehat{\mu}(\chi)| < \varepsilon$ for all $\chi \in H$ and $\mu \in \mathscr{Q}$;*

(iv) *for each $\varepsilon > 0$ there exists a symmetric subset H of X with $0 < \theta(H) < \infty$ such that $\frac{1}{\theta(H)} \int_H (1 - \widehat{\mu}(\chi))\, \mathrm{d}\theta(\chi) < \varepsilon$ for all $\mu \in \mathscr{Q}$.*

Proof. (i) $\iff$ (ii) follows from Prokhorov's theorem.

(iii) $\implies$ (iv) is trivial.

(iv) $\implies$ (ii) follows from Corollary 6.5.

(ii) $\implies$ (iii). By (ii), for each $\varepsilon > 0$ there exists a compact subset $K \subset G$ such that $\mu(G \setminus K) < \varepsilon/4$ for all $\mu \in \mathscr{Q}$. Consider the set

$$H := \{\chi \in X : |\chi(x) - 1| < \varepsilon \text{ for all } x \in K\}.$$

Then H is a neighborhood of e_X (see, for example, Hewitt and Ross [4, 23.15]), and we have

$$|1 - \widehat{\mu}(\chi)| \leqslant \int_G \big|1 - \chi(x)\big|\, \mathrm{d}\mu(x) \leqslant 2\mu(G \setminus K) + \int_K \big|1 - \chi(x)\big|\, \mathrm{d}\mu(x) < \varepsilon,$$

hence the assertion. □

REFERENCES

[1] R. M. DUDLEY, *Real Analysis and Probability.* The Wadsworth & Brooks/Cole Mathematics Series, Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1989.

[2] R. DURRETT, *Probability: Theory and Examples,* Second edition, Duxbury Press, Belmont, CA, 1996.

[3] B. FRISTEDT and L. GRAY, *A Modern Approach to Probability Theory,* Probability and its Applications, Birkhäuser Boston, Inc., Boston, MA, 1997.

[4] E. HEWITT and K. A. ROSS, *Abstract Harmonic Analysis,* Springer–Verlag, Berlin, Göttingen, Heidelberg, 1963.

[5] J. JACOD and PH. PROTTER, *Probability Essentials,* Universitext, Springer-Verlag, Berlin, 2000.

[6] M. LOÈVE, *Probability Theory. Foundations. Random Sequences,* D. Van Nostrand Company, Inc., Toronto-New York-London, 1955.

[7] K. R. PARTHASARATHY, *Probability Measures on Metric Spaces,* Probability and Mathematical Statistics, No. 3. Academic Press, Inc., New York, London, 1967.

[8] K. R. PARTHASARATHY, *Introduction to Probability and Measure,* The Macmillan Co. of India, Ltd., Delhi, 1977.

[9] A. N. SHIRYAYEV, *Probability,* Graduate Texts in Mathematics, 95. Springer-Verlag, New York, 1984.
[10] N. N. VAKHANIA, V. I. TARIELADZE and S. A. CHOBANYAN, *Probability distributions on Banach spaces,* Mathematics and its Applications (Soviet Series), 14. D. Reidel Publishing Co., Dordrecht, 1987.

(Received March 18, 2007)

Herbert Heyer
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
D–72076 Tübingen
Germany
e-mail: herbert.heyer@uni-tuebingen.de

Gyula Pap
Faculty of Informatics
University of Debrecen
Pf.12, H–4010 Debrecen
Hungary
e-mail: papgy@inf.unideb.hu