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A CROSS–RATIO INEQUALITY FOR QUASICIRCLES

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Abstract. In this note we give a geometric characteristic for quasicircles by making use of cross-ratios, which is analogous to quadruple points identity of circles.

1. Introduction

We denote by $\overline{R}^2 = R^2 \bigcup \{\infty\}$ the one point compactification of R^2. The cross-ratio (a,b,c,d) is defined by

$$(a,b,c,d) = \frac{(a-c)(b-d)}{(a-d)(b-c)} \tag{1.1}$$

for distinct points $a,b,c,d \in \overline{R}^2$, with the understanding that $|\infty - x|/|\infty - y| = 1$ for all $x,y \in R^2$.

It is well-known that cross-ratios are invariant under Möbius mappings. Cross-ratios play a very crucial role in many mathematics fields, such as geometric function theory [1], discrete geometry [2], ring geometry [3] and Teichmüller space theory [4].

We denote by B^2 and ∂B^2 the unit disk and its boundary. In this paper, let circles be the images of ∂B^2 under Möbius transformations. Hence a line is a circle with radius infinity or through infinity. A quasicircle is the image of a circle under a quasiconformal self map $\overline{R}^2 \to \overline{R}^2$, that is a quasiconformal map of the extended plane onto itself. Quasicircles, with many properties analogous to those of circles, are very important in quasiconformal mapping theory, complex dynamics, fuchsian groups, Teichmüller space theory and low dimensional topology, see [5–9], etc. In 1963, L. V. Ahlfors obtained the three-point property of quasicircles as follows:

THEOREM A. (see [10]) *A Jordan curve $\Gamma \subset \overline{R}^2$ is a quasicircle if and only if there exists a constant $c > 1$, for any two distinct points $z_1, z_2 \in \Gamma \cap R^2$ such that*

$$\min_{i=1,2} \operatorname{dia}(\gamma_i) \leqslant c|z_1 - z_2|,$$

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where γ_1 *and* γ_2 *are the two components of* $\Gamma \setminus \{z_1, z_2\}$, *and* $\mathrm{dia}\,(\gamma_i)$ *is the Euclidean diameter of* γ_i.

Later, F. W. Gehring [11], B. G. Osgood [12], J. G. Krzyż [13], Y. M. Chu and J. F. Cheng [14] and many other researchers studied quasicircles deeply, and obtained some different geometrical and analytical conditions.

In this short note, we shall prove a cross-ratio characteristic for quasicircles. That is the following theorem:

THEOREM. *A Jordan curve* $\Gamma \subset \overline{R}^2$ *is a quasicircle if and only if there exists a constant* $c \geqslant 1$, *for any four ordered distinct points* $z_1, z_2, z_3, z_4 \in \Gamma$ *such that*

$$|(z_1, z_2, z_4, z_3)| + |(z_1, z_4, z_2, z_3)| \leqslant c. \tag{1.2}$$

2. Proof of the theorem

We shall use the relatively standard notation and terminology of [15]. For $0 < r < 1$, $R_G(r) = B^2 \setminus [0, r]$ is the Grötzsch ring. Denote by $\mu(r)$ the modulus of the Grötzsch ring and let $\mu(0) = \infty$ and $\mu(1) = 0$. It is well-known that $\mu(r)$ is a continuous decreasing function and can be represented as

$$\mu(r) = \frac{\pi}{2}\frac{K(\sqrt{1-r^2})}{K(r)}, \tag{2.1}$$

where $K(r) = \int_0^1 [(1-x^2)(1-r^2x^2)]^{-\frac{1}{2}}\, dx$ is the first kind of elliptic integral.

For $0 < r < 1$ and $K > 0$, we define

$$\varphi_K(r) = \mu^{-1}\left(\frac{1}{K}\mu(r)\right) \tag{2.2}$$

with $\varphi_K(0) = 0$ and $\varphi_K(1) = 1$. Then $\varphi_K(r)$ is increasing on $[0, 1]$.

The following two lemmas are useful in the proof of the theorem.

LEMMA 1. *For* $0 < r < 1$,

$$\varphi_K(r)^2 + \varphi_{1/K}(\sqrt{1-r^2})^2 = 1.$$

Proof. For $0 < r < 1$, by (2.1) we have

$$\mu(r)\mu(\sqrt{1-r^2}) = \frac{\pi^2}{4}, \quad \mu(\varphi_K(r))\mu(\sqrt{1-\varphi_K^2(r)}) = \frac{\pi^2}{4}. \tag{2.3}$$

By the definition of $\varphi_K(r)$ in (2.2), we have

$$\mu(\varphi_K(r)) = \frac{1}{K}\mu(r) \tag{2.4}$$

and

$$\mu(\varphi_{1/K}(\sqrt{1-r^2})) = K\mu(\sqrt{1-r^2}). \tag{2.5}$$

Combining (2.3), (2.4) and (2.5), we obtain

$$\mu(\varphi_K(r))\mu(\varphi_{1/K}(\sqrt{1-r^2})) = \frac{\pi^2}{4}. \tag{2.6}$$

Hence the identity

$$\varphi_K(r)^2 + \varphi_{1/K}(\sqrt{1-r^2})^2 = 1$$

follows from (2.3), (2.6) and the monotonicity of $\mu(r)$. The proof is completed. □

LEMMA 2. (see [16]) *Let $f : \overline{R}^2 \to \overline{R}^2$ be a $K-$quasiconformal mapping with $f(\infty) = \infty$, then for any three distinct points $z_1, z_2, z_3 \in R^2$,*

$$\sin\frac{\alpha}{2} \geqslant \varphi_{1/K}\left(\sin\frac{\beta}{2}\right), \tag{2.7}$$

where

$$\alpha = \arcsin\left(\frac{|z_1 - z_3|}{|z_1 - z_2| + |z_2 - z_3|}\right)$$

and

$$\beta = \arcsin\left(\frac{|f(z_1) - f(z_3)|}{|f(z_1) - f(z_2)| + |f(z_2) - f(z_3)|}\right).$$

Now we are about to prove the theorem.

Proof of the theorem. For the necessity, we may assume that $z_4 = \infty$, since the cross-ratio is invariant under Möbius transformation. Since Γ is a quasicircle, there must be a $K-$quasiconformal mapping $f : \overline{R}^2 \to \overline{R}^2$ such that $f(\Gamma)$ is a line in $\overline{R}^2$ with $f(\infty) = \infty$. Let

$$\alpha = \arcsin\left(\frac{|z_1 - z_3|}{|z_1 - z_2| + |z_2 - z_3|}\right)$$

and

$$\beta = \arcsin\left(\frac{|f(z_1) - f(z_3)|}{|f(z_1) - f(z_2)| + |f(z_2) - f(z_3)|}\right) = \frac{\pi}{2}.$$

Then, by lemma 2, we obtain

$$\sin\frac{\alpha}{2} \geqslant \varphi_{1/K}(\sin\frac{\beta}{2}) = \varphi_{1/K}\left(\frac{1}{\sqrt{2}}\right).$$

Combined with lemma 1, it yields that

$$\sin\alpha = 2\sin\frac{\alpha}{2}\sqrt{1-\sin^2\frac{\alpha}{2}} \geqslant 2\varphi_{1/K}\left(\frac{1}{\sqrt{2}}\right)\varphi_K\left(\frac{1}{\sqrt{2}}\right),$$

That is

$$\frac{|z_1 - z_3|}{|z_1 - z_2| + |z_2 - z_3|} \geqslant 2\varphi_{1/K}\left(\frac{1}{\sqrt{2}}\right)\varphi_K\left(\frac{1}{\sqrt{2}}\right).$$

Hence we have

$$|(z_1,z_4,z_2,z_3)|+|(z_1,z_2,z_4,z_3)|=\frac{|z_1-z_2|+|z_2-z_3|}{|z_1-z_3|}\leqslant\frac{1}{2\varphi_{1/K}\left(\frac{1}{\sqrt{2}}\right)\varphi_K\left(\frac{1}{\sqrt{2}}\right)}.$$

For the sufficiency, we could let $z_4=\infty$. Then for any ordered three distinct points $z_1,z_2,z_3\in R^2$,

$$|(z_1,z_4,z_2,z_3)|+|(z_1,z_2,z_4,z_3)|=\frac{|z_1-z_2|+|z_2-z_3|}{|z_1-z_3|}\leqslant c. \tag{2.8}$$

For any two distinct points $x,y\in\Gamma\cap R^2$, let γ be the bounded component of $\Gamma\setminus\{x,y\}$. Let $z,w\in\gamma$ with $\operatorname{dia}(\gamma)=|z-w|$. Then

$$\begin{aligned}\operatorname{dia}(\gamma)=|z-w|&\leqslant|z-y|+|y-x|+|x-w|\\&\leqslant(|z-y|+|z-x|)+|y-x|+(|w-x|+|w-y|)\\&\leqslant c|x-y|+|y-x|+c|x-y|\\&\leqslant(2c+1)|x-y|.\end{aligned}\tag{2.9}$$

Here the second inequality follows from (2.8). Hence it follows that Γ is a quasicircle from theorem A.

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