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I-CONVERGENT SEQUENCE SPACES ASSOCIATED WITH MULTIPLIER SEQUENCES

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Abstract. In this article we introduce the sequence spaces $c^I(\Lambda)$, $c_0^I(\Lambda)$, $m^I(\Lambda)$ and $m_0^I(\Lambda)$ associated with the multiplier sequence $\Lambda = (\lambda_k)$ of non-zero scalars. We study the different algebraic and topological properties of these sequence spaces like solidness, symmetricity, sequence algebra, convergence free etc. Also we characterize the multiplier problem and obtain some inclusion relation involving these sequence spaces.

1. Introduction

The works on I-convergence was studied at the initial stage by Kostyrko, Macaj and Šalát [5]. Later on it was further investigated by Kostyrko, Šalát and Wilczynski [4], Šalát, Tripathy and Ziman ([6], [7]), Demirci [1] and others. The idea depends on the notion of ideals of subsets of N.

The scope for the studies on sequence spaces was extended by using the notion of associated multiplier sequences. Goes and Goes [2] defined the differentiated sequence space dE and integrated sequence space $\int E$ for a given sequence space E,by using multiplier sequences (k^{-1}) and (k) respectively. Kamthan [3] used the multiplier sequence $(k!)$. Tripathy and Sen [11], Tripathy [9] and Tripathy and Mahanta [10] used a general multiplier sequence $\Lambda = (\lambda_k)$ of non-zero scalars for their studies on sequence spaces associated with multiplier sequences. In this paper we shall consider a general multiplier sequence $\Lambda = (\lambda_k)$ of non-zero scalars.

Let $\Lambda = (\lambda_k)$ be a sequence of non-zero scalars. Then for a sequence space E, the multiplier sequence space $E(\Lambda)$, associated with the multiplier sequence Λ is defined as

$$E(\Lambda) = \{(x_k) \in w : (\lambda_k x_k) \in E\}.$$

Throughout the paper w, ℓ_∞, c, c_0 denote the classes of *all*, *bounded*, *convergent*, *null* sequence spaces respectively.

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2. Definitions and Preliminaries

Let X be a non-empty set. Then a family of sets $I \subset 2^X$, (power sets of X) is said to be an *ideal*, if A, $B \in I$ then $A \cup B \in I$ and for each $A \in I$ with $B \subset A$ implies $B \in I$.

A non-empty family of sets $\mathscr{F} \subset 2^X$ is said to be a *filter* on X if and only if $\phi \notin \mathscr{F}$, for each $A, B \in \mathscr{F}$ we have $A \cup B \in \mathscr{F}$ and for each $A \in \mathscr{F}$ and $B \supset A$, implies $B \in \mathscr{F}$.

For each ideal I, there is a filter $\mathscr{F}(I)$ corresponding to I *i.e.* $\mathscr{F}(I) = \{K \subseteq N : K^c \in N\}$, with $K^c = N - K$.

The usual convergence is a particular case of I-convergence. In this case $I = I_f$ (the ideal of all finite subsets of N).

Let $A \subset N$, then A is said to have *asymptotic density* $\delta(A)$, if $\delta(A) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{\infty}\chi_A(k)$ exists.

A sequence (x_n) is said to be *statistically convergent* to L, if for every $\varepsilon > 0$, $\delta(\{k \in N : |x_k - L| \geqslant \varepsilon\}) = 0$.

The statistical convergence is a particular case of I-convergence. In this case $I = I_\delta$ (the ideal of all subsets of N of zero *asymptotic density*).

Let $A \subset N$ and $d_n = \frac{1}{s_n}\sum_{k=1}^{n}\frac{\chi_A(k)}{k}$, for $n \in N$, where $s_n = \sum_{k=1}^{n}\frac{1}{k}$. If $\lim_{n\to\infty} d_n(A)$ exists, then it is called as the *logarithmic density* of A. $I_d = \{A \subset N : d(A) = 0\}$ is an ideal.

Let $T = (t_{nk})$ be a *regular non-negative matrix* (one may refer to Hardy [5]).

For $A \subset N$, define $d_T^{(n)}(A) = \sum_{k=1}^{\infty} t_{n,k}\chi_A(k)$ for $n \in N$. If $\lim_{n\to\infty} d_T^{(n)}(A) = d_T(A)$ exists, then $d_T(A)$ is called as T-*density* of A. Clearly $I_{d_T} = \{A \subset N : d_T(A) = 0\}$ is an ideal. Further I_δ and I_d are particular cases of I_{d_T}.

Let $I_c = \{A \subset N : \sum_{a\in A}\frac{1}{a} < \infty\}$. Then I_c is an ideal of N.

For other examples of I-convergence, one may refer to Kostyrko, Šalát and Wilczynski [4]. The notion of I-*monotonic* sequence was studied by Šalát, Tripathy and Ziman [6].

A sequence $x = (x_k)$ of complex terms is said to be I-*convergent* to L if for every $\varepsilon > 0, \{k \in N : |x_k - L| \geqslant \varepsilon\} \in I$.

A sequence $x = (x_k)$ of complex terms is said to be I-*Cauchy* if for every $\varepsilon > 0$, there exists a number $m = m(\varepsilon)$ such that $\{k \in N : |x_k - x_m| \geqslant \varepsilon\} \in I$.

A sequence space E is said to be *solid* (or *normal*) if $(x_k) \in E$, and $(\alpha_k) \in E$, a sequence of scalars with $|\alpha_k| \leqslant 1$, for all $k \in N$, then $(\alpha_k x_k) \in E$.

A sequence space E is said to be *symmetric* if $(x_{\pi(k)}) \in E$, whenever $(x_k) \in E$, where π is a permutation of N.

A sequence space E is said to be a *sequence algebra* if $(x_k) * (y_k) = (x_k y_k) \in E$, whenever $(x_k), (y_k) \in E$.

A sequence space E is said to be *convergence free*, if $(x_k) \in E$, and if $y_k = 0$, whenever $x_k = 0$, then $(y_k) \in E$.

A sequence $x = (x_k)$ is said to be *I-bounded* if there exists an $M > 0$, such that $\{k \in N : |x_k| > M\} \in I$.

Throughout ℓ_∞^I, c^I, c_0^I denote the classes of all *I-bounded*, *I-convergent* and *I-null* sequences respectively.

We write $m^I = c^I \cap \ell_\infty$ and $m_0^I = c_0^I \cap \ell_\infty$.

The spaces $\ell_\infty^I(\Lambda)$, $c^I(\Lambda)$, $c_0^I(\Lambda)$ are defined as follows:

$\ell_\infty^I(\Lambda) = \{(x_k) \in w : \sup_k |\lambda_k x_k| < \infty\}$.

$c^I(\Lambda) = \{(x_k) \in w : I - \lim(\lambda_k x_k - L) = 0, \text{ for some } L \in C\}$.

$c_0^I(\Lambda) = \{(x_k) \in w : I - \lim(\lambda_k x_k) = 0\}$.

Also we write $m^I(\Lambda) = c^I(\Lambda) \cap \ell_\infty(\Lambda)$ and $m_0^I(\Lambda) = c_0^I(\Lambda) \cap \ell_\infty(\Lambda)$.

LEMMA 1. (Tripathy and Mahanta [10], Proposition 4). *If a space E is bounded and solid then $(\lambda_k) \in M(E,E)$ if and only if $(\lambda_k) \in \ell_\infty$.*

LEMMA 2. (Šalát, Tripathy and Ziman [7], Lemma 2.5). *Let $K \in \mathscr{F}(I)$ and $M \subseteq N$. If $M \notin I$, then $M \cap K \notin I$.*

3. Main Results

The proof of the following result is easy, so omitted.

THEOREM 1. *For $\Lambda = (\lambda_k)$ a given multiplier sequence, $c^I(\Lambda)$, $c_0^I(\Lambda)$, $m^I(\Lambda)$ and $m_0^I(\Lambda)$ are linear spaces.*

THEOREM 2. *The sequence spaces $m^I(\Lambda)$ and $m_0^I(\Lambda)$ are Banach spaces with the norm*

$$||x||_\Lambda = \sup_k |\lambda_k x_k|. \tag{1}$$

Proof. Clearly $m^I(\Lambda)$ is a normed linear space with the norm (1). Now we show that $m^I(\Lambda)$ is complete with respect to the norm (1).

Let $(x^{(i)})$ be a Cauchy sequence in $m^I(\Lambda)$. Then for each $\varepsilon > 0$, there exists a number n_0 such that

$$\begin{aligned} &||x_k^{(i)} - x_k^{(j)}||_\Lambda < \varepsilon, \text{ for all } i,j \geqslant n_0 \\ &\quad \Rightarrow \sup_k |\lambda_k (x_k^{(i)} - x_k^{(j)})| < \varepsilon, \text{ for all } i,j \geqslant n_0 \\ &\quad \Rightarrow |\lambda_k (x_k^{(i)} - x_k^{(j)})| < \varepsilon, \text{ for all } i,j \geqslant n_0, \text{ for each } k \in N. \qquad (2) \\ &\quad \Rightarrow (\lambda_k x_k^{(i)}) \text{ is a Cauchy sequence in } C. \end{aligned}$$

Let $(y_k^{(i)}) = (\lambda_k x_k^{(i)})$, for each $k \in N$. Since C is complete, there exists $y_k \in C$ such that $y_k^{(i)} \to y_k$, as $i \to \infty$, for each $k \in N$. Since $m^I(\Lambda)$ is a linear space, we can write y_k as $y_k = \lambda_k x_k$, for each $k \in N$. Then $(y_k) \in m^I(\Lambda)$. Therefore $\lim_{i\to\infty} (\lambda_k x_k^{(i)}) = y_k$, for each $k \in N$.

From (2),

$$\lim_{j\to\infty} |\lambda_k(x_k^{(i)} - x_k^{(j)})| < \varepsilon, \text{ for all } i \geqslant n_0, \text{ for each } k \in N$$
$$\Rightarrow \sup_k |\lambda_k(x_k^{(i)} - x_k)| < \varepsilon, \text{ for all } i \geqslant n_0$$
$$\Rightarrow ||x^{(i)} - x|| < \varepsilon, \text{ for all } i \geqslant n_0.$$

For all $i \geqslant n_0$, $x = x^{(i)} - (x^{(i)} - x) \in m^I(\Lambda)$, because $m^I(\Lambda)$ is a linear space.

Therefore $m^I(\Lambda)$ is complete and hence it is a Banach space.

Similarly it can be shown that $m_0^I(\Lambda)$ is a Banach space.

THEOREM 3. *The sequence space* $m_0^I(\Lambda)$ *is solid.*

Proof. Let $(x_k) \in m_0^I(\Lambda)$ and (α_k) be a sequence of scalars with $|\alpha_k| \leqslant 1$, for all $k \in N$.

The result follows from the following inclusion relation

$$\{k \in N : |\lambda_k x_k| \geqslant \varepsilon\} \supseteq \{k \in N : |\alpha_k \lambda_k x_k| \geqslant \varepsilon\}.$$

PROPOSITION 4. *The sequence space* $m^I(\Lambda)$ *is not solid in general.*

Proof. The result follows from the following example.

EXAMPLE 1. Let $I = I_\delta$. Consider the sequence (x_k) defined by

$$x_k = \begin{cases} k^{-1}, & \text{if } k = i^3,\ i \in N \\ 0, & \text{otherwise} \end{cases}$$

Let $\alpha_k = (-1)^k$, for all $k \in N$ and $\lambda_k = 1$, for all $k \in N$.

Then $(\lambda_k x_k) \in c^I(\Lambda)$, but $(\alpha_k \lambda_k x_k) \notin c^I(\Lambda)$.

THEOREM 5. *Let* $I \neq I_f$, *then the sequence spaces* $c^I(\Lambda)$ *and* $c_0^I(\Lambda)$ *are not symmetric in general.*

Proof. The result follows from the following examples.

EXAMPLE 2. Let $I = I_\delta$. Consider the sequence (x_k) defined by

$$x_k = \begin{cases} k, & \text{if } k = i^2,\ i \in N \\ \frac{1}{k^3}, & \text{otherwise} \end{cases}$$

Let $\lambda_k = 1$, for all $k \in N$. Consider the rearrangement (y_k) of (x_k) defined by

$$(y_k) = (x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, \dots)$$

Then $(\lambda_k y_k)$ neither belongs to $c^I(\Lambda)$ nor to $c_0^I(\Lambda)$; where as $(\lambda_k x_k)$ belongs to both $c^I(\Lambda)$ as well as $c_0^I(\Lambda)$.

We may consider the following example too.

EXAMPLE 3. For $I = I_c$. Let $A = \{k : k = i^2 \text{ or } j^3, \text{ for } i,j \in N\}$, then $\sum_{a \in A} a^{-1} < \infty$. Consider the sequence (x_k) defined by

$$x_k = \begin{cases} k, & \text{if } k \in A, \\ 1, & \text{otherwise.} \end{cases}$$

Let $\lambda_k = 1$, for each $k \in N$. Consider the rearrangement (z_k) of (x_k) defined by

$$(z_k) = (x_1, x_2, x_8, x_4, x_5, x_{27}, x_6, x_7, x_{64}, x_9, x_{10}, \dots)$$

Then $(\lambda_k z_k)$ neither belongs to $c^I(\Lambda)$ nor to $c_0^I(\Lambda)$; but $(\lambda_k x_k)$ belongs to both $c^I(\Lambda)$ and $c_0^I(\Lambda)$.

THEOREM 6. *The sequence spaces* $c^I(\Lambda)$ *and* $c_0^I(\Lambda)$ *are not sequence algebra in general.*

Proof. The result follows from the following example.

EXAMPLE 4. Consider the sequences $x_k = k^2$, $y_k = k^2$ and $\lambda_k = k^{-4}$, for all $k \in N$, then $(x_k), (y_k) \in Z(\Lambda)$, but $(x_k * y_k) \notin Z(\Lambda)$, where $Z = c^I, c_0^I$.

THEOREM 7. *The sequence spaces* $c^I(\Lambda)$ *and* $c_0^I(\Lambda)$ *are not convergence free.*

Proof. The result follows from the following example.

EXAMPLE 5. Consider $P \in I_\delta$ and define the sequence (x_k) by

$$x_k = \begin{cases} \frac{1}{k}, & \text{if } k \notin P, \\ k, & \text{if } k \in P. \end{cases}$$

Let $Q \subseteq N$, $Q \notin I_\delta$ and define the sequence (y_k) by

$$y_k = \begin{cases} k, & \text{if } k \in Q, \\ \frac{1}{k}, & \text{if } k \notin Q. \end{cases}$$

Then $(x_k) \in c_0^I(\Lambda) \subset c^I(\Lambda)$, but (y_k) neither belongs to $c^I(\Lambda)$ nor to $c_0^I(\Lambda)$.

THEOREM 8. *If* $I \neq I_f$, *then the sequence spaces* $c^I(\Lambda)$ *and* $c_0^I(\Lambda)$ *are not separable.*

Proof. The proof of this result is easy, so omitted.

THEOREM 9. $(\lambda_k) \in M(c_0^I, c_0^I) = M(m_0^I, m_0^I)$ *if and only if* $(\lambda_k u) \in \ell_\infty^I$.

Proof. Suppose $(\lambda_k) \in \ell_\infty^I$. Then there exists a $J > 0$ such that

$$R = \{k \in N : |\lambda_k| \geqslant J\} \in I,$$
$$S = \{k \in N : |x_k| \geqslant \frac{\varepsilon}{J}\} \in I.$$

Then $R \cup S = \{k \in N : |\lambda_k x_k| \geqslant \varepsilon\} \in I$.
Then $(\lambda_k x_k) \in c_0^I (= m_0^I)$. Hence $(\lambda_k) \in M(c_0^I, c_0^I) = M(m_0^I, m_0^I)$.
The converse part is easy, so omitted.

THEOREM 10. *If the sequence space* c^I *is not solid, then* $(\lambda_k) \notin (c^I, c^I)$.

Proof. The result follows from Lemma 1.

THEOREM 11. *If* $\Omega\Lambda^{-1} = (\omega_k \lambda_k^{-1}) \in \ell_\infty^I$, *then* $Z(\Omega) \subset Z(\Lambda)$ *and the inclusion is proper, where* $Z = c^I$, c_0^I, m_0^I, m^I.

Proof. The proof is a routine verification and the inclusion is proper follows from the following example.

EXAMPLE 6. Consider the sequence $(x_k) \in Z(\Omega)$ and $\omega_k = k^{-1}$, $\lambda_k = k^{-3}$ for all $k \in N$. Then $(x_k) \in Z(\Omega)$ and $(\omega_k \lambda_k^{-1})$ is bounded, but $(x_k) \in Z(\Lambda)$.

REFERENCES

[1] K. DEMIRCI, *I-limit superior and limit inferior*, Math. Commun., **6**, (2001), 165–172.

[2] G. GOES, S. GOES, *Sequences of bounded variation and sequences of Fourier coefficients*, Math. Zeift., **118**, (1970), 93–102.

[3] P. K. KAMTHAN, *Bases in certain classes of Frechet spaces*, Tamkang Jour. Math., (1976), 41–49.

[4] P. KOSTYRKO, T. ŠALÁT, W. WILCZYNSKI, *I-convergence*, Real Analysis Exchange, **26(2)**, (2000–2001), 669–686.

[5] P. KOSTYRKO, M. MACAJ, T. ŠALÁT, *Statistical convergence and I-convergence*, Real Analysis Exchange (in Press).

[6] T. ŠALÁT, B. C. TRIPATHY, M. ZIMAN, *On some properties of I-convergence*, Tatra Mt. Math. Publ., **28**, (2004), 279–286.

[7] T. ŠALÁT, B. C. TRIPATHY, M. ZIMAN, *On I-convergence field*, Italian Jour. Pure Appl. Math., **17**, (2005), 45–54.

[8] B. C. TRIPATHY, *Matrix transformations between some classes of sequences*, J. Math. Anal. Appl., **206(2)**, (1997), 448–450.

[9] B. C. TRIPATHY, *On some classes of difference paranormed sequence spaces associated with multiplier sequences*, International Jour. Math. Sci., **2(1)**, (2003), 159–166.

[10] B. C. TRIPATHY, S. MAHANTA, *On a class of vector valued sequences associated with multiplier sequences*, Acta Math. Appl. Sinica., **20(3)**, (2004), 487–494.

[11] B. C. TRIPATHY, M. SEN, *Vector valued paranormed bounded and null sequences associated with multiplier sequences*, Soochow Jour. Math., **29**(3), (2003), 313–325.

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