Mathematical
Inequalities
& Applications
Volume 11, Number 3 (2008), 585–597

THE APPROXIMATION OF POWER FUNCTION BY THE q–BERNSTEIN POLYNOMIALS IN THE CASE $q > 1$

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Dedicated to
Professor Viktor Solomonovich Videnskii
on the occasion of his 85-th birthday

(*communicated by Z. Ditzian*)

Abstract. Since for $q > 1$, q-Bernstein polynomials are not positive linear operators on $C[0,1]$, the investigation of their convergence properties turns out to be much more difficult than that in the case $0 < q < 1$.

It is known that, in the case $q > 1$, the q-Bernstein polynomials approximate the entire functions and, in particular, polynomials uniformly on any compact set in $\mathbb{C}$. In this paper, the possibility of the approximation for the function $(z+a)^{\alpha}$, $a \geqslant 0$, with a non-integer $\alpha > -1$ is studied. It is proved that for $a > 0$, the function is uniformly approximated on any compact set in $\{z : |z| < a\}$, while on any Jordan arc in $\{z : |z| > a\}$, the uniform approximation is impossible. In the case $a = 0$, the results of the paper reveal the following interesting phenomenon: the power function z^{α}, $\alpha > 0$, is approximated by its q-Bernstein polynomials either on any (when $\alpha \in \mathbb{N}$) or no (when $\alpha \notin \mathbb{N}$) Jordan arc in $\mathbb{C}$.

1. Introduction

The importance of Bernstein polynomials gave rise to further studies of their different generalizations and related topics. Due to the intensive development of q-Calculus, some generalizations involving q-integers have emerged. In 1987, a q-analogue of the Bernstein operator was introduced by A. Lupaş [7]. Note that the operators defined by A. Lupaş are given by rational functions rather than polynomials. In 1997, another generalization of Bernstein polynomials based on the q-integers, called *q-Bernstein polynomials*, was introduced by G. M. Phillips [14]. These polynomials have recently been brought to the spotlight and studied from different angles by a number of researchers. Reviews of the results on the q-Bernstein polynomials along with extensive bibliography on this matter are given in [15], Ch. 7 (results obtained in 1997–1999) and [10] (results obtained in 2000–2004). More recent results can be found, for example, in [11], [12], [13], [17]–[22]. A generalization of the Bernstein-Durrmeyer operator related to q-Bernstein polynomials has been considered in [3].

For the sequel, we need the following definitions (cf., e.g. [15], Ch. 8, §8.1):

Mathematics subject classification (2000): 41A10, 30E10.

Key words and phrases: q-integers, q-binomial coefficients, q-Bernstein polynomials, uniform convergence.

Let $q>0$. For any $n\in\mathbb{Z}_+$, the *q-integer* $[n]_q$ is defined by

$$[n]_q := 1+q+\dots+q^{n-1} \ \ (n\in\mathbb{N}), \ \ [0]_q := 0;$$

and the *q-factorial* $[n]_q!$ by

$$[n]_q! := [1]_q[2]_q\dots[n]_q \ \ (n\in\mathbb{N}), \ \ [0]_q! := 1.$$

For integers $0\leqslant k\leqslant n$, the *q-binomial* coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Clearly, for $q=1$,

$$[n]_1 = n, \quad [n]_1! = n!, \quad \begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}.$$

DEFINITION 1.1. Let $f:[0,1]\to\mathbb{C}$. The *q-Bernstein polynomials* of f are defined by:

$$B_{n,q}(f;z) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) p_{nk}(q;z), \ \ n\in\mathbb{N},$$

where

$$p_{nk}(q;z) := \begin{bmatrix} n \\ k \end{bmatrix}_q z^k \prod_{j=0}^{n-k-1}\left(1-q^jz\right), \ \ k=0,1,\dots n. \tag{1.1}$$

Note that for $q=1$, we recover the classical Bernstein polynomials.

It has been shown by G. M. Phillips *et. al.* (see [15], Ch. 7) that some properties of the classical Bernstein polynomials are taken after by the q-Bernstein polynomials. For example, the q-Bernstein polynomials possess the end-point interpolation property, leave invariant linear functions, admit representation with the help of q-differences, and they are degree-reducing on polynomials. Apart from that, basic q-Bernstein polynomials (1.1) admit a probabilistic interpretation via q-binomial distribution, see [1] and [4].

In the case $0<q<1$, the resemblance between the classical Bernstein and q-Bernstein polynomials goes even further. In this case, q-Bernstein polynomials are positive linear operators on $C[0,1]$ with $\|B_{n,q}\|=1$. Moreover, if a function f is increasing (decreasing) on [0,1], then $B_{n,q}(f;x)\,(0<q<1)$ is also increasing (decreasing) on [0,1]; and if f is convex (concave) on $[0,1]$, then so is $B_{n,q}(f;x)\,(0<q<1)$. In addition, q-Bernstein polynomials of a convex function f in the case $0<q<1$ have the same monotonicity properties as those of the classical Bernstein polynomials, namely, $B_{n-1,q}(f;x)\geqslant B_{n,q}(f;x)\geqslant f(x)$, $x\in[0,1]$, $n=2,3,\dots$. A quantitative estimate of the difference between $B_{n,q}(f;x)$, $0<q<1$, and $B_{n,1}(f;x)$ is given by V. S. Videnskii in [18].

The approximation with q-Bernstein polynomials has been studied in quite a few papers, starting from [14], see also [5], [9], [12], [15], [17]–[22]. The study reveals that the convergence properties of the q-Bernstein polynomials are basically different from those of the classical ones. Moreover, the behavior of the q-Bernstein polynomials in terms of convergence for $0 < q < 1$ is quite different from that for $q > 1$. In the case of $0 < q < 1$, the q-Bernstein polynomials are *positive* linear operators on $C[0,1]$, while, for $q > 1$, the positivity does not persist any longer. This appears to be vital for the investigation of convergence. We notice here that despite being positive linear operators for $0 < q < 1$, q-Bernstein polynomials do not satisfy the conditions of Korovkin's Theorem, because

$$B_{n,q}(t^2;x) = x^2 + \frac{x(1-x)}{[n]_q} \to x^2 + (1-q)x(1-x) \neq x^2, \quad n \to \infty.$$

However, they satisfy the conditions of H. Wang's Korovkin-type Theorem ([19], Theorem 2) and serve as a model for this theorem. H. Wang's theorem guarantees the existence of the limit operator $B_{\infty,q}$ for the sequence $\{B_{n,q}\}$ which, unlike the situation in the classical case, is not the identity operator. Results related to the properties of $B_{\infty,q}$ may be found in the references listed above.

For the time being, contrary to a large number of papers dedicated to the investigation of convergence in the case $0 < q < 1$, there are only two papers, namely [9] and [22], treating systematically the case $q > 1$. However, the results of [9] show that, for $q > 1$, the approximation with the q-Bernstein polynomials may be *faster* than with the classical ones (see [9], Theorem 6). For instance, the rate of approximation in $C[0,1]$ for functions analytic in $\{z : |z| < q + \varepsilon\}$ is q^{-n} versus $1/n$ for the classical Bernstein polynomials. On the other hand, for some infinitely differentiable functions and even analytic on $[0,1]$, their sequences of q-Bernstein polynomials $(q > 1)$ may be divergent (see [9], Theorem 2). This situation is totally impossible for $0 < q \leqslant 1$. In general, the problem to describe the class of functions in $C[0,1]$ which are uniformly approximated by their q-Bernstein polynomials in the case $q > 1$ is yet to be solved.

It is exactly an unexpected behavior of q-Bernstein polynomials with respect to convergence that makes the study of their convergence properties interesting.

It is known (see [9], Theorem 1) that entire functions and, in particular, polynomials are uniformly approximated by their q-Bernstein polynomials $(q > 1)$ on any compact set in $\mathbb{C}$. The aim of this paper is to examine the possibility of the uniform approximation for the function $f(z) = (z+a)^\alpha$, $a > 0$ with a non-integer $\alpha > -1$. It has been shown that the uniform approximation occurs on any compact set in $\{z : |z| < a\}$, while on any Jordan arc lying in $\{z : |z| > a\}$, the sequence $\{B_{n,q}(f;z)\}$ is not even uniformly bounded. In particular, our results imply that the approximation of $(x+a)^\alpha$ in $C[0,1]$ takes place if and only if $a \geqslant 1$. At the same time, it is proven that z^α with a non-integer $\alpha > 0$ cannot be uniformly approximated by its q-Bernstein polynomials $(q > 1)$ on any interval.

We discover, therefore, the following phenomenon worthy of note: if the function $(z+a)^\alpha$, with $a \geqslant 0, \alpha > 0$ is uniformly approximated by its q-Bernstein polynomials $(q > 1)$ on some interval in $\mathbb{C}$, then this function is approximated on any interval that is closer to the origin. In particular, for $z^\alpha, \alpha > 0$, we observe an interesting polarity in the range of possibilities for the uniform approximation: z^α, $\alpha > 0$, is approximated either on each (in the case $\alpha \in \mathbb{N}$) or no (in the case $\alpha \notin \mathbb{N}$) Jordan arc in $\mathbb{C}$.

2. Statement of results

We start with the following lemma, which gives explicitly higher coefficients of the q-Bernstein polynomials of the function $(z+a)^\alpha$. From here on, whenever we consider the function $f(z)=(z+a)^\alpha$, we assume that $a \geqslant 0$, $\alpha > -1$, and the branch of $f(z)$ analytic in $\mathbb{C} \setminus (-\infty, -a]$ so that $f(x) > 0$ for $x > -a$ is chosen.

LEMMA 2.1. *Let $f(z)=(z+a)^\alpha$ and*

$$B_{n,q}(f;z)=\sum_{k=0}^{n} c_{kn} z^k.$$

Then, for $k > \alpha$, we have:

$$c_{kn} = \frac{(-1)^{k+1}\lambda_{kn}\sin\pi\alpha}{\pi}\int_a^\infty \frac{(x-a)^\alpha dx}{x\left(x+\frac{1}{[n]_q}\right)\left(x+\frac{[2]_q}{[n]_q}\right)\cdots\left(x+\frac{[k]_q}{[n]_q}\right)}, \tag{2.1}$$

where

$$\lambda_{0n}=\lambda_{1n}=1, \quad \lambda_{kn}=\prod_{j=1}^{k-1}\left(1-\frac{[j]_q}{[n]_q}\right), \quad k=2,\ldots,n. \tag{2.2}$$

REMARK 2.1. If $q=1$, then λ_{kn} $(k=0,1,\ldots,n)$ are eigenvalues of the classical Bernstein operator, whose eigenstructure together with applications had been studied in [2]. Some results of [2] have been extended to q-Bernstein polynomials in [9].

The next theorem addresses the possibility of the uniform approximation for the power function by its q-Bernstein polynomials in the case $q>1$.

THEOREM 2.2. *Let $q>1$, $a>0$, and $f(z)=(z+a)^\alpha$. Then, for any compact set $K \subset \{z : |z| < a\}$,*

$$B_{n,q}(f;z) \to f(z), \quad n\to\infty,$$

uniformly on K.

COROLLARY 2.3. *If $q>1, a>0$, then for any $0<c<a$,*

$$B_{n,q}\left((t+a)^\alpha; x\right) \to (x+a)^\alpha, \quad n\to\infty,$$

uniformly on $[0,c]$.

In particular, we obtain the following result which also can be derived from Theorem 7 of [9]:

If $q>1, a>1$, then

$$B_{n,q}\left((t+a)^\alpha; x\right) \to (x+a)^\alpha, \quad n\to\infty,$$

uniformly on $[0,1]$.

The following statement shows that Theorem 2.2 is sharp in the following sense: the function $(x+a)^\alpha$ cannot be approximated by its q-Bernstein polynomials on any interval beyond $[-a,a]$. More precisely, the following assertion is true.

THEOREM 2.4. *Let $q > 1$, $a > 0$. If $\alpha > -1$ is not an integer, then for any Jordan arc $J \subset \{z : |z| > a\}$, the sequence $\{B_{n,q}(f;z)\}$ is not uniformly bounded on J.*

COROLLARY 2.5. *Let $q > 1$, $0 < a < 1$. If $\alpha > -1$ is not an integer, then $(x+a)^\alpha$ cannot be uniformly approximated by its q-Bernstein polynomials in $C[0,1]$.*

To investigate the approximation of $(x+a)^\alpha$ with $a = 1$, we use the following general statement.

THEOREM 2.6. *If $f(x) = \sum_{k=0}^{\infty}(-1)^k a_k x^k$, $a_k \geqslant 0$, where $a_k \to 0$ and $a_{k+1} \leqslant a_k$ for $k \geqslant k_0$, then*

$$B_{n,q}(f;x) \to f(x),\ n \to \infty,$$

uniformly on $[0,1]$.

COROLLARY 2.7. *Let $q > 1$. Then for any $\alpha > -1$,*

$$B_{n,q}((t+1)^\alpha;x) \to (x+1)^\alpha,\ n \to \infty,$$

uniformly on $[0,1]$.

REMARK 2.2. For $\alpha > 0$,

$$B_{n,q}\left((t+1)^\alpha;z\right) \to (z+1)^\alpha,\ \ n \to \infty,$$

uniformly in $\{z : |z| \leqslant 1\}$.

Theorems 2.2 and 2.4 show that as a decreases the interval where the function $(x+a)^\alpha$, $a > 0$, $0 < \alpha \notin \mathbb{N}$, is uniformly approximated by its q-Bernstein polynomials $(q > 1)$ gets narrower. What happens if $a = 0$? Theorem 2.8 demonstrates that x^α with a non-integer α is not approximated by its q-Bernstein polynomials on any interval.

THEOREM 2.8. *Let $q > 1$,*

$$f(z) = A_1 z^{\alpha_1} + \cdots + A_m z^{\alpha_m},\ \ A_i \in \mathbb{C},\ \ \alpha_i \geqslant 0\ \ (i = 1,\ldots,m). \tag{2.3}$$

If at least one of α_i is not an integer, then the sequence $\{B_{n,q}(f;z)\}$ is not uniformly bounded on any Jordan arc.

COROLLARY 2.9. *If $f(x) = A_1 x^{\alpha_1} + \ldots A_m x^{\alpha_m}$, $A_i \in \mathbb{C}$, $\alpha_i \geqslant 0$ $(i = 1,\ldots,m)$, is not a polynomial, then in the case $q > 1$, it cannot be uniformly approximated by its q-Bernstein polynomials on any interval.*

We conclude, therefore, that a function of the form (2.3) can be uniformly approximated by its q-Bernstein $(q > 1)$ polynomials on a Jordan arc if and only if f is a polynomial, in which case the uniform approximation occurs on any compact set in $\mathbb{C}$. This is an astonishing fact to observe that a function of the form (2.3) can be approximated by its q-Bernstein polynomials either on any or no Jordan arc in $\mathbb{C}$.

3. Proofs of the theorems

Proof of Lemma 2.1. We need the following representation of the q-Bernstein polynomials given in [9], formulae (6) and (7):

$$B_{n,q}(f;z)=\sum_{k=0}^{n}\lambda_{kn}f\left[0;\frac{1}{[n]_q};\ldots;\frac{[k]_q}{[n]_q}\right]z^k, \tag{3.1}$$

where $f\,[x_0;x_1;\ldots;x_k]$ denotes the divided differences of f, that is

$$f\,[x_0]=f(x_0),\;\; f\,[x_0;x_1]=\frac{f(x_1)-f(x_0)}{x_1-x_0},\ldots,$$

$$f\,[x_0;x_1;\ldots;x_k]=\frac{f\,[x_1;\ldots;x_k]-f\,[x_0;\ldots;x_{k-1}]}{x_k-x_0},$$

and λ_{kn} are given by (2.2).

If f is analytic on $[0,1]$, then by ([6], §2.7, page 44) the divided differences of can be expressed as follows:

$$f\left[0;\frac{1}{[n]_q};\ldots;\frac{[k]_q}{[n]_q}\right]=\frac{1}{2\pi i}\oint_L\frac{f(\zeta)\,d\zeta}{\zeta\left(\zeta-\frac{1}{[n]_q}\right)\ldots\left(\zeta-\frac{[k]_q}{[n]_q}\right)}, \tag{3.2}$$

where L is a contour encircling $[0,1]$ so that f is analytic on and within L. Therefore, for $f(z)=(z+a)^\alpha$, we have:

$$f\left[0;\frac{1}{[n]_q};\ldots;\frac{[k]_q}{[n]_q}\right]=\frac{1}{2\pi i}\oint_L\frac{(\zeta+a)^\alpha\,d\zeta}{\zeta\left(\zeta-\frac{1}{[n]_q}\right)\ldots\left(\zeta-\frac{[k]_q}{[n]_q}\right)}.$$

Taking into account that $(z+a)^\alpha$ is analytic in $\mathbb{C}\setminus(-\infty,-a]$, and for $k>\alpha$, the integrand in (3.2) is $o(|\zeta|^{-1})$ as $\zeta\to\infty$, $k\geqslant 1$, we obtain by the Cauchy Theorem:

$$\begin{aligned}
&\oint_L\frac{(\zeta+a)^\alpha\,d\zeta}{\zeta\left(\zeta-\frac{1}{[n]_q}\right)\ldots\left(\zeta-\frac{[k]_q}{[n]_q}\right)}\\
&\quad=(-1)^{k+1}e^{i\pi\alpha}\int_a^\infty\frac{(x-a)^\alpha\,dx}{x\left(x+\frac{1}{[n]_q}\right)\ldots\left(x+\frac{[k]_q}{[n]_q}\right)}\\
&\qquad-(-1)^{k+1}e^{-i\pi\alpha}\int_a^\infty\frac{(x-a)^\alpha\,dx}{x\left(x+\frac{1}{[n]_q}\right)\ldots\left(x+\frac{[k]_q}{[n]_q}\right)}\\
&\quad=(-1)^{k+1}2i\sin(\pi\alpha)\int_a^\infty\frac{(x-a)^\alpha\,dx}{x\left(x+\frac{1}{[n]_q}\right)\ldots\left(x+\frac{[k]_q}{[n]_q}\right)}.
\end{aligned}$$

□

Proof of Theorem 2.2. First, we prove that the coefficients c_{kn} satisfy the following estimates:

$$|c_{kn}| \leqslant \frac{C}{a^k}, \quad k = 0,1,\ldots,n, \quad n = 1,2,\ldots \tag{3.3}$$

where C is a positive constant independent from both k and n.

Indeed, for $k \leqslant \alpha$, we have:

$$\begin{aligned}|c_{kn}| &= \lambda_{kn} f\left[0; \frac{1}{[n]_q}; \ldots; \frac{[k]_q}{[n]_q}\right] \\ &\leqslant f\left[0; \frac{1}{[n]_q}; \ldots; \frac{[k]_q}{[n]_q}\right] = \frac{f^{(k)}(\xi)}{k!}, \quad \xi \in \left(0, \frac{[k]_q}{[n]_q}\right).\end{aligned}$$

Since, for $k \leqslant \alpha$,

$$f^{(k)}(\xi) \leqslant f^{(k)}(1) = \alpha(\alpha-1)\ldots(\alpha-k+1)(1+a)^{\alpha-k},$$

we obtain:

$$|c_{kn}| \leqslant \binom{\alpha}{k}(1+a)^{\alpha}.$$

We set

$$C_1 = C_1(a,\alpha) := \max_{0\leqslant k\leqslant\alpha} \binom{\alpha}{k} a^k (1+a)^{\alpha}$$

and derive

$$|c_{kn}| \leqslant \frac{C_1}{a^k}, \quad k \leqslant \alpha, \quad n = 1,2,\ldots \tag{3.4}$$

For $k > \alpha$, formula (2.1) implies:

$$\begin{aligned}|c_{kn}| &\leqslant \frac{|\sin(\pi\alpha)|}{\pi}\int_a^{\infty}\frac{(x-a)^{\alpha}}{x^{k+1}}dx = \frac{|\sin(\pi\alpha)|a^{\alpha}}{\pi a^k}\int_0^1 t^{\alpha}(1-t)^{k-\alpha-1}\,dt \\ &\leqslant \frac{|\sin(\pi\alpha)|a^{\alpha}}{\pi a^k(k-\alpha)}\Gamma(\alpha+1) \leqslant \frac{C_2}{a^k}.\end{aligned} \tag{3.5}$$

Combining (3.4) and (3.5), we conclude that

$$|c_{kn}| \leqslant \frac{C}{a^k}, \quad k = 0,1,\ldots,n; \quad n = 1,2,\ldots$$

with $C = \max\{C_1, C_2\}$.

It follows that the sequence $\{B_{n,q}(f;z)\}$ is uniformly bounded in any disc $\{z : |z| \leqslant \rho\}$ with $\rho < a$. Indeed, in virtue of (3.3), we have:

$$|B_{n,q}(f;z)| \leqslant \sum_{k=0}^{n}|c_{kn}|\rho^k \leqslant \frac{C}{1-\rho/a}, \quad |z| \leqslant \rho.$$

Apart from that, Lemma 1 of [9] shows that the sequence $\{B_{n,q}(f;z)\}$ converges to $f(z)$ on the set $\{q^{-j}\}_{j=0}^{\infty}$.

The statement now follows from the Vitali Theorem (cf., e.g., [16], Ch. V, §5.2). □

Proof of Theorem 2.4. Assume that the sequence $\{B_{n,q}(f;z)\}$ is uniformly bounded on a Jordan arc $J \subset \{z : |z| > a\}$, that is for some $M > 0$,

$$|B_{n,q}(f;z)| \leqslant M, \ z \in J. \tag{3.6}$$

We denote

$$c := \min\{|z| : z \in J\}, \quad d := \max\{|z| : z \in J\}.$$

By the conditions of the theorem, we have $0 < a < c < d < \infty$. Consider the auxiliary polynomials:

$$Q_n(f;z) := z^n B_{n,q}(f;a/z) = \sum_{k=0}^{n} c_{n-k,n} a^{n-k} z^k. \tag{3.7}$$

We set:

$$J_1 := \{z : z = a/w, w \in J\}.$$

Clearly,

$$J_1 \subset \left\{z : \frac{a}{d} \leqslant |z| \leqslant \frac{a}{c}\right\}$$

and (3.6) implies:

$$|Q_n(f;z)| \leqslant M \cdot \left(\frac{a}{c}\right)^n \quad \text{for} \ \ z \in J_1.$$

On the other hand, estimate (3.3) indicates that

$$|Q_n(f;z)| \leqslant \frac{C}{1-\rho} =: C_1 \ \ \text{for} \ \ |z| \leqslant \rho < 1.$$

We fix $\rho \in (a/c, 1)$ and apply the Two-constants Theorem (cf., e.g., [8], p. 41) to arrive at an estimate for $Q_n(f;z)$ in $\{z : |z| < \rho\}$. As a result, we obtain:

$$|Q_n(f;z)| \leqslant \left[M\left(\frac{a}{c}\right)^n\right]^{\omega(z)} \cdot C_1^{1-\omega(z)}, \ \ z \in \{z : |z| < \rho\} \setminus J_1, \tag{3.8}$$

where $\omega(z)$ is the harmonic measure of J_1 with respect to $\{z : |z| < \rho\} \setminus J_1$. We note that $0 \in \{z : |z| < a/d\} \subset \{z : |z| < \rho\} \setminus J_1$. Therefore, $\omega(0) > 0$ and (3.8) implies that

$$|Q_n(f;0)| \leqslant C_1^{1-\omega(0)} \cdot M^{\omega(0)} \left(\frac{a}{c}\right)^{n\omega(0)} =: M_1 \left(\frac{a}{c}\right)^{n\omega(0)}. \tag{3.9}$$

Also, it follows directly from (3.7) that

$$|Q_n(f;0)| = |c_{nn}| a^n.$$

For $n > \alpha$, Lemma 2.1 yields:

$$|c_{nn}| \geqslant \frac{\lambda_{nn}|\sin(\pi\alpha)|}{\pi\left(1+\frac{1}{a[n]_q}\right)\left(1+\frac{[2]_q}{a[n]_q}\right)\cdots\left(1+\frac{[n]_q}{a[n]_q}\right)} \int_a^\infty \frac{(x-a)^\alpha \, dx}{x^{n+1}}.$$

Since the sequence λ_{nn} is decreasing and

$$\lim_{n\to\infty} \lambda_{nn} = \prod_{j=1}^{\infty}\left(1-\frac{1}{q^j}\right) =: \lambda > 0,$$

we have

$$\lambda_{nn} > \lambda > 0 \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.10}$$

Taking into account that

$$\frac{[n-j]_q}{[n]_q} \leqslant \frac{1}{q^j},$$

we conclude:

$$\prod_{j=1}^{n}\left(1+\frac{[j]_q}{a[n]_q}\right) \leqslant \prod_{j=0}^{\infty}\left(1+\frac{1}{aq^j}\right) =: C_2 > 0. \tag{3.11}$$

Therefore,

$$\begin{aligned}|c_{nn}| &\geqslant \frac{\lambda|\sin(\pi\alpha)|}{\pi C_2}\int_a^\infty \frac{(x-a)^\alpha}{x^{n+1}}\,dx \\ &=: C_3 a^{\alpha-n}\int_0^1 t^\alpha(1-t)^{n-\alpha-1}dt = C_3 a^{\alpha-n}B(\alpha+1, n-\alpha) \geqslant \frac{C_4}{a^n n^{1+\alpha}}\end{aligned}$$

with $C_4 := C_3 a^\alpha \Gamma(\alpha+1)$. We note that $C_4 \neq 0$, because $\alpha \notin \mathbb{Z}$. Consequently, we obtain:

$$|Q_n(f;0)| = |c_{nn}|a^n \geqslant \frac{C_4}{n^{1+\alpha}},$$

contrary to (3.9). □

Proof of Theorem 2.6. Without loss of generality, we may take $k_0 = 0$ and consider

$$f(x) = \sum_{k=0}^{\infty}(-1)^k a_k x^k, \quad \text{where} \quad a_k \downarrow 0, \quad a_{k+1} \leqslant a_k, \quad k = 0, 1, \dots \tag{3.12}$$

The series in (3.12) converges uniformly on $[0,1]$ and

$$\left|\sum_{k=n}^{\infty}(-1)^k a_k x^k\right| \leqslant a_n, \quad x \in [0,1], \quad n = 0, 1, \dots$$

Since for each fixed n, the operator $B_{n,q}$ is bounded in $C[0,1]$, we have:

$$B_{n,q}(f;x) = \sum_{k=0}^{\infty}(-1)^k a_k B_{n,q}(t^k;x). \tag{3.13}$$

Formula (3.1) implies that all $B_{n,q}(t^k;x)$ are polynomials with non-negative coefficients and hence

$$0 \leqslant B_{n,q}(t^k;x) \leqslant 1 \quad \text{for} \quad x \in [0,1], \quad n = 1, 2, \dots, \quad k = 0, 1, \dots$$

In addition, Videnskii's recurrence formula ([17], formula (3.1)) shows that for each $n = 1, 2, \dots,$ we have:

$$B_{n,q}(t^{k+1};x) \leqslant B_{n,q}(t^k;x), \quad x \in [0,1], \quad k = 0, 1, \dots$$

Therefore,

$$\left|\sum_{k=n}^{\infty}(-1)^k a_k B_{n,q}(t^k;x)\right| \leqslant a_n, \quad x\in[0,1].$$

Given $\varepsilon>0$, we choose N so that $a_N<\varepsilon$. Then

$$|B_{n,q}(f;x)-f(x)| \leqslant \sum_{k=0}^{N} a_k \left|B_{n,q}(t^k;x)-x^k\right| + 2\varepsilon, \quad n>N.$$

By Theorem 5 of [9],

$$B_{n,q}(t^k;z)\to z^k,\ n\to\infty$$

uniformly on any compact set in $\mathbf{C}$. We choose $n_0>N$ in such a way that for $n>n_0$,

$$\left|B_{n,q}(t^k;x)-x^k\right| \leqslant \frac{\varepsilon}{N}, \quad x\in[0,1], \quad k=0,1,\ldots,N.$$

Then $n>n_0$ implies

$$|B_{n,q}(f;x)-f(x)|<3\varepsilon, \quad x\in[0,1]. \qquad \square$$

The proof of Theorem 2.8 is based upon the following observation.

REMARK 3.1. Let $P(z)=\sum_{k=0}^{n} c_k z^k$ be a polynomial so that for a Jordan arc J,

$$|P(z)|\leqslant M, \quad z\in J.$$

Then there exists a positive constant D depending only on J such that

$$|c_n|\leqslant M\cdot D^n.$$

Indeed, by the Riemann Conformal Mapping Theorem, there exists a conformal mapping $w=\varphi(z)$ of $\overline{\mathbb{C}}\setminus J$ on $\{w:|w|>1\}$ so that $\varphi(\infty)=\infty$,

$$\lim_{z\to\infty}\frac{\varphi(z)}{z} =: d\neq 0. \tag{3.14}$$

Consider the function

$$\frac{P(z)}{[\varphi(z)]^n}$$

analytic in $\mathbb{C}\setminus J$. In virtue of (3.14), we have:

$$\lim_{z\to\infty}\left|\frac{P(z)}{[\varphi(z)]^n}\right| = \frac{|c_n|}{D^n}, \quad D:=|d|,$$

and it follows that the function is analytic at ∞ as well. By the condition, its modulus is bounded by M on J. Applying the Maximum Modulus Principle we obtain that

$$\frac{|c_n|}{D^n} = \lim_{z\to\infty}\left|\frac{P(z)}{[\varphi(z)]^n}\right| \leqslant M,$$

as mentioned.

Proof of Theorem 2.8. Without loss of generality we assume that

$$0 < \alpha_1 < \cdots < \alpha_m \ \text{ with } \ \alpha_i \notin \mathbf{N} \ \ (i = 1, \ldots, m).$$

Suppose that the sequence $\{B_{n,q}(f\,;z)\}$ is uniformly bounded on a Jordan arc $J,$ that is for some $M > 0,$

$$|B_{n,q}(f\,;z)| \leqslant M, \quad z \in J.$$

By the above remark, the coefficients c_{nn} satisfy the following estimate:

$$|c_{nn}| \leqslant M \cdot D^n, \quad n = 1, 2, \ldots, \tag{3.15}$$

where $D > 0$ depends only on $J.$

On the other hand, we estimate coefficients $c_{nn},$ for $n > \alpha,$ with the help of (2.1). We obtain, for any $\delta > 0,$

$$|c_{nn}| = \frac{\lambda_{nn}}{\pi} \left| \sum_{j=1}^{m} \int_0^\infty \frac{A_j \sin(\pi\alpha_j) x^{\alpha_j} dx}{x\left(x + \frac{1}{[n]_q}\right) \ldots \left(x + \frac{[n]_q}{[n]_q}\right)} \right| \geqslant \frac{\lambda}{\pi} \left| \sum_{j=1}^{m} \left\{ \int_0^\delta + \int_\delta^\infty \right\} \right|,$$

where $\lambda > 0$ is the same constant as in (3.10).

Hence

$$|c_{nn}| \geqslant \frac{\lambda}{\pi} \left| \sum_{j=1}^{m} \int_0^\delta \right| - \frac{\lambda}{\pi} \sum_{j=1}^{m} \int_\delta^\infty \frac{|A_j \sin(\pi\alpha_j)|\, x^{\alpha_j} dx}{x\left(x + \frac{1}{[n]_q}\right) \ldots \left(x + \frac{[n]_q}{[n]_q}\right)} =: \sigma_1 - \sigma_2.$$

First, we choose $\delta > 0$ in such a way that

$$\left| 1 + \sum_{j=2}^{m} \frac{A_j \sin(\pi\alpha_j)}{A_1 \sin(\pi\alpha_1)} x^{\alpha_j - \alpha_1} \right| \geqslant \frac{1}{2} \ \text{ for } \ x \in [0, \delta].$$

With this choice of $\delta,$ we have, for n large enough:

$$\begin{aligned}
\sigma_1 &\geqslant \frac{\lambda\,|A_1 \sin(\pi\alpha_1)|}{2\pi} \int_0^\delta \frac{x^{\alpha_1} dx}{x\left(x + \frac{1}{[n]_q}\right) \ldots \left(x + \frac{[n]_q}{[n]_q}\right)} \\
&\geqslant \frac{\lambda\,|A_1 \sin(\pi\alpha_1)|}{2\pi} \int_0^{1/[n]_q} \frac{x^{\alpha_1} dx}{x\left(\frac{1}{[n]_q} + \frac{1}{[n]_q}\right) \ldots \left(\frac{[n]_q}{[n]_q} + \frac{[n]_q}{[n]_q}\right)} \\
&= \frac{\lambda\,|A_1 \sin(\pi\alpha_1)|\,[n]_q^n}{2\pi 2^n [n]_q!} \cdot \frac{1}{\alpha_1 [n]_q^{\alpha_1}} \geqslant C_1 \cdot q^{n(n+1)/2} 2^{-n} q^{-n\alpha_1} \geqslant C_2 \cdot q^{n^2/3}.
\end{aligned}$$

Now we estimate σ_2, for $n > \alpha_m$, as follows:

$$\sigma_2 \leqslant \frac{\lambda}{\pi} \sum_{j=1}^{m} |A_j \sin(\pi\alpha_j)| \int_{\delta}^{\infty} \frac{x^{\alpha_j} dx}{x^{n+1}} = \frac{\lambda}{\pi} \sum_{j=1}^{m} |A_j \sin(\pi\alpha_j)| \cdot \frac{\delta^{\alpha_j - n}}{n - \alpha_j} \leqslant C_3\, \delta^{-n}.$$

As a result, we derive that, for n large enough,

$$|c_{nn}| \geqslant \sigma_1 - \sigma_2 \geqslant C_4\, q^{n^2/4}$$

contrary to (3.15). $\square$

Acknowledgement. I would like to express my gratitude to Dr. P. Danesh from Atilim University Departmental English Language Studies Unit for his assistance in the preparation of the manuscript. I am sincerely grateful to the anonymous referee for his/her thorough reading of the manuscript and very helpful remarks.

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(Received November 21, 2006)

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