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ON GRONWALL–BELLMAN–BIHARI TYPE INTEGRAL INEQUALITIES IN SEVERAL VARIABLES WITH RETARDATION FOR DISCONTINUOUS FUNCTIONS

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Abstract. In this work some new nonlinear inequalities of the Gronwall-Bellman-Bihari type in n-independent variables with delay for discontinuous functions are presented. They have non-Lipschitz type discontinuities in some set of points for a n-dimensional Euclidean space. The results r extend recent results of R. Bellman, I. Bihari, B. K. Bondge, V. Lakshmikantham, B. G. Pachpatte for continuous case of integral inequalities and its generalization for integro-sum inequalities (discontinuous case) presented in the investigations of D. Bainov, S. Borysenko, V. Lakshmikantham, S. Leela, A. Samoilenko.

1. Introduction

In the context of the theory of integral inequalities for continuous functions, basic results were obtained by T. H. Gronwall [17], R. Bellman [7] I. Bihari [8]. The development of this theory is well described in the monographs by R. P. Agarwal [2], D. Bainov, P. Simeonov [4], E. F. Bechenbach, R. Bellman [5], R. Bellman, K. L. Cooke [7], V. Lakshmikantham, S. Leela [18], A. Martynyuk, V. Lakshmikantham, S. Leela [21], A. N. Filatov, L. V. Sharova [16], W. Walter [30]. Numerous generalizations of Gronwall-Bellman-Bihari type inequalities for functions of n-independent variables were investigated by R. P. Agarwal [1], Akinyele Olusola [3], B. K. Bondge, B. G. Pachpatte [9, 10], B. G. Pachpatte [24], M. H. Shih, C. C. Yeh [28], E. Thandapani, R. P. Agarwal [29], C. C. Yeh [31], E. C. Young [32], A. I. Zahariev, D. D. Bainov [32] and many others (see also [22]).

Firstly starting from the works [12], [27] the theory of integral inequalities for discontinuous functions (integro-sum inequalities [25], [26]) considered such type of inequalities:

$$u(t) \leqslant \varphi(t) + \int_{t_0}^{t} \psi(\tau) u^m(\tau) d\tau + \sum_{t_0 < t_i < t} \beta_i u(t_i - 0),$$

with $\{t_i\}-$ points of discontinuities of the function $u(t)$.

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The strongest results in the theory of integral inequalites for discontinuous functions were described in the monographs by D. Bainov, P. Simeonov [4], V. Lakshmikantham, D. Bainov, P. S. Simeonov [20], S. D. Borysenko [14], A. M. Samoilenko, S. D. Borysenko et al [26], S. Borysenko, G. Iovane [15]. The history of the development of the integro-sum inequalities was properly described in the work of Yu. A. Mitropolskiy, G. Iovane, S. Borysenko [23].

In this paper we generalize some results obtained in [13], [15], [23] where mainly integro-sum inequalities of Bellman-Bihari type with discontinuities of non-Lipschitz type were investigated:

$$\varphi(t) \leqslant \varphi(t) + \int_{t_0}^{t} K(t,\tau,u(\tau))d\tau + \sum_{t_0<t_i<t} \beta_i u^m(t_i-0), \quad m>0.$$

In Section 2 we find a new analogy of the Gronwall-Bellman inequality for discontinuous functions of $n-$ independent variables.

In Section 3 we investigate a new analogy of the Bihari type inequality with discontinuities of the Hölder type for functions of several variables.

2. Analogy of Bellman result

Let us consider a Euclidean space R^n with the points $x=(x^1,...,x^n)$, $x^0=(x^{10},...x^{n0})$, with the order $x^0 \leqslant x(x^{i0} \leqslant x^i)$, $\forall i=\overline{1,n}$.

We denote

$$\int_{x^0}^{x} ...d\sigma = \int_{x^{10}}^{x^1} ... \int_{x^{n0}}^{x^n} ...d\sigma_1...d\sigma_n,$$

$$\sum_{x^0<x_k<x} \beta_k = \sum_{x^{10}<x_{k_1}<x^1,...,x^{n0}<x_{k_n}<x^n} \beta_k,$$

$$D_j = \frac{\partial}{\partial x_j}, \quad j=\overline{1,n}.$$

Let $\mathscr{J}$ be a space of continuous functions such that:

1. $f: R^n \to R^n$;
2. $f=(f_1(x),...,f_n(x)) : f_j : R^n \to R$, $j=\overline{1,n}$;
3. $f(x) \leqslant x$, $\lim_{|x|\to\infty} f_j(x) = \infty$, $\forall j=\overline{1,n}$.

The next proposition is valid.

THEOREM 2.1. *Consider the integro-sum inequality of the following form*

$$u(x) \leqslant \psi(x) + g(x)\int_{x^0}^{x} f(\tau)u(P(\tau))d\tau + + \sum_{x^0<x_i<x} \beta_i u^m(x_i-0), \qquad m>0 \quad (1)$$

where nonnegative function $u(x)$ is determined in the domain

$$\Omega = \Omega_{k_1,...,k_n} = \{x : x^1 \in [x_{k_1-1}, x_{k_1}[, ..., x^n \in [x_{k_n-1}, ..., x_{k_n}[, \ \ k_j=\overline{1,n}\} \quad (2)$$

and $\{x_k\} = \{x_{k_1}, ..., x_{k_n}\}$ *are the points of finite jump of the function* $u(x) : x_1 < x_2 < ...., \lim_{k\to\infty} x_k = \infty, \quad u(x_i - 0) \neq u(x_i + 0), \quad \forall i \in N.$

Suppose, that $g \geqslant 1, \ P(v) \in \mathscr{J}, \ \beta_i = const \geqslant 0,$ *the function* $\psi(x) > 0$ *and non-decreasing to* $x \in \Omega$; $f \geqslant 0 : f = 0, \ \forall x \in \Omega_{k_1,...,k_n}$ *at* $k_i \neq k_j, \ i,j = \overline{1,n}.$

Then in domain Ω *for the function* $u(x)$ *the following estimate is justified*

$$u(x) \leqslant \psi(x)g(x) \prod_{x^0<x_i<x} (1 + \beta_i \psi^{m-1}(x_i) g^m(x_i)) \times \\ \times \exp\left[\int_{x^0}^{x} f(\sigma) g(P(\sigma)) \frac{\psi(P(\sigma))}{\psi(\sigma)} d\sigma\right], \\ if \quad 0 < m \leqslant 1, \forall x \geqslant x^0; \tag{3}$$

$$u(x) \leqslant \psi(x)g(x) \prod_{x^0<x_i<x} (1 + \beta_i \psi^{m-1}(x_i) g^m(x_i)) \times \\ \times \exp\left[m \int_{x^0}^{x} f(\sigma) g(P(\sigma)) \frac{\psi(P(\sigma))}{\psi(\sigma)} d\sigma\right], \\ if \quad m \geqslant 1, \forall x \geqslant x^0. \tag{4}$$

Proof. It is obvious that

$$\frac{u(x)}{\psi(x)} \leqslant 1 + g(x) \int_{x^0}^{x} f(s) \frac{u(P(s))}{\psi(s)} ds + \sum_{x^0<x_i<x} \beta_i [\frac{u(x_i - 0)}{\psi(x_i - 0)}]^m \psi^{m-1}(x_i) \\ \leqslant g(x) \left\{ 1 + \int_{x^0}^{x} f(s) \frac{u(P(s))}{\psi(s)} ds + \sum_{x^0<x_i<x} \beta_i \left[\frac{u(x_i - 0)}{\psi(x_i - 0)}\right]^m \psi^{m-1}(x_i) \right\}.$$

Denote

$$\widetilde{u}(x) = 1 + \int_{x^0}^{x} f(s) \frac{u(P(s))}{\psi(s)} ds + + \sum_{x^0<x_i<x} \beta_i [\frac{u(x_i - 0)}{\psi(x_i - 0)}]^m \psi^{m-1}(x_i),$$

$$\widetilde{u}(x^0) = 1.$$

Then

$$u(P(x)) \leqslant u(x)$$
$$u(P(x)) \leqslant g(P(x))\psi(P(x))\widetilde{u}(P(x)) \leqslant g(P(x))\psi(P(x))\widetilde{u}(x).$$

Consider the domain $D_{k_1,...,k_1}$. It is evident that

$$\widetilde{u}(x) = 1 + \int_{x^0}^{x} f(s) \frac{u(P(s))}{\psi(s)} ds.$$

By using the scheme of [3], [4] we obtained such inequalities:

$$D_1...D_n[\widetilde{u}(x)] = f(x) \frac{u(P(x))}{\psi(x)} \leqslant f(x) g(P(x)) \frac{\psi(P(x))}{\psi(x)} \widetilde{u}(x) \\ \Longrightarrow D_1...D_n[\widetilde{u}(x)] \leqslant A(x)\widetilde{u}(x).$$

Here $A(x) = f(x)g(x)\frac{\psi(P(x))}{\psi(x)}$.
Then

$$\frac{\widetilde{u}(x)D_1...D_n[\widetilde{u}(x)]}{[\widetilde{u}(x)]^2} \leqslant A(x) + \frac{D_n[\widetilde{u}(x)]D_1...D_{n-1}[\widetilde{u}(x)]}{[\widetilde{u}(x)]^2}$$
$$\Longrightarrow D_n(\frac{D_1...D_n[\widetilde{u}(x)]}{u(x)}) \leqslant A(x)$$
$$\Longrightarrow \frac{D_1...D_{n-1}[\widetilde{u}(x)]}{\widetilde{u}(x)} \leqslant \int_{x^{n0}}^{x^n} A(x^1,...,x^{n-1},\sigma^n)d\sigma^n$$
$$\Longrightarrow \frac{\widetilde{u}(x)D_1...D_{n-1}[\widetilde{u}(x)]}{[\widetilde{u}(x)]^2} \leqslant \int_{x^{n0}}^{x^n} A(x^1,...,x^{n-1},\sigma^n)d\sigma^n + + \frac{D_{n-1}[\widetilde{u}(x)]D_1...D_{n-2}[\widetilde{u}(x)]}{[\widetilde{u}(x)]^2} \quad (5)$$
$$\Longrightarrow D_{n-1}(\frac{D_1...D_{n-2}[\widetilde{u}(x)]}{\widetilde{u}(x)}) \leqslant \int_{x^{n0}}^{x^n} A(x^1,...,x^{n-1},\sigma^n)d\sigma^n$$
$$\Longrightarrow \frac{D_1...D_{n-2}[\widetilde{u}(x)]}{\widetilde{u}(x)} \leqslant \int_{x^{n-10}}^{x^{n-1}}\int_{x^{n0}}^{x^n} A(x^1,...,x^{n-2},\sigma^{n-1},\sigma^n)d\sigma^n d\sigma^{n-1}$$
$$\Longrightarrow \frac{D_1[\widetilde{u}(x)]}{\widetilde{u}(x)} \leqslant \int_{x^{20}}^{x^2}\cdots\int_{x^{n0}}^{x^n} A(x^1,\sigma^2,...,\sigma^n)d\sigma^n...d\sigma^2$$
$$\Longrightarrow \ln[\widetilde{u}(x)] \leqslant \int_{x^0}^{x^n} A(\sigma)d\sigma \Longrightarrow \widetilde{u}(x) \leqslant \exp(\int_{x^0}^{x} A(\sigma)d\sigma).$$

By using the results in [11], [23] and continuing the estimate (5) for the whole domain Ω as in [15], we obtain such estimates

$$\widetilde{u}(x) \leqslant \prod_{x^0<x_i<x}(1+\beta_i\psi^{m-1}(x_i)g^m(x_i))\exp\left[\int_{x^0}^{x} f(\sigma)g(P(\sigma))\frac{\psi(P(\sigma))}{\psi(\sigma)}d\sigma\right],$$

if $0 < m \leqslant 1,\ \forall x \geqslant x_0$;

$$\widetilde{u}(x) \leqslant \prod_{x^0<x_i<x}(1+\beta_i\psi^{m-1}(x_i)g^m(x_i))\exp\left[m\int_{x^0}^{x} f(\sigma)g(P(\sigma))\frac{\psi(P(\sigma))}{\psi(\sigma)}d\sigma\right],$$

if $m \geqslant 1,\quad \forall x \geqslant x^0$.

Since $u(x) \leqslant \psi(x)g(x)\widetilde{u}(x)$, the inequalities (3), (4) are finally proved.

3. Analogy of Bihari result

The next result can be proved.

THEOREM 3.1. *Let an integro-sum functional inequality be presented*

$$u(x) \leqslant \psi(x) + g(x)\int_{x^0}^{x} f(\tau)u^m(P(\tau))d\tau + \sum_{x^0<x_i<x}\beta_i u^m(x_i - 0), \quad (6)$$

with $m \neq 1,\ \ m > 0$, *where the functions* $u,\ \psi,\ g,\ f$ *and constants* β_i *satisfy the conditions of the theorem* 2.1.

Then

$$u(x) \leqslant \psi(x)g(x) \prod_{x^0<x_i<x} (1+\beta_i\psi^{m-1}(x_i)g^m(x_i))\times \\ \times \left[1+(m-1)\int_{x^0}^{x} f(\tau)\psi^{m-1}(\tau)g^m(P(\tau))\left[\frac{\psi(P(\tau))}{\psi(\tau)}\right]^m d\tau\right]^{1/1-m}, \tag{7}$$

if $0 < m < 1$;

$$u(x) \leqslant \psi(x)g(x) \prod_{x^0<x_i<x} (1+m\beta_i\psi^{m-1}(x_i)g^m(x_i))\times \\ \times [1-(m-1)[\prod_{x^0<x_i<x} (1+m\beta_i\psi^{m-1}(x_i)g^m(x_i))]^{m-1}\times \\ \times \int_{x^0}^{x} f(\tau)\psi^{m-1}(\tau)g^m(P(\tau))[\frac{\psi(P(\tau))}{\psi(\tau)}]^m d\tau]^{-1/m-1}, \tag{8}$$

for $m > 1$, *only if*

$$\int_{x^0}^{x} f(\tau)\psi^{m-1}(\tau)g^m(P(\tau))[\frac{\psi(P(\tau))}{\psi(\tau)}]^m d\tau] \leqslant 1/m \tag{9}$$

$$\prod_{x^0<x_i<x} (1+m\beta_i\psi^{m-1}(x_i)g^m(x_i)) < (1+\frac{1}{m-1})^{1/m-1}. \tag{10}$$

Proof. From the inequality (6) it follows

$$\frac{u(x)}{\psi(x)} \leqslant g(x)[1+\int_{x^0}^{x} f(s)\psi^{m-1}(s)[\frac{u(P(s))}{\psi(s)}]^m ds \\ + \sum_{x^0<x_i<x} \beta_i[\frac{u(x_i-0)}{\psi(x_i-0)}]^m \psi^{m-1}(x_i-0). \tag{11}$$

By denoting

$$u^*(x) = 1+\int_{x^0}^{x} f(s)\psi^{m-1}(s)[\frac{u(P(s))}{\psi(s)}]^m ds + \sum_{x^0<x_i<x} \beta_i[\frac{u(x_i-0)}{\psi(x_i-0)}]^m \psi^{m-1}(x_i-0)$$

$u^*(x) = 1,\ x^i = x^{i0},\ 1 \leqslant i \leqslant n,$ the following inequality is fulfilled

$$\frac{u(x)}{\psi(x)} \leqslant g(x)u^*(x).$$

It is evident that

$$\begin{aligned} u(P(x)) &\leqslant u(x), \\ u(P(x)) &\leqslant g(P(x))\psi(P(x))u^*(P(x)) \\ &\leqslant g(P(x))\psi(P(x))u^*(x), \\ u(x_i-0) &\leqslant \psi(x_i-0)g(x_i-0)u^*(x_i-0). \end{aligned}$$

Consider the first domain of continuity $\Omega_{k_1,\dots,k_n}$ of the function $u(x)$ (for $k_i = 1$, $\forall i = \overline{1,n}$).

Then

$$D_1 D_2 \dots D_n[u^*(x)] = f(x)\psi^{m-1}(x)[\frac{\psi(P(x))}{\psi(x)}]^m$$

$$\leqslant f(x)\psi^{m-1}(x)g^m(P(x))[\frac{\psi(P(x))}{\psi(x)}]^m[u^*(x)]^m.$$

Let $W(x) = f(x)\psi^{m-1}(x)g^m(P(x))[\frac{\psi(P(x))}{\psi(x)}]^m$.

Then

$$\frac{D_1 \dots D_n[u^*(x)]}{[u^*(x)]^m} \leqslant W(x)$$

.

Let us consider the trivial ratios

$$D_n(\frac{D_1 \dots D_{n-1}[u^*(x)]}{[u^*(x)]^m}) \leqslant W(x)$$

$$\frac{D_1 \dots D_{n-1}[u^*(x)]}{[u^*(x)]^m} \leqslant \int_{x^{n0}}^{x^n} W(x^1, \dots, x^{n-1}, \tau^n) d\tau^n,$$

$$\vdots$$

$$\frac{D_1[u^*(x)]}{[u^*(x)]^m} \leqslant \int_{x^{20}}^{x^2} \dots \int_{x^{n0}}^{x^n} W(x^1, t^1, \dots, t^n) dt^n \dots dt^2.$$

Then

$$u^*(x) \leqslant \{1 + (1-m)\int_{x^0}^{x} W(t)dt\}^{1/1-m}$$

with $x \in \Omega_{k_1,\dots,k_1}$. The estimates (7), (8) hold in the first domain of continuity for the function $u(x)$ (i.e. (9), (10) take place for $m > 1$).

It is easy to see that $\forall x \in \Omega_{k_p,\dots,k_p}$, $\forall p \leqslant n$

$$u^*(x) \leqslant \prod_{i=1}^{p-1}(1 + \beta_i \psi^{m-1}(x_i - 0)g^m(x_i - 0)[1 + (1-m)\times$$
$$\times \int_{x^0}^{x} f(\tau)\psi^{m-1}(\tau)g^m(P(\tau))[\frac{\psi(P(\tau))}{\psi(\tau)}]^m d\tau]^{1/1-m}, \quad (12)$$

if $0 < m < 1$;

$$u^*(x) \leqslant \prod_{i=1}^{p-1}(1 + m\beta_i \psi^{m-1}(x_i - 0)g^m(x_i - 0)[1 - (m-1)\times$$
$$\times [\prod_{i=1}^{p-1}(1 + m\beta_i \psi^{m-1}(x_i - 0)g^m(x_i - 0)]^{m-1}\times \quad (13)$$
$$\times \int_{x^0}^{x} f(\tau)\psi^{m-1}(\tau)g^m(P(\tau))[\frac{\psi(P(\tau))}{\psi(\tau)}]^m d\tau]^{-1/m-1},$$

for $m > 1$ if only the ratios (9), (10) take place.

Thanks to it the proof of theorem 3.1 is completed.

REMARKS. For the case $n = 1$, the results obtained in the theorem 2.1, theorem 3.1 coincide with the results in [15, theorem 2.1, p. 26, theorem 2.2 p. 28], [23, proposition 2.13 p. 17; proposition 2.14, p. 17].

If $n = 2$ from the results presented in the theorems 2.1, 3.1 the result in [15, proposition 3.5, p. 66, proposition 3.6 p. 66] follows.

If $n = 1$, $g(x) = 1$, from the results of theorems 2.1, 3.1 the results in [11, lemma p. 323] follows. If $P(s) = s$, $\beta_i = 0$, $n = 2$, we obtain the first result in the theory of integral inequalities for functions of 2-independent variables, an unpublished theorem of Wendroff, which was mentioned in the monograph by Bechenbach, Bellman [5].

If $n = 1$, $m = 1$, $\psi = cost$, $g = 1$ we obtain the first result in the theory of integro-sum inequalities [27].

For $n = 2$, $g = 1$, $m = 1$ the result of the theorem 2.1 coincide with the result in [13, lemma 1, p. 1639]; if $m \neq 1$, an independent generalization of Wendroff's result for discontinuous functions is obtained.

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