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UNIFORM NON–SQUARENESS, UNIFORM NORMAL STRUCTURE AND GAO'S CONSTANTS

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Abstract. We consider two geometric constants, $f(X)$ and $E(X)$, introduced by Gao recently. An estimate concerning the James and Gao's constants is obtained. This estimate therefore allow us to get: (1) A Banach space X is uniformly non-square if and only if $f(X) > 2$ or $E(X) < 8$; (2) X has uniform normal structure provided $f(X) > 4(3-\sqrt{5})$ or $E(X) < 3+\sqrt{5}$.

1. Introduction

We shall assume throughout this paper that X is a real Banach space. The non-trivial space will mean later on that X is a real space with $\dim X \geqslant 2$. Denote by S_X, B_X the unit sphere and the unit ball of X, respectively.

There are two geometric properties of Banach spaces which are widely studied recently. One is normal structure. A Banach space X is said to have *(weak) normal structure* if whenever C is a (weak compact) bounded convex subset of X with $\operatorname{diam} C > 0$, then $\operatorname{rad} C < \operatorname{diam} C$, where $\operatorname{diam} C = \sup\{\|x-y\| : x, y \in C\}$ and $\operatorname{rad} C = \inf\{\sup\{\|x-y\| : x \in C\} : y \in C\}$ are the diameter and radius of the set C. X is said to have *uniform normal structure* if $\inf\{\operatorname{diam} C/\operatorname{rad} C\} > 1$, where the infimum is taken over all bounded closed convex subsets C of X with $\operatorname{diam} C > 0$. Another is uniform non-squareness. A Banach space X is called *uniformly non-square* if for any $x, y \in S_X$ there exists a $\delta > 0$, such that either $\|x-y\|/2 \leqslant 1-\delta$, or $\|x+y\|/2 \leqslant 1-\delta$.

Based on the famous work of James concerning the non-square spaces, the James constant

$$\begin{aligned} J(X) &= \sup\{\|x+y\| \wedge \|x-y\| : x, y \in S_X\} \\ &= \sup\{\|x+y\| \wedge \|x-y\| : x, y \in B_X\}, \end{aligned}$$

was studied by several authors (see for example [2, 5, 7]). It is worth noting that the second equality above only holds when X is non-trivial. The James constant is

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also called non-square constant because its extreme value characterize uniform non-squareness, i.e., X is uniformly non-square if and only if $J(X) < 2$.

From Pythagorean theorem, Gao [4] introduced two quadratic constants

$$E(X) = \sup\{\|x+y\|^2 + \|x-y\|^2 : x, y \in S_X\}$$

and

$$f(X) = \inf\{\|x+y\|^2 + \|x-y\|^2 : x, y \in S_X\}.$$

In this paper, we will state two inequalities between the James and Gao's constants and therefore get some useful information from Gao's constants.

2. Equivalent forms

We begin this section with a modulus

$$\gamma_X(t) = \sup\left\{\frac{\|x+ty\|^2 + \|x-ty\|^2}{2} : x, y \in S_X\right\} \quad (t \geqslant 0),$$

originally introduced concerning the von Neumann-Jordan constant. By Proposition 2.2 in [11], one can easily get

$$E(X) = \sup\{\|x+y\|^2 + \|x-y\|^2 : x, y \in B_X\}.$$

By the convexity of $\gamma_X(t)$ and Krein-Milman Theorem, it has been shown that the supremum in the definition of $\gamma_X(t)$ may as well be taken from the extreme sets of the unit ball provided X is finite dimensional (see [11, Corollary 2.4]). This fact enable us to deduce that if X is a Banach space with a finite dimension, then

$$E(X) = \sup\{\|x+y\|^2 + \|x-y\|^2 : x, y \in ex(B_X)\},$$

where $ex(B_X)$ denotes the set of the extreme points of B_X.

PROPOSITION 1. *Let X be a Banach space. Then*

$$\begin{aligned} f(X) &= \inf\{\|x+y\|^2 + \|x-y\|^2 : x \in S_X, \|y\| \geqslant 1\} \\ &= \inf\{\|x+y\|^2 + \|x-y\|^2 : \|x\|, \|y\| \geqslant 1\}. \end{aligned}$$

Proof. Write $f_1(X) = \inf\{\|x+y\|^2 + \|x-y\|^2 : x \in S_X, \|y\| \geqslant 1\}$ and $f_2(X) = \inf\{\|x+y\|^2 + \|x-y\|^2 : \|x\|, \|y\| \geqslant 1\}$, respectively. Obviously, $f(X) \geqslant f_1(X) \geqslant f_2(X)$. To show the converse inequalities, assume that $x \in S_X, \|y\| \geqslant 1$. Let $z = y/\|y\|, t = \|y\|$, then $z \in S_X, t \geqslant 1$ and also that

$$\|x+y\|^2 + \|x-y\|^2 = \|x+tz\|^2 + \|x-tz\|^2.$$

Consider the even convex function $g(t) := \|x+tz\|^2 + \|x-tz\|^2$. Hence,

$$\begin{aligned} \|x+tz\|^2 + \|x-tz\|^2 = g(t) &= \frac{t+1}{2t} g(t) + \frac{t-1}{2t} g(-t) \\ &\geqslant g(1) = \|x+z\|^2 + \|x-z\|^2 \\ &\geqslant f(X), \end{aligned}$$

which implies that $f(X) \leqslant f_1(X)$.

To show $f_1(X) \leqslant f_2(X)$, we may assume $\|y\| \geqslant \|x\| \geqslant 1$ without loss generality. Hence,

$$\begin{aligned}\|x+y\|^2+\|x-y\|^2 &= \|x\|^2\left(\left\|\frac{x}{\|x\|}+\frac{y}{\|x\|}\right\|^2+\left\|\frac{x}{\|x\|}-\frac{y}{\|x\|}\right\|^2\right)\\ &\geqslant \left\|\frac{x}{\|x\|}+\frac{y}{\|x\|}\right\|^2+\left\|\frac{x}{\|x\|}-\frac{y}{\|x\|}\right\|^2\\ &\geqslant f_1(X),\end{aligned}$$

which implies $f_1(X) \leqslant f_2(X)$. □

COROLLARY 1. *Let X be a Banach space. Then*

$$E(X) = \sup\left\{\frac{\|x+y\|^2+\|x-y\|^2}{\max(\|x\|^2,\|y\|^2)} : x,y \in X, \|x\|+\|y\| \neq 0\right\} \tag{2.1}$$

and

$$f(X) = \inf\left\{\frac{\|x+y\|^2+\|x-y\|^2}{\min(\|x\|^2,\|y\|^2)} : x,y \in X, \|x\|+\|y\| \neq 0\right\}. \tag{2.2}$$

Proof. Let $x,y \in X, (x,y) \neq (0,0)$, and assume that $\max(\|x\|,\|y\|) = \|x\| > 0$. Thus,

$$\begin{aligned}E(X) &\geqslant \left\|\frac{x}{\|x\|}+\frac{y}{\|x\|}\right\|^2+\left\|\frac{x}{\|x\|}-\frac{y}{\|x\|}\right\|^2\\ &= \frac{\|x+y\|^2+\|x-y\|^2}{\max(\|x\|^2,\|y\|^2)}.\end{aligned}$$

Since the opposite inequality holds obviously, we get (2.1). The proof of (2.2) is similar to that of (2.1). □

THEOREM 1. *Let X be the* $L_p[0,1]$ *space,* $p,\ q \geqslant 1$ *and* $1/p+1/q=1$. *Then*

$$E(X) = 2^{1+2/r} \quad \text{and} \quad f(X) = 2^{1+2/r'},$$

where $r=\min(p,q), r'=\max(p,q)$.

Proof. We only consider $E(L_p)$ for $p \in [1,2]$ and $f(L_p)$ for $p \geqslant 2$, since the rest cases were discussed by Gao in [4]. Recall Clarkson's inequalities (cf. [8, 9]):

$$(\|x+y\|^q+\|x-y\|^q)^{1/q} \leqslant 2^{1/q}(\|x\|^p+\|y\|^p)^{1/p} \quad (1 \leqslant p \leqslant 2)$$

and

$$(\|x+y\|^q+\|x-y\|^q)^{1/q} \geqslant 2^{1/q}(\|x\|^p+\|y\|^p)^{1/p} \quad (p \geqslant 2).$$

Let $x,y \in X$ with $(x,y) \neq (0,0)$ in the rest proof and consider two cases for $p \geqslant 1$.

(1) $1 \leqslant p \leqslant 2$. By Hölder and Clarkson inequalities,

$$\frac{\|x+y\|^2+\|x-y\|^2}{\max(\|x\|^2,\|y\|^2)} \leqslant \frac{2^{1-2/q}(\|x+y\|^q+\|x-y\|^q)^{2/q}}{2^{-2/p}(\|x\|^p+\|y\|^p)^{2/p}} \leqslant 2^{1+2/p},$$

which implies that $E(X) \leqslant 2^{1+2/p}$. To show the converse inequality, let

$$x(t) = \begin{cases} 2^{1/p}, & t \in [0,1/2] \\ 0, & t \in [1/2,1] \end{cases}, \qquad y(t) = \begin{cases} 0, & t \in [0,1/2] \\ 2^{1/p}, & t \in [1/2,1] \end{cases},$$

then $\|x(t)+y(t)\|^2 + \|x(t)-y(t)\|^2 = 2^{1+2/p}$.

(2) $p \geqslant 2$. Similarly, we have

$$\frac{\|x+y\|^2 + \|x-y\|^2}{\min(\|x\|^2, \|y\|^2)} \geqslant \frac{2^{1-2/q}(\|x+y\|^q + \|x-y\|^q)^{2/q}}{2^{-2/p}(\|x\|^p + \|y\|^p)^{2/p}} \geqslant 2^{1+2/p},$$

which implies $f(X) \geqslant 2^{1+2/p}$. Put $x(t), y(t)$ as above, and so $f(X) = 2^{1+2/p}$. □

REMARK 1. Note that Clarkson's inequalities play import role in the above proof. It is well-known that Clarkson's inequalities hold in various Banach spaces, such as ℓ_p, $L_p(L_q)$, Sobolev and Logarithmic spaces (see for example [8, 10]). So we can also obtain $E(X)$ and $f(X)$ in such spaces using the same method.

3. Some properties

In this section, we will investigate some geometric concepts, such like uniform non-squareness, reflexivity and uniform normal structure, in terms of $E(X)$ and $f(X)$. Let us begin with some inequalities between the James and Gao's constants. The idea for these estimates partly comes from Theorem 3 in [7].

THEOREM 2. *Let X be a non-trivial Banach space. Then*

$$2[J(X)]^2 \leqslant E(X) \leqslant [J(X)]^2 + 4 \tag{3.1}$$

and

$$\frac{[J(X)]^2 - 2J(X) + 2}{[J(X)]^2} \leqslant \frac{f(X)}{4} \leqslant \frac{2}{[J(X)]^2}. \tag{3.2}$$

Proof. Note that the following inequalities

$$2(\min(a,b))^2 \leqslant a^2 + b^2 \leqslant (\min(a,b))^2 + 4,$$

hold for any $0 \leqslant a, b \leqslant 2$, so (3.1) is trivial.

Now let us prove (3.2). Fix $\epsilon > 0$ sufficiently small and choose $x, y \in S_X$ such that $\|x+y\|, \|x-y\| > J(X) - \epsilon$. Hence

$$f(X) \leqslant \frac{\|u+v\|^2 + \|u-v\|^2}{\min(\|u\|^2, \|v\|^2)} \leqslant \frac{4(\|x\|^2 + \|y\|^2)}{\min(\|x+y\|^2, \|x-y\|^2)} < \frac{8}{[J(X)-\epsilon]^2},$$

where $u = (x+y)/2, v = (x-y)/2$. Letting $\epsilon \to 0$, we get the right side of (3.2).

To show the left side of (3.2), put $t_0 = 2(J(X)-1)/J(X) \in [2-\sqrt{2}, 2]$. We consider two cases for any $x, y \in S_X$ in the following.

CASE 1. $\min(\|x+y\|, \|x-y\|) = t \in [0, t_0]$. Hence

$$\begin{aligned}\|x+y\|^2 + \|x-y\|^2 &= (\|x+y\| + \|x-y\| - t)^2 + t^2 \\ &\geqslant (2-t)^2 + t^2 \geqslant (2-t_0)^2 + t_0^2 \\ &= \frac{4[J(X)^2 - 2J(X) + 2]}{[J(X)]^2}.\end{aligned}$$

CASE 2. $\min(\|x+y\|, \|x-y\|) = t \in [t_0, 2]$. We first observe that

$$\max(\|x+y\|, \|x-y\|) \geqslant 2/J(X), \tag{3.3}$$

for any $x, y \in S_X$. Indeed, we may assume without loss of generality that $x, y \in S_X, x \neq \pm y$ and $\|x+y\| \geqslant \|x-y\|$. Put $u = (x+y)/2, v = (x-y)/2$, then $\|u\| \geqslant \|v\|$ and

$$\begin{aligned}J(X) &\geqslant \frac{1}{\|u\|} \min(\|u+v\|, \|u-v\|) = \frac{\min(\|u+v\|, \|u-v\|)}{\max(\|u\|, \|v\|)} \\ &= \frac{2\min(\|x\|, \|y\|)}{\max(\|x+y\|, \|x-y\|)} = \frac{2}{\max(\|x+y\|, \|x-y\|)},\end{aligned}$$

which implies the inequality (3.3). Thus,

$$\begin{aligned}\|x+y\|^2 + \|x-y\|^2 &= \min(\|x+y\|^2, \|x-y\|^2) + \max(\|x+y\|^2, \|x-y\|^2) \\ &\geqslant t_0^2 + [J(X)/2]^2 = \frac{4[J(X)^2 - 2J(X) + 2]}{[J(X)]^2}.\end{aligned}$$

Therefore, from both cases, we get (3.2). □

REMARK 2. If $X = (\mathbb{R}, |\cdot|)$, then $J(X) = 0, f(X) = E(X) = 4$. So the above inequalities hold only for X being non-trivial.

It is well-known that uniform non-squareness, as well as uniform normal structure, has been proved very useful in Metric Fixed Point Theory (see for example [6]). In a recent paper [4], Gao proved that X is uniformly non-square if $f(X) > 2$ or $E(X) < 8$; X has normal structure if $f(X) > 32/9$ or $E(X) < 5$. The above estimates therefore allow us to clearly obtain the following.

COROLLARY 2. *A Banach space X is uniformly non-square if and only if* $E(X) < 8$ *or* $f(X) > 2$.

COROLLARY 3. *A Banach space X has uniform normal structure if* $E(X) < 3+\sqrt{5}$ *or* $f(X) > 4(3-\sqrt{5})$.

Proof. To verify this assertion, it suffices to note that the above assumption implies $J(X) < (1+\sqrt{5})/2$, which in turn implies X having uniform normal structure from Theorem 2.1 in [2]. □

REMARK 3. Obviously, Corollary 2 and 3 extend the corresponding results in [4].

Next, let us discuss Gao's constant for its dual space. It follows from Theorem 1 that $E(X) = E(X^*)$ for X being the L_p space. This conclusion for general case however is not true. There is a simple example which shows that $E(X) \neq E(X^*)$.

EXAMPLE 1. ([7], Example 2) Consider $\ell_2-\ell_1$ space i.e., $X=\mathbb{R}^2$ with the norm

$$\|x\| = \begin{cases} \|x\|_2 & x_1x_2 \geqslant 0, \\ \|x\|_1 & x_1x_2 \leqslant 0. \end{cases}$$

Its dual is $\ell_2-\ell_\infty$ space i.e., $X=\mathbb{R}^2$ with the norm

$$\|x\| = \begin{cases} \|x\|_2 & x_1x_2 \geqslant 0, \\ \|x\|_\infty & x_1x_2 \leqslant 0. \end{cases}$$

Then $E(\ell_2-\ell_1)=6>3+2\sqrt{2}=E(\ell_2-\ell_\infty)$.

In fact, the equality $E(\ell_2-\ell_1)=6$ follows from the fact $\gamma_{\ell 2-\ell_1}(t)=1+t+t^2$ (cf. [11]). Note that $ex(B_{\ell_2-\ell_\infty})=\{(x,y);x^2+y^2=1,xy\geqslant 0\}\cup(-1,1)\cup(1,-1)$. Elementary computation shows

$$\|x+y\|^2+\|x-y\|^2 \leqslant 3+2\sqrt{2},$$

for any $x,y\in ex(B_{\ell_2-\ell_\infty})$. Put $x=(1/\sqrt{2},1/\sqrt{2})$ and $y=(-1,1)$. Then $x,y\in ex(B_{\ell_2-\ell_\infty})$, $\|x+y\|^2+\|x-y\|^2=3+2\sqrt{2}$ and the second equality follows.

We now establish an inequality between Gao's constant for a space and for its dual as the following:

THEOREM 3. *Let X be a Banach space and* X^* *its dual. Then*

$$2E(X^*)-8 \leqslant E(X) \leqslant E(X^*)/2+4. \tag{3.4}$$

Proof. To be convinced that (3.4) is true, consider a constant

$$A_2(X)=\sup\left\{\frac{\|x+y\|+\|x-y\|}{2} : x,y\in S_X\right\}$$

introduced by Baronti et al. in [1] and note that $A_2(X)=A_2(X^*)$. Observe first that for any $a,b\in[0,2]$,

$$\frac{(a+b)^2}{2} \leqslant a^2+b^2 \leqslant \left(\frac{a+b}{2}\right)^2+4,$$

which gives that

$$2[A_2(X)]^2 \leqslant E(X) \leqslant [A_2(X)]^2+4.$$

Thus, on one hand, we have

$$E(X) \leqslant [A_2(X)]^2+4=[A_2(X^*)]^2+4 \leqslant E(X^*)/2+4.$$

On the other hand,

$$E(X) \geqslant 2[A_2(X)]^2=2[A_2(X^*)]^2 \geqslant 2[E(X^*)-4].$$

Thus the proof is complete. □

Consequently, from the (3.4), we obtain the following.

COROLLARY 4. *$E(X) < 8 \Leftrightarrow X^*$ is uniformly non-square $\Leftrightarrow X$ is uniformly non-square.*

It follows from Corollary 2 that if $E(X) < 8$ or $f(X) > 2$, then X is uniformly non-square and therefore super-reflexive. The converse however is not true from the following example. The method of computation below is taken from Example 6 in [7].

EXAMPLE 2. ([7], Example 6) Let $X_{p,\lambda}$ $(1 < p \leqslant 2, \lambda > 1)$ be the $\ell_{p'}$ space with the norm

$$\|x\|_{p,\lambda} = \max(\|x\|_{p'}, \lambda\|x\|_\infty) \qquad (1/p + 1/p' = 1),$$

then $E(X_{p,\lambda}) = \min(2^{1+2/p}\lambda^2, 8)$, $f(X_{p,\lambda}) = \max(2^{1+2/p'}/\lambda^2, 2)$.

Proof. Since the inequalities

$$\|x\|_{p'} \leqslant \|x\|_{p,\lambda} \leqslant \lambda\|x\|_{p'}$$

hold for any $x \in X_{p,\lambda}$, then for any $x, y \in X$ with $\|x\| + \|y\| \neq 0$, we have

$$\frac{\|x+y\|_{p,\lambda}^2 + \|x-y\|_{p,\lambda}^2}{\max(\|x\|_{p,\lambda}^2, \|y\|_{p,\lambda}^2)} \leqslant \frac{\lambda^2(\|x+y\|_{p'}^2 + \|x-y\|_{p'}^2)}{\max(\|x\|_{p'}^2, \|y\|_{p'}^2)} \leqslant \lambda^2 E(\ell_p).$$

Thus $E(X_{p,\lambda}) \leqslant \lambda^2 E(\ell_p) = 2^{1+2/p}\lambda^2$. We now consider two cases for $\lambda > 1$.

CASE 1. If $1 < \lambda \leqslant 2^{1/p'}$, let $x = (2^{-1/p'}, 2^{-1/p'}, 0, \cdots)$, $y = (2^{-1/p'}, -2^{-1/p'}, 0, \cdots)$, then $\|x\|_{p,\lambda} = \|y\|_{p,\lambda} = 1$ and $\|x+y\|_{p,\lambda} = \|x-y\|_{p,\lambda} = 2^{1/p}\lambda$. Thus $E(X_{p,\lambda}) = 2^{1+2/p}\lambda^2$.

CASE 2. If $\lambda \geqslant 2^{1/p'}$, let $x = (1/\lambda, 1/\lambda, 0, \cdots), y = (1/\lambda, -1/\lambda, 0, \cdots)$, then $\|x\|_{p,\lambda} = \|y\|_{p,\lambda} = 1$ and $\|x+y\|_{p,\lambda} = \|x-y\|_{p,\lambda} = 2$. Therefore $E(X_{p,\lambda}) = 8$.

Similarly, we have $f(X) \geqslant \max(2^{1+2/p'}/\lambda^2, 2)$. Put $x = (1/\lambda, 0, \cdots), y = (0, 1/\lambda, 0, \cdots)$. Then $x, y \in S_X$ and also $\|x+y\|_{p,\lambda}^2 + \|x-y\|_{p,\lambda}^2 = \max(2^{1+2/p'}/\lambda^2, 2)$. □

Finally, let us present a characterization for super-reflexivity. Denote by $\widetilde{E}(X)$ $(\widetilde{f}(X))$ the infimum (supremum) of all $E(X)$ $(f(X))$ for the equivalent norms of a Banach space X.

THEOREM 4. *The following conditions are equivalent.*

$$(1)\ \widetilde{E}(X) < 8; \quad (2)\ \widetilde{f}(X) > 2; \quad (3)\ X \textit{ is super} - \textit{reflexive}.$$

Proof. Since $(1) \Rightarrow (2)$ and $(2) \Rightarrow (3)$ are trivial, it suffices to show $(3) \Rightarrow (1)$. It's well-known (see for example [3]) that if X is super-reflexive, then there exists an equivalent uniformly convex norm $\|\cdot\|$ on X such that $(X, \|\cdot\|)$ is uniformly convex, which in turn implies uniform non-squareness, thus $E(X) < 8$. This completes the proof since $\widetilde{E}(X) \leqslant E(X)$. □

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