Mathematical
Inequalities
& Applications
Volume 11, Number 4 (2008), 615–628

BASIC INDEXES AND ALUTHGE TRANSFORMATION FOR 2 BY 2 MATRICES

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(*communicated by T. Furuta*)

Abstract. For 2×2 matrices we provide precise forms of their basic properties such as norms, polar decompositions and singular decompositions. Using these facts, then, we show the exact form of the Aluthge transform $\Delta(A)$ of a matrix A and give an alternative proof of Ando-Yamazaki's result of the convergence of iterations $\{\Delta^n(A)\}$. In case $|\lambda| = |\mu|$ for the eigenvalues λ, μ of A we determine the precise forms of their limits (Theorem 3).

1. Introduction

The first noncommutative linear system with topologies is the algebra of all 2×2 matrices, M_2. Therefore, in many fields of mathematics, particularly in those fields connected with linear operators and their operator algebras, we often make experimental investigations in this system for the problems as our first trials. Among them, we notice that there are even results which are remained only at the level of 2 by 2 matrices such as the convergence problem of the iterations of Aluthge transforms $\{\Delta^n(A)\}$ of A (Ando-Yamazaki). Thus it would be of some importance to show precise forms of those basic items mentioned above.

In this paper, by using these results, we show first the precise form of the Aluthge transform $\Delta(A)$ and give an alternative proof of the above Ando-Yamazaki's convergence theorem. We furthermore show the precise forms of their limiting point when $|\lambda| = |\mu|$ for the eigenvalues λ, μ of A.

2. Exact forms of polar decompositions and singular values

Write a 2×2 matrix as $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and put

$$\alpha = |a|^2 + |b|^2 + |c|^2 + |d|^2 + 2\,|ad-bc|\,,\ \beta = |a|^2 + |b|^2 + |c|^2 + |d|^2 - 2\,|ad-bc|\,.$$

THEOREM 1. *Let $A = V|A|$ be the polar decomposition of A.*

Mathematics subject classification (2000): 15A60, 15A18.

Key words and phrases: Aluthge transform, polar decompositions, singular decompositions, singular values.

Paper MIA-11-51

Then the absolute value $|A|$ is written as

$$|A| = \frac{1}{\sqrt{\alpha}}\left(A^*A + |ad-bc|\,I\right)$$

and if A is invertible, i.e. if $ad-bc \neq 0$, the unique partial isometry (in fact a unitary) V is written as

$$V = \frac{1}{\sqrt{\alpha}}\left(A + |ad-bc|\,(A^*)^{-1}\right).$$

If A is not invertible, then $V = \frac{1}{\sqrt{\alpha}}A$. Moreover if we put

$$V' = \frac{1}{\sqrt{\alpha}}\begin{pmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix},$$

then V' is a partial isometry from $\ker(A)$ onto $(ran(A))^{\perp}$, and $Z = V + V'$ becomes a unitary with $A = Z|A|$.

Proof. We have

$$A^*A = \begin{pmatrix} |a|^2+|c|^2 & \bar{a}b+\bar{c}d \\ a\bar{b}+c\bar{d} & |b|^2+|d|^2 \end{pmatrix}.$$

Hence, a straightforward calculation shows that

$$\begin{aligned} AA^*A &= \left(|a|^2+|b|^2+|c|^2+|d|^2\right)A - (ad-bc)\begin{pmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix} \\ &= \alpha A - 2\,|ad-bc|\,A - (ad-bc)\begin{pmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix}. \end{aligned}$$

Therefore if A is invertible, we have

$$AA^*A = \alpha A - 2\,|ad-bc|\,A - |ad-bc|^2\,(A^*)^{-1}.$$

Now put

$$H = \frac{1}{\sqrt{\alpha}}\left(A^*A + |ad-bc|\,I\right),$$

then H is positive and the above equality simply leads us that

$$H^2 = A^*A.$$

Hence, H is the absolute value of A, that is, $H = |A|$. Moreover putting

$$V = \frac{1}{\sqrt{\alpha}}\left(A + |ad-bc|\,(A^*)^{-1}\right),$$

we have that

$$\begin{aligned} V\,|A| &= \frac{1}{\alpha}\left(AA^*A + 2\,|ad-bc|\,A + |ad-bc|^2\,(A^*)^{-1}\right) \\ &= \frac{1}{\alpha}\left(\alpha A - 2\,|ad-bc|\,A - |ad-bc|^2\,(A^*)^{-1} + 2\,|ad-bc|\,A + |ad-bc|^2\,(A^*)^{-1}\right) \\ &= A. \end{aligned}$$

Since A is invertible, V is the unique unitary matrix, and this is the polar decomposition of A.

If A is not invertible, we have the equalities $AA^*A = \alpha A$, and $H = \frac{1}{\sqrt{\alpha}}A^*A$. Moreover, putting $V = \frac{1}{\sqrt{\alpha}}A$, we see that $V|A| = A$.

Now by the equalities $V^*V = \frac{1}{\alpha}A^*A$ and

$$(V^*V)^2 = \frac{1}{\alpha^2}A^*AA^*A = \frac{1}{\alpha^2}A^*(AA^*A) = \frac{1}{\alpha^2}A^*\alpha A = \frac{1}{\alpha}A^*A = V^*V,$$

V^*V is a projection on $ran\,(|A|)$. Therefore, V is a partial isometry from $ran\,(|A|)$ onto $ran\,(A)$.

Moreover, since $ker(V) = ker(A) = ker(|A|)$, V is the unique partial isometry of the polar decomposition of A.

Next, let

$$V^{'} = \frac{1}{\sqrt{\alpha}}\begin{pmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix},$$

then we see that

$$V^{'*}V^{'} = \frac{1}{\alpha}\begin{pmatrix} |b|^2 + |d|^2 & -(\bar{a}b + \bar{c}d) \\ -(a\bar{b} + c\bar{d}) & |a|^2 + |c|^2 \end{pmatrix} = I - V^*V.$$

Hence, $V^{'*}V^{'}$ is a projection on $ker(A)$, and $V^{'}$ is a partial isometry from $ker(A)$ onto $(ran(A))^{\perp}$.

Thus putting $Z = V + V'$ we obtain a unitary operator Z such that $Z|A| = A$. This completes the proof. □

The above arguments are done by almost bare hand. There is however another a little sophiscated way to find the form of the absolute value of A by using the Cayley-Hamilton theorem. In fact, from the equation

$$(A^*A)^2 - tr(A^*A)A^*A + |ad-bc|^2 I = 0$$

we have

$$|A|^4 + 2|ad-bc||A|^2 + |ad-bc|^2 I = \alpha|A|^2,$$

hence

$$\sqrt{\alpha}|A| = A^*A + |ad-bc|I,$$

and we have the form of $|A|$.

Here since the case $det(A) = 0$ means that A is a degenerate matrix becoming a one dimensional operator, in what follows we shall mainly assume that A is invertible.

The following theorem shows precise forms of the singular value decomposition of 2×2 matrices. Here unitary matrices U and W in the theorem have however naturally many choices. We denote the singular values of A by $s_1(A)$ and $s_2(A)$.

THEOREM 2. *Keep the same notations as above.*

(1) Then we have that

$$s_1(A) = \frac{\sqrt{\alpha}+\sqrt{\beta}}{2} \quad \textit{and } s_2(A) = \frac{\sqrt{\alpha}-\sqrt{\beta}}{2}.$$

Hence, the norm of A is written as

$$\|A\| = s_1(A) = \frac{\sqrt{\alpha}+\sqrt{\beta}}{2}.$$

(2) An exact form of the singular value decomposition of A is

$$A = W \begin{pmatrix} \frac{\sqrt{\alpha}+\sqrt{\beta}}{2} & 0 \\ 0 & \frac{\sqrt{\alpha}-\sqrt{\beta}}{2} \end{pmatrix} U^*,$$

where W and U are unitary such that

$$U = \frac{1}{\sqrt{2}\,\alpha^{\frac14}\beta^{\frac14}} \begin{pmatrix} \sqrt{\sqrt{\alpha}\sqrt{\beta}+(|a|^2+|c|^2-|b|^2-|d|^2)} & -\frac{\bar{a}b+\bar{c}d}{|\bar{a}b+\bar{c}d|}\sqrt{\sqrt{\alpha}\sqrt{\beta}-(|a|^2+|c|^2-|b|^2-|d|^2)} \\ \frac{a\bar{b}+c\bar{d}}{|a\bar{b}+c\bar{d}|}\sqrt{\sqrt{\alpha}\sqrt{\beta}-(|a|^2+|c|^2-|b|^2-|d|^2)} & \sqrt{\sqrt{\alpha}\sqrt{\beta}+(|a|^2+|c|^2-|b|^2-|d|^2)} \end{pmatrix}$$

$$W = \frac{1}{\sqrt{2}\,\alpha^{\frac14}\beta^{\frac14}\sqrt{\sqrt{\alpha}\sqrt{\beta}+(|a|^2+|c|^2-|b|^2-|d|^2)}}$$

$$\times \begin{pmatrix} a(\sqrt{\alpha}+\sqrt{\beta}) - \frac{ad-bc}{|ad-bc|}\bar{d}(\sqrt{\alpha}-\sqrt{\beta}) & -\frac{ad-bc}{|ad-bc|}\bar{c}(\sqrt{\alpha}+\sqrt{\beta}) - b(\sqrt{\alpha}-\sqrt{\beta}) \\ c(\sqrt{\alpha}+\sqrt{\beta}) + \frac{ad-bc}{|ad-bc|}\bar{b}(\sqrt{\alpha}-\sqrt{\beta}) & \frac{ad-bc}{|ad-bc|}\bar{a}(\sqrt{\alpha}+\sqrt{\beta}) - d(\sqrt{\alpha}-\sqrt{\beta}) \end{pmatrix}$$

if A is invertible, and $\bar{a}b+\bar{c}d \neq 0$.

If $\bar{a}b+\bar{c}d = 0$,

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad W = \frac{1}{\sqrt{\alpha}} \begin{pmatrix} a + \frac{ad-bc}{|ad-bc|}\bar{d} & b - \frac{ad-bc}{|ad-bc|}\bar{c} \\ c - \frac{ad-bc}{|ad-bc|}\bar{b} & d + \frac{ad-bc}{|ad-bc|}\bar{a} \end{pmatrix}$$

in case $|a|^2+|c|^2 \geqslant |b|^2+|d|^2$, *and*

$$U=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad W=\frac{1}{\sqrt{\alpha}}\begin{pmatrix} b-\dfrac{ad-bc}{|ad-bc|}\bar{c} & -a-\dfrac{ad-bc}{|ad-bc|}\bar{d} \\ d+\dfrac{ad-bc}{|ad-bc|}\bar{a} & -c+\dfrac{ad-bc}{|ad-bc|}\bar{b} \end{pmatrix}$$

in case $|a|^2+|c|^2 < |b|^2+|d|^2$.

Since singular values of A are just ordered square roots of the eigenvalues of A^*A, the following diagonalization process for a selfadjoint matrix B leads us to the assertion of the theorem. Thus we state the exact form of the diagonalization in the following.

Proof of this proposition is however a series of routine calculi, and we leave detailed checkes to the readers.

PROPOSITION 1. *Write*

$$B=\begin{pmatrix} u & v \\ \bar{v} & w \end{pmatrix}$$

and put

$$\gamma=(u-w)^2+4|v|^2.$$

Then a diagonalizing form of B *is written as follows with a unitary* U.
Here we may naturally assume that $v \neq 0$ *for a non-trivial case.*

$$B=U\begin{pmatrix} \dfrac{u+w+\sqrt{\gamma}}{2} & 0 \\ 0 & \dfrac{u+w-\sqrt{\gamma}}{2} \end{pmatrix}U^*$$

where

$$U=\frac{1}{\sqrt{2}\gamma^{\frac{1}{4}}}\begin{pmatrix} \sqrt{\sqrt{\gamma}+(u-w)} & -\dfrac{v}{|v|}\sqrt{\sqrt{\gamma}-(u-w)} \\ \dfrac{\bar{v}}{|v|}\sqrt{\sqrt{\gamma}-(u-w)} & \sqrt{\sqrt{\gamma}+(u-w)} \end{pmatrix}.$$

Proof of Theorem 2. For a given matrix A we know the entries of A^*A as before, and we see that

$$\begin{aligned} \gamma &= (|a|^2+|c|^2-|b|^2-|d|^2)^2+4\,|\bar{a}b+\bar{c}d|^2 \\ &= (|a|^2+|c|^2+|b|^2+|d|^2)^2-4\,|ad-bc|^2 \\ &= \alpha\beta. \end{aligned}$$

Therefore, we have the eigenvalues of A^*A as

$$\begin{aligned} \frac{(u+w)\pm\sqrt{\gamma}}{2} &= \frac{\dfrac{\alpha+\beta}{2}\pm\sqrt{\alpha\beta}}{2} \\ &= \frac{\alpha+\beta\pm 2\sqrt{\alpha\beta}}{4}=\left(\frac{\sqrt{\alpha}\pm\sqrt{\beta}}{2}\right)^2. \end{aligned}$$

Hence, by the above proposition, if $v = \bar{a}b + \bar{c}d \neq 0$ we obtain the precise form of a diagonalization unitary U for A^*A such that

$$A^*A = U \begin{pmatrix} \left(\dfrac{\sqrt{\alpha}+\sqrt{\beta}}{2}\right)^2 & 0 \\ 0 & \left(\dfrac{\sqrt{\alpha}-\sqrt{\beta}}{2}\right)^2 \end{pmatrix} U^*.$$

Therefore the eigenvalues of $|A|$ are $\dfrac{\sqrt{\alpha} \pm \sqrt{\beta}}{2}$ and we heve

$$|A| = U \begin{pmatrix} \dfrac{\sqrt{\alpha}+\sqrt{\beta}}{2} & 0 \\ 0 & \dfrac{\sqrt{\alpha}-\sqrt{\beta}}{2} \end{pmatrix} U^*.$$

Now put

$$V = \frac{1}{\sqrt{\alpha}} \begin{pmatrix} a + \dfrac{ad-bc}{|ad-bc|}\bar{d} & b - \dfrac{ad-bc}{|ad-bc|}\bar{c} \\ c - \dfrac{ad-bc}{|ad-bc|}\bar{b} & d + \dfrac{ad-bc}{|ad-bc|}\bar{a} \end{pmatrix},$$

assuming that A is invertible. Then, by Theorem 1 V becomes a unitary with $A = V|A|$. Hence, putting $W = VU$, we obtain a precise form of the singular value decomposition of A, that is,

$$A = W \begin{pmatrix} \dfrac{\sqrt{\alpha}+\sqrt{\beta}}{2} & 0 \\ 0 & \dfrac{\sqrt{\alpha}-\sqrt{\beta}}{2} \end{pmatrix} U^*,$$

where the unitaries U and W have matrix expressions stated in the theorem. This completes the proof. □

We shall make use of these forms in the proof of our main result, Theorem 3.

3. Exact forms of Aluthge transformation and its iterations

For a given matrix A, its Aluthge transform is defined as

$$\Delta(A) = |A|^{\frac{1}{2}} V |A|^{\frac{1}{2}}.$$

This transformation is playing an important role in the operator theory, and among those problems related to this transformation the problem of convergence of iterations is only known for the case of 2 by 2 matrices [3, P.300]. Moreover the precise form of their limit points is not known. In this section we shall first show the exact form of $\Delta(A)$, and then give another proof of the convergence theorem making use of this exact form of the transformation. Furthermore, we shall show the precise form of limit points provided that $|\lambda| = |\mu|$ where λ and μ are eigenvalues of A. So far, we have been unable to find a general answer for the precise form, but we believe that our method would bring up a new insight for further investigation.

To begin with, note that we may assume $det(A) = 1$ because $\Delta(\xi A) = \xi\Delta(A)$, and the case $det(A) = 0$ is treated rather easily. Thus we keep this assumption throughout our arguments.

PROPOSITION 2. *With the assumption that $det(A) = 1$ the Aluthge transformation of A is written as*

$$\Delta(A) = \frac{(\sqrt{\alpha}+1)tr(A)+tr(A^*)}{\left(\sqrt{\alpha}+2\right)\alpha}A^*A + \frac{1}{\sqrt{\alpha}}\left(A-A^*\right) + \frac{(\sqrt{\alpha}+1)tr(A^*)+tr(A)}{\left(\sqrt{\alpha}+2\right)\alpha}I.$$

Proof. Now put the eigenvalues of $|A|^{\frac{1}{2}}$

$$\sigma = \frac{\sqrt{\sqrt{\alpha}+\sqrt{\beta}}}{\sqrt{2}} \quad \text{and} \quad \tau = \frac{\sqrt{\sqrt{\alpha}-\sqrt{\beta}}}{\sqrt{2}},$$

then we have

$$\sigma\tau = \sqrt{|ad-bc|} = 1$$

and

$$\sigma^2+\tau^2 = \sqrt{\alpha}.$$

Recall that in our case

$$|A| = \frac{1}{\sqrt{\alpha}}(A^*A+I), \quad V = \frac{1}{\sqrt{\alpha}}(A+(A^*)^{-1}) = \frac{1}{\sqrt{\alpha}}(A+tr(A^*)I-A^*).$$

Moreover, by Cayley-Hamilton Theorem

$$(|A|^{\frac{1}{2}}-\sigma I)(|A|^{\frac{1}{2}}-\tau I) = |A| - (\sigma+\tau)\,|A|^{\frac{1}{2}} + I = 0$$

and

$$|A|^{\frac{1}{2}} = \frac{1}{\sigma+\tau}(|A|+I).$$

Then we have that

$$\begin{aligned}\Delta(A) &= |A|^{\frac{1}{2}}V|A|^{\frac{1}{2}} \\ &= \frac{1}{(\sigma+\tau)^2}(|A|+I)V(|A|+I) \\ &= \frac{1}{\left(\sqrt{\alpha}+2\right)}(|A|+I)(A+V).\end{aligned}$$

Replacing $|A|$ and V by the above expressions, this is written as

$$\frac{1}{\left(\sqrt{\alpha}+2\right)\alpha}\{(\sqrt{\alpha}+1)A^*A^2+tr(A^*)A^*A-A^*AA^*+(\sqrt{\alpha}+1)^2A$$
$$+tr(A^*)(\sqrt{\alpha}+1)I-(\sqrt{\alpha}+1)A^*\}.$$

Here

$$A^2 = tr(A)A - I$$

and

$$A^*AA^* = (\alpha - 2)A^* - tr(A)I + A$$

by the same arguments as in the proof of Theorem 1.
Therefore it follows that

$$\Delta(A) = \frac{(\sqrt{\alpha}+1)tr(A)+tr(A^*)}{(\sqrt{\alpha}+2)\ \alpha}A^*A + \frac{1}{\sqrt{\alpha}}\ (A - A^*) + \frac{(\sqrt{\alpha}+1)tr(A^*)+tr(A)}{(\sqrt{\alpha}+2)\ \alpha}I.$$

This completes the proof. □

By the above proposition we can check that A is normal if $\Delta(A) = A$, using the concrete form of A.

COROLLARY 1. *If* $tr(A) = -tr(A^*)$ *in particular when* $tr(A) = 0$, $\Delta(A)$ *becomes normal. i.e.*

$$\Delta(A) = \frac{tr(A)}{(\sqrt{\alpha}+2)\sqrt{\alpha}}\,(A^*A - I) + \frac{1}{\sqrt{\alpha}}\,(A - A^*)\,.$$

Hence

$$\lim_{n\to\infty} \Delta^n(A) = \Delta(A).$$

With this result we shall show next an alternative proof of Ando-Yamazaki's theorem of the convergence of iterations for the Aluthge transformation.

Let $r(A)$ be the spectral radius of A. Write

$$\Delta^n(A) = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$$

and note that

$$\alpha_n = |a_n|^2 + |b_n|^2 + |c_n|^2 + |d_n|^2 + 2 = tr\left\{\Delta^n(A)^*\Delta^n(A)\right\} + 2$$

because $\det(\Delta^n(A)) = \det(A) = 1$.

LEMMA 1. $tr\left\{\Delta^n(A)^*\Delta^n(A)\right\} \geqslant tr\left\{\Delta^{n+1}(A)^*\Delta^{n+1}(A)\right\} \geqslant \cdots \geqslant 2$ *and*

$$\lim_{n\to\infty} tr\left\{\Delta^n(A)^*\Delta^n(A)\right\} = r(A)^2 + \frac{1}{r(A)^2}.$$

Proof. By Theorem 2

$$\|\ \Delta^n(A)\ \| = \frac{\sqrt{tr\left\{\Delta^n(A)^*\Delta^n(A)\right\} + 2} + \sqrt{tr\left\{\Delta^n(A)^*\Delta^n(A)\right\} - 2}}{2},$$

which converges to $r(A)$ by [4, Theorem 1]. Now since $\|\ \Delta^n(A)\ \| \geqslant \|\ \Delta^{n+1}(A)\ \|$, we obtain that

$$tr\left\{\Delta^n(A)^*\Delta^n(A)\right\} \geqslant tr\left\{\Delta^{n+1}(A)^*\Delta^{n+1}(A)\right\} \geqslant \cdots \geqslant 2.$$

Therefore we finally see that

$$\lim_{n\to\infty} tr\{\Delta^n(A)^*\Delta^n(A)\} = r(A)^2 + \frac{1}{r(A)^2}.$$

This completes the proof. □

By the above lemma we see that

$$\sqrt{\alpha} \geqslant \sqrt{\alpha_1} \geqslant \cdots \geqslant \sqrt{\alpha_n} \geqslant \sqrt{\alpha_{n+1}} \geqslant \cdots \geqslant 2.$$

Let α_∞ be the limit point of the sequence $\{\alpha_n\}$, then we have

$$\sqrt{\alpha_\infty} = \lim_{n\to\infty} \sqrt{\alpha_n} = r(A) + \frac{1}{r(A)}.$$

LEMMA 2. *For a* 2×2 *complex matrix* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, *we have an estimation of the norm*

$$\| \Delta(A) - \Delta^2(A) \| \leqslant \frac{2}{\sqrt{\alpha_\infty}} \| A - \Delta(A) \| .$$

Proof. We recall that exact form of the singular value decomposition of A in Theorem 2

$$A = W \begin{pmatrix} s_1(A) & 0 \\ 0 & s_2(A) \end{pmatrix} U^*.$$

Hence

$$U^*AU = U^*W \begin{pmatrix} s_1(A) & 0 \\ 0 & s_2(A) \end{pmatrix}.$$

Put $Z = U^*W$, then Z is a unitary and by the construction of U and W $\det(U^*) = \det(W) = 1$, hence $\det(Z) = \det(U^*)\det(W) = 1$. Therefore we can write

$$Z = \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix} \text{ such that } |u|^2 + |v|^2 = 1 .$$

Put $B = U^*AU$, then

$$B = Z \begin{pmatrix} s_1(A) & 0 \\ 0 & s_2(A) \end{pmatrix}$$

is its polar decompositin and

$$B = \begin{pmatrix} us_1(A) & -\bar{v}s_2(A) \\ vs_1(A) & \bar{u}s_2(A) \end{pmatrix}.$$

Therefore since $s_1(A)\cdot s_2(A) = 1$ by definitions,

$$\begin{aligned} \Delta(B) &= \begin{pmatrix} \sqrt{s_1(A)} & 0 \\ 0 & \sqrt{s_2(A)} \end{pmatrix} \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix} \begin{pmatrix} \sqrt{s_1(A)} & 0 \\ 0 & \sqrt{s_2(A)} \end{pmatrix} \\ &= \begin{pmatrix} us_1(A) & -\bar{v} \\ v & \bar{u}s_2(A) \end{pmatrix}. \end{aligned}$$

It follows that

$$\begin{aligned} \| B-\Delta(B) \| &= \left\| \begin{pmatrix} 0 & -\bar{v}(s_2(A)-1) \\ v(s_1(A)-1) & 0 \end{pmatrix} \right\| \\ &= Max(|v|\,(s_1(A)-1), |v|\,(1-s_2(A)) \\ &= |v|\,(s_1(A)-1) \end{aligned}$$

because $s_1(A)-1 > 1-s_2(A)$. Hence

$$\begin{aligned} \| A-\Delta(A) \| &= \| U^*AU - U^*\Delta(A)U \| \\ &= \| B-\Delta(B) \| \\ &= |v|\,(s_1(A)-1). \end{aligned}$$

Now for $Z = U^*W$, by the concrete form of U^* and W if $\bar{a}b+\bar{c}d \neq 0$

$$\begin{aligned} v = &-\frac{2\left(a\bar{b}+c\bar{d}\right)\left\{a\left(\sqrt{\alpha}+\sqrt{\beta}\right)-\bar{d}\left(\sqrt{\alpha}-\sqrt{\beta}\right)\right\}}{2\sqrt{\alpha}\sqrt{\beta}\left(\sqrt{\alpha}\sqrt{\beta}+(|a|^2+|c|^2-|b|^2-|d|^2)\right)} \\ &+\frac{c\left(\sqrt{\alpha}+\sqrt{\beta}\right)+\bar{b}\left(\sqrt{\alpha}-\sqrt{\beta}\right)}{2\sqrt{\alpha}\sqrt{\beta}} \\ = &-\frac{2\left(a\bar{b}+c\bar{d}\right)\left\{a\left(\sqrt{\alpha}+\sqrt{\beta}\right)-\bar{d}\left(\sqrt{\alpha}-\sqrt{\beta}\right)\right\}}{2\sqrt{\alpha}\sqrt{\beta}\left(\sqrt{\alpha}\sqrt{\beta}+(|a|^2+|c|^2-|b|^2-|d|^2)\right)} \\ &\times\frac{\left(\sqrt{\alpha}\sqrt{\beta}-(|a|^2+|c|^2-|b|^2-|d|^2)\right)}{\left(\sqrt{\alpha}\sqrt{\beta}-(|a|^2+|c|^2-|b|^2-|d|^2)\right)}+\frac{(c+\bar{b})\sqrt{\alpha}+(c-\bar{b})\sqrt{\beta}}{2\sqrt{\alpha}\sqrt{\beta}} \\ = &-\frac{\left(a\bar{b}+c\bar{d}\right)\left\{(a-\bar{d})\sqrt{\alpha}+(a+\bar{d})\sqrt{\beta}\right\}\left(\sqrt{\alpha}\sqrt{\beta}-(|a|^2+|c|^2-|b|^2-|d|^2)\right)}{4\sqrt{\alpha}\sqrt{\beta}\left(a\bar{b}+c\bar{d}\right)\left(\bar{a}b+\bar{c}d\right)} \\ &+\frac{(c+\bar{b})\sqrt{\alpha}+(c-\bar{b})\sqrt{\beta}}{2\sqrt{\alpha}\sqrt{\beta}} \\ = &-\frac{\left(a\bar{b}+c\bar{d}\right)(\bar{c}+b)\sqrt{\alpha}-\left(a\bar{b}+c\bar{d}\right)(\bar{c}-b)\sqrt{\beta}}{2\sqrt{\alpha}\sqrt{\beta}\,(\bar{a}b+\bar{c}d)}+\frac{c+\bar{b}}{2\sqrt{\beta}}+\frac{c-\bar{b}}{2\sqrt{\alpha}} \\ = &-\frac{a\bar{b}+c\bar{d}}{\bar{a}b+\bar{c}d}\cdot\frac{\bar{c}+b}{2\sqrt{\beta}}+\frac{a\bar{b}+c\bar{d}}{\bar{a}b+\bar{c}d}\cdot\frac{\bar{c}-b}{2\sqrt{\alpha}}+\frac{c+\bar{b}}{2\sqrt{\beta}}+\frac{c-\bar{b}}{2\sqrt{\alpha}}. \end{aligned}$$

We apply the above form for the matrix

$$\Delta(B) = \begin{pmatrix} us_1(A) & -\bar{v} \\ v & \bar{u}s_2(A) \end{pmatrix}$$

in place of A and obtain the following value of v' coressponding to v,

$$\begin{aligned} v' &= \frac{-uvs_1(A)+uvs_2(A)}{-\bar{u}\bar{v}s_1(A)+\bar{u}\bar{v}s_2(A)} \cdot \frac{2\bar{v}}{2\sqrt{\alpha_1}} + \frac{2v}{2\sqrt{\alpha_1}} \\ &= \left(\frac{u}{\bar{u}}+1\right) \cdot \frac{v}{\sqrt{\alpha_1}}. \end{aligned}$$

When $\bar{a}b+\bar{c}d=0$, we see that

$$v = \frac{c-\bar{b}}{\sqrt{\alpha}} \quad \text{or} \ = \frac{\bar{c}-b}{\sqrt{\alpha}}.$$

Hence

$$v' = \frac{2v}{\sqrt{\alpha_1}} \quad \text{or} \ = \frac{2\bar{v}}{\sqrt{\alpha_1}}.$$

Thus in any case

$$|v'| \leqslant \frac{2\,|v|}{\sqrt{\alpha_1}},$$

we have

$$\begin{aligned} \| \Delta(A) - \Delta^2(A) \| &= \| U^*\Delta(A)U - U^*\Delta^2(A)U \| \\ &= \| \Delta(B) - \Delta^2(B) \| \\ &= |v'|\,(s_1(\Delta(A)) - 1) \\ &\leqslant |v'|\,(s_1(A) - 1) \\ &\leqslant \frac{2\,|v|\,(s_1(A)-1)}{\sqrt{\alpha_1}} \\ &\leqslant \frac{2}{\sqrt{\alpha_\infty}} \ \| A - \Delta(A) \| . \end{aligned}$$

This completes the proof. □

Let λ and μ be the eigenvalues of A.

THEOREM 3. *The iterations* $\{\Delta^n(A)\}$ *converge for any* 2×2 *matrix* A. *In particular when* $|\lambda| = |\mu| = 1$ *and* $det(A) = 1$

$$\lim_{n\to\infty} \Delta^n(A) = \frac{tr(A)}{2} I + \frac{\sqrt{4-\{tr(A)\}^2}}{2\sqrt{\alpha-\{tr(A)\}^2}}(A-A^*),$$

where $\alpha = tr(A^*A) + 2|det(A)|$.

Proof. By Lemma 2

$$\| \Delta(A) - \Delta^2(A) \| \leqslant \frac{2}{\sqrt{\alpha_\infty}} \ \| A - \Delta(A) \| .$$

Moreover by Lemma 1 $|\lambda| = |\mu| = 1$ if and only if $\dfrac{2}{\sqrt{\alpha_\infty}} = 1$.
Therefore, if $|\lambda| \neq |\mu|$ then

$$\frac{2}{\sqrt{\alpha_\infty}} < 1.$$

Put $k = \dfrac{2}{\sqrt{\alpha_\infty}}$ then

$$\| \Delta(A) - \Delta^2(A) \| \leqslant k \| A - \Delta(A) \| .$$

Therefore we have

$$\| \Delta^n(A) - \Delta^{n+1}(A) \| \leqslant k \| \Delta^{n-1}(A) - \Delta^n(A) \| \leqslant \cdots \leqslant k^n \| A - \Delta(A) \| .$$

For $n > m$ we have

$$\begin{aligned} \| \Delta^m(A) - \Delta^n(A) \| &\leqslant \sum_{l=m}^{n-1} \| \Delta^l(A) - \Delta^{l+1}(A) \| \\ &\leqslant \sum_{l=m}^{n-1} k^l \| A - \Delta(A) \| \\ &= \frac{k^m (1 - k^{n-m})}{1-k} \| A - \Delta(A) \| \\ &\leqslant \frac{k^m}{1-k} \| A - \Delta(A) \| . \end{aligned}$$

Therefore $\{\Delta^n(A)\}$ is a Cauchy sequence, and it converges.

When $|\lambda| = |\mu| = 1$, we have that $\mu = \bar{\lambda}$, and $tr(A) = \lambda + \mu = tr(A^*)$ i.e. a real number. Therefore

$$\Delta(A) = \frac{tr(A)}{\alpha}(A^*A + I) + \frac{1}{\sqrt{\alpha}}(A - A^*),$$

and

$$\Delta^{n+1}(A) = \frac{tr(A)}{\alpha_n}(\Delta^n(A)^*\Delta^n(A) + I) + \frac{1}{\sqrt{\alpha_n}}(\Delta^n(A) - \Delta^n(A)^*).$$

Hence

$$\begin{aligned} \Delta^{n+1}(A) - \Delta^{n+1}(A)^* &= \frac{2}{\sqrt{\alpha_n}}(\Delta^n(A) - \Delta^n(A)^*) \\ &= \frac{2}{\sqrt{\alpha_n}} \cdot \frac{2}{\sqrt{\alpha_{n-1}}} \cdots \frac{2}{\sqrt{\alpha_1}} \cdot \frac{2}{\sqrt{\alpha}}(A - A^*) \\ &= p_n(A - A^*). \end{aligned}$$

Here the sequence

$$p_n = \frac{2}{\sqrt{\alpha_n}} \cdot \frac{2}{\sqrt{\alpha_{n-1}}} \cdots \frac{2}{\sqrt{\alpha_1}} \cdot \frac{2}{\sqrt{\alpha}}$$

converges because $\{p_n\}$ is positive and decreasing. Let p be the limit of p_n. Then the sequence $\left\{\Delta^{n+1}(A) - \Delta^{n+1}(A)^*\right\}$ converges to $p(A - A^*)$.
Next we shall show the sequence $\{\Delta^n(A)^*\Delta^n(A)\}$ converges to the identity.
In fact,

$$
\begin{aligned}
&\| \Delta^n(A)^*\Delta^n(A) - I \| \\
&= \frac{\sqrt{(|a_n|^2 + |c_n|^2 - 1)^2 + (|b_n|^2 + |d_n|^2 - 1)^2 + 2\,|\overline{a_n}b_n + \overline{c_n}d_n|^2 + 2\,|2 - tr\,\{\Delta^n(A)^*\Delta^n(A)\}|}}{2} \\
&\quad + \frac{\sqrt{(|a_n|^2 + |c_n|^2 - 1)^2 + (|b_n|^2 + |d_n|^2 - 1)^2 + 2\,|\overline{a_n}b_n + \overline{c_n}d_n|^2 - 2\,|2 - tr\,\{\Delta^n(A)^*\Delta^n(A)\}|}}{2} \\
&= \frac{\sqrt{\{tr\,\{\Delta^n(A)^*\Delta^n(A)\}\}^2 - 4} + \sqrt{\{tr\,\{\Delta^n(A)^*\Delta^n(A)\}\}^2 - 4tr\,\{\Delta^n(A)^*\Delta^n(A)\} + 4}}{2} \\
&\to 0
\end{aligned}
$$

because

$$\lim_{n\to\infty} tr\,\{\Delta^n(A)^*\Delta^n(A)\} = 2$$

and

$$|\overline{a_n}b_n + \overline{c_n}d_n|^2 = \left(|a_n|^2 + |c_n|^2\right)\left(|b_n|^2 + |d_n|^2\right) - 1.$$

It follows that

$$
\begin{aligned}
\lim_{n\to\infty} \Delta^{n+1}(A) &= \lim_{n\to\infty} \left\{ \frac{tr(A)}{\alpha_n}(\Delta^n(A)^*\Delta^n(A) + I) + \frac{1}{\sqrt{\alpha_n}}(\Delta^n(A) - \Delta^n(A)^*) \right\} \\
&= \frac{tr(A)}{4}(I + I) + \frac{1}{2}p(A - A^*) \\
&= \frac{tr(A)}{2}I + \frac{p}{2}(A - A^*).
\end{aligned}
$$

Since

$$\det\left(\frac{tr(A)}{2}I + \frac{p}{2}(A - A^*)\right) = 1,$$

it follows that

$$\frac{p^2}{4}\left[\alpha - \{tr(A)\}^2\right] + \frac{\{tr(A)\}^2}{4} = 1.$$

Hence

$$p = \frac{\sqrt{4 - \{tr(A)\}^2}}{\sqrt{\alpha - \{tr(A)\}^2}},$$

and

$$\lim_{n\to\infty} \Delta^n(A) = \frac{tr(A)}{2}I + \frac{\sqrt{4 - \{tr(A)\}^2}}{2\sqrt{\alpha - \{tr(A)\}^2}}(A - A^*).$$

This completes the proof. □

By same argument as in the proof of Theorem 3, if λ and μ are distinct real numbers, it follows that the sequence $\{\Delta^n(A) - \Delta^n(A)^*\}$ converges to 0. Therefore we see that the limit point of iterations $\{\Delta^n(A)\}$ is selfadjoint. But even in this case we have been unable to obtain the precise form of the limit point of $\{\Delta^n(A)\}$ because we have been unable to find the precise form of the limit point of $\{\Delta^n(A)^*\Delta^n(A)\}$.

Acknowlegment. The author is deeply indebted to Professor J.Tomiyama for many valuable advices in the preparation of this paper and would like to express his thanks to T.Yamazaki for fruitful comments about this paper. The author also would like to express his cordial thanks to the referee for careful reading with many comments, which has brought considerable improvements of the author's first draft.

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(Received March 14, 2007)

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