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ON THE MATRIX NORMS OF A GCD RELATED MATRIX

ERCAN ALTINIŞIK

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Abstract. In this paper we investigate the matrix norms of a GCD related matrix, i.e., $(S_f) = \left(f(i,j)/(i^r j^r)\right)$ for multiplicative arithmetical functions f. In particular, we obtain upper bounds for the ℓ_p norms of (S_f) for $f = \varphi$, σ_α, and ψ in terms of infinite prime products. Furthermore, we give lower and upper bounds for these infinite prime products by using particular norm inequalities.

1. Introduction

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers. Let (S) be the $n\times n$ matrix of which the ij-entry is $s_{ij} = (x_i, x_j)$, the greatest common divisor of x_i and x_j. In 1876 H. J. S. Smith [16] calculated the determinant of the matrix (S) when S is factor closed. In 1989 Beslin and Ligh [4] called the matrix (S) the greatest common divisor (GCD) matrix on S and triggered the study of GCD matrices. Since Smith's paper a large number of results on GCD matrices and many generalizations of them have been presented in the literature. For general accounts see e.g. [2, 8].

In the study of GCD matrices, the structures of these matrices are investigated, and some particular properties are tried to be presented in terms of some number-theoretical tools. Recently, some authors have investigated the matrix norms and eigenvalues of the GCD and related matrices. It is the first time that Taşcı [19] gave a lower bound for the Perron root of the GCD matrix defined on $S = \{1, 2, \ldots, n\}$. Lindqvist and Seip [12] obtained sharp lower and upper bounds for the eigenvalues of the matrix $\left(\frac{(i,j)^{2s}}{i^s j^s}\right)$. In [6, 17, 18, 20] some inequalities for the ℓ_p norms of the GCD matrix and some particular related matrices by the help of similar techniques were presented. Altınışık, Tuğlu and Haukkanen [1] gave a sharp upper bound for the ℓ_p norm of the $n\times n$ matrix $\left(\frac{(i,j)^r}{i^r j^r}\right)$, that is

$$\lim_{n\to\infty} \left\| \left(\frac{(i,j)^r}{i^r j^r}\right)_{n\times n} \right\|_p = \frac{\zeta(rp)^{3/p}}{\zeta(2rp)^{1/p}}. \tag{1.1}$$

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In [10] Haukkanen generalized the result in (1.1) as follow:

$$\lim_{n\to\infty}\left\|\left(\frac{(i,j)^s}{[i,j]^r}\right)_{n\times n}\right\|_p=\frac{\zeta(rp)^{2/p}\zeta(rp-ps)^{1/p}}{\zeta(2rp)^{1/p}}. \tag{1.2}$$

He also presented some estimates of the ℓ_p norm of the matrix $\left(\frac{(i,j)^s}{[i,j]^r}\right)$ in another paper [9].

Let f be an arithmetical function and r a positive real number. Consider the $n\times n$ matrix

$$\left(S_f\right)=\left(\frac{f\left((i,j)\right)}{i^r j^r}\right), \tag{1.3}$$

where $f\left((i,j)\right)$ denotes the value of f evaluated at the greatest common divisor of i and j. In this paper we mainly investigate some certain matrix norms of the matrix $\left(S_f\right)$. We generalize all results given in (1.1) and (1.2) which were presented in [1, 9, 10]. The structure of the paper is as follows. In Section 2, we summarize some basic tools from number theory and matrix theory we need throughout the paper. In Section 3, we give the main theorem of the paper and sharp upper bounds for ℓ_p norms of the matrices $\left(S_\varphi\right)$, $\left(S_{\sigma_\alpha}\right)$, and $\left(S_\psi\right)$ in terms of some infinite prime products. In Section 4, we study on the reverse direction. Namely, we present some lower and upper bounds for infinite prime products obtained in Section 3. In the last section discuss the intersection some different tools of mathematics which are used in the study of GCD matrices.

2. Preliminaries

We firstly review the basic tools of arithmetical functions and Dirichlet series in the light of the text of Apostol [3]. For general and detailed accounts the reader can see e.g. [3, 7, 13, 14, 15].

Let f and g be arithmetical functions. The Dirichlet convolution $f*g$ of f and g is defined by

$$(f*g)(n)=\sum_{d\mid n}f(d)g(n/d).$$

Let I and u be the arithmetical functions defined as $I(n)=\left[\frac{1}{n}\right]$ and $u(n)=1$ for all $n\in\mathbb{Z}^+$, respectively. Under the Dirichlet convolution the inverse of u is the Möbius function μ. Let N^α be the arithmetical function defined as $N^\alpha(n)=n^\alpha$ for a given $\alpha\in\mathbb{R}$ and all $n\in\mathbb{Z}^+$. Jordan's totient function J_α is defined as

$$J_\alpha(n)=(N^\alpha*\mu)(n)=\sum_{d\mid n}d^\alpha\mu(n/d)=n^\alpha\prod_{p\mid n}(1-\frac{1}{p^\alpha})$$

for a given $\alpha\in\mathbb{R}$. Then it is obvious that

$$n^\alpha=\sum_{d\mid n}J_\alpha(d)$$

for all $n\in\mathbb{Z}^+$. In particular, $J_1=\varphi$ is Euler's totient function.

For $\alpha \in \mathbb{R}$, the arithmetical function σ_α defined as

$$\sigma_\alpha(n) = \sum_{d|n} d^\alpha, \tag{2.1}$$

the sum of the αth powers of the divisors of n, is called divisor functions. It is clear that $\sigma_\alpha = u * N^\alpha$. Dedekind's ψ-function is defined by

$$\psi(n) = n\prod_{p|n}\left(1+\frac{1}{p}\right). \tag{2.2}$$

An arithmetical function f is said to be multiplicative if f is not identically zero and if $f(mn) = f(m)f(n)$ whenever $(m,n) = 1$. It is clear that $f(1) = 1$ if f is multiplicative. If $f(mn) = f(m)f(n)$ for all m and n, then f is said to be completely multiplicative. All the functions given above are multiplicative functions but only I, u, and N^α are completely multiplicative functions among them.

Let s be a complex number, and let f be an arithmetical function. The Dirichlet series corresponding to f is the series

$$\mathscr{L}_f(s) = \sum_{n=1}^{\infty}\frac{f(n)}{n^s}. \tag{2.3}$$

When we take $f = u$ in (2.3) we obtain the so-famous Dirichlet series ($s > 1$)

$$\mathscr{L}_u(s) = \zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s},$$

the Riemann zeta function. Since we will deal with the Dirichlet series with real s throughout the paper we assume $s \in \mathbb{R}$. Let the Dirichlet series $\mathscr{L}_f(s)$ and $\mathscr{L}_g(s)$ be convergent absolutely for $s > s_0$. Then we have $\mathscr{L}_f(s)\mathscr{L}_g(s) = \mathscr{L}_{f*g}(s)$ and also $\mathscr{L}_{f*g}(s)$ converges absolutely for $s > s_0$. Moreover, $\mathscr{L}_{f^{-1}}(s) = (\mathscr{L}_f(s))^{-1}$, where f^{-1} is the inverse of f under the Dirichlet convolution, and $\mathscr{L}_{f^{-1}}(s)$ converges for $s > s_0$. On the other hand, if f is multiplicative the Dirichlet series corresponding to f can be rewritten as

$$\mathscr{L}_f(s) = \prod_{\wp}\left\{1+\frac{f(\wp)}{\wp^s}+\frac{f(\wp^2)}{\wp^{2s}}+\frac{f(\wp^3)}{\wp^{3s}}+\cdots\right\}. \tag{2.4}$$

We now give some tools of matrix theory, in particular matrix norms, in the light of the text of Horn and Johnson [11]. Let $M_n(\mathbb{C})$ denote the set of the $n\times n$ matrices with entries from $\mathbb{C}$, and $1 \leqslant p < \infty$. The ℓ_p norm of a matrix $A \in M_n(\mathbb{C})$ is defined as

$$\|A\|_p = \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^p\right)^{\frac{1}{p}}.$$

The maximum column sum matrix norm and the maximum row sum matrix norm of A are defined as

$$|||A|||_1 = \max_{1 \leqslant j \leqslant n} \sum_{i=1}^{n} |a_{ij}|$$

and

$$|||A|||_\infty = \max_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} |a_{ij}|,$$

respectively. For a $A \in M_n(\mathbb{C})$ we have the inequality

$$|||A|||_1 \leqslant \|A\|_1 \text{ and } |||A|||_\infty \leqslant \|A\|_1. \tag{2.5}$$

The spectral norm of A is

$$|||A|||_2 = \max\{\sqrt{\lambda} : \lambda \text{ is an eigenvalue of } A^*A\},$$

where A^* is the conjugate transpose of A. Let $A \in M_n(\mathbb{C})$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the eigenvalues of A. The spectral radius $\rho(A)$ of A is defined as

$$\rho(A) = \max_{1 \leqslant i \leqslant n} |\lambda_i|.$$

For any matrix norm $|||\cdot|||$, we have

$$\rho(A) \leqslant |||A|||. \tag{2.6}$$

3. Matrix Norms of The Matrix (S_f)

Let f be an arithmetical function and (S_f) denote the $n \times n$ matrix which is given in (1.3). We will use the notation $f^p(n)$ for $(f(n))^p$ for the sake of simplicity. We now present the main theorem of the paper.

THEOREM 1. [THE MAIN THEOREM] *Let r and $1 \leqslant p < \infty$ be reals, and let $rp > 1$. If f is multiplicative then*

$$\lim_{n\to\infty} \left\| (S_f) \right\|_p = \frac{\zeta(rp)^{2/p}}{\zeta(2rp)^{1/p}} \left(\sum_{i=1}^{\infty} \frac{f^p(i)}{i^{2rp}} \right)^{1/p} = \frac{\zeta(rp)^{2/p}}{\zeta(2rp)^{1/p}} \left(\mathscr{L}_{f^p}(2rp) \right)^{1/p}.$$

Moreover, if $\mathscr{L}_f(rp)$ is convergent then $\lim_{n\to\infty} \left\| (S_f) \right\|_p < \infty$.

Proof. For the ℓ_p norm of the matrix (S_f), we have

$$\begin{aligned}
\lim_{n\to\infty} \left\| (S_f) \right\|_p^p &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{f^p(i,j)}{i^{rp} j^{rp}} \\
&= \sum_{i=1}^{\infty} \frac{1}{i^{rp}} \sum_{j=1}^{\infty} \frac{1}{j^{rp}} \sum_{\substack{d|i \\ d|j}} F(d),
\end{aligned}$$

where $f^p = F * u$. By an easy manipulation between the indices d and j, we have

$$\begin{aligned}
\lim_{n\to\infty} \left\| \left(S_f\right) \right\|_p^p &= \sum_{i=1}^{\infty} \frac{1}{i^{rp}} \sum_{d|i} F(d) \sum_{j=1}^{\infty} \frac{1}{(dj)^{rp}} \\
&= \zeta(rp) \sum_{i=1}^{\infty} \frac{1}{i^{rp}} \sum_{d|i} \frac{F(d)}{d^{rp}} \\
&= \zeta(rp) \sum_{i=1}^{\infty} \frac{\left(F * N^{rp}\right)(i)}{i^{2rp}} \\
&= \zeta(rp) \sum_{i=1}^{\infty} \frac{F(i)}{i^{2rp}} \sum_{i=1}^{\infty} \frac{N^{rp}(i)}{i^{2rp}} \\
&= \zeta(rp)^2 \sum_{i=1}^{\infty} \frac{\left(f^p * \mu\right)(i)}{i^{2rp}} \\
&= \frac{\zeta(rp)^2}{\zeta(2rp)} \sum_{i=1}^{\infty} \frac{f^p(i)}{i^{2rp}} \\
&= \frac{\zeta(rp)^2}{\zeta(2rp)} \mathscr{L}_{f^p}(2rp).
\end{aligned}$$

This is the first part of the theorem. We now consider $\mathscr{L}_{f^p}(2rp)$. If $\mathscr{L}_f(rp)$ is convergent then by the comparison test $\mathscr{L}_{f^p}(2rp)$ converges. Thus the ℓ_p norm of $\left(S_f\right)$ is bounded above by

$$\left(\frac{\zeta(rp)^2}{\zeta(2rp)} \mathscr{L}_{f^p}(2rp) \right)^{1/p}.$$

□

The series $\mathscr{L}_{f^p}(2rp)$ in Theorem 1 can be nicely factored in terms of the Riemann zeta function when f is a completely multiplicative function. It should be noted that there are no general methods for such a factorization when f is not completely multiplicative. In the literature, there are some elegant formulae for particular multiplicative arithmetical functions and particular exponents p. We will discuss such formulae in the last section.

In [1, 10] using this nice factorization the authors obtained sharp upper bounds for some GCD related matrices. As a consequence of Theorem 1 we can give the upper bound in (1.2) (Theorem 3.1 in [10]) and the upper bound in (1.1) (Theorem 3 in [1]). Let $f = N^{(r+s)}$ in Theorem 1. Then we have

$$\begin{aligned}
\lim_{n\to\infty} \left\| \left(S_f\right) \right\|_p^p &= \frac{\zeta(rp)^2}{\zeta(2rp)} \mathscr{L}_{(N^{(r+s)})^p}(2rp) \\
&= \frac{\zeta(rp)^2}{\zeta(2rp)} \sum_{i=1}^{\infty} \frac{1}{i^{(rp-sp)}}
\end{aligned}$$

$$= \frac{\zeta(rp)^2\ \zeta(rp-sp)}{\zeta(2rp)}$$

Taking $s=0$ in the last equality we have the upper bound in (1.1).

We continue with the following theorems as corollaries of the main theorem. We will give sharp upper bounds for the ℓ_p norms of $\left(S_f\right)$ for f being Euler's totient φ, the divisor functions σ_α, and Dedekind's totient ψ, respectively.

THEOREM 2. *Let φ be Euler's totient. Then we have*

$$\lim_{n\to\infty} \left\|\left(S_\varphi\right)\right\|_p = \frac{\zeta(rp)^{2/p}}{\zeta(2rp)^{1/p}} \prod_{\wp}\left(1+\frac{(\wp-1)^p}{\wp^{2rp}-\wp^p}\right)^{1/p},$$

where the product is over all the primes $\wp$.

Proof. Take $f=\varphi$ in Theorem 1. By the fact that φ is multiplicative and the identity (2.4) we have

$$\begin{aligned}
\mathscr{L}_{\varphi^p}(2rp) &= \prod_{\wp}\left\{1+\frac{\varphi^p(\wp)}{\wp^{2rp}}+\frac{\varphi^p(\wp^2)}{\wp^{4rp}}+\frac{\varphi^p(\wp^3)}{\wp^{6rp}}+\cdots\right\}\\
&= \prod_{\wp}\left\{1+\frac{(\wp-1)^p}{\wp^{2rp}}+\frac{\wp^p(\wp-1)^p}{\wp^{4rp}}+\frac{\wp^{2p}(\wp-1)^p}{\wp^{6rp}}+\cdots\right\}\\
&= \prod_{\wp}\left\{1+\frac{(\wp-1)^p}{\wp^{2rp}}+\frac{(\wp-1)^p}{\wp^{(4r-1)p}}+\frac{(\wp-1)^p}{\wp^{(6r-2)p}}+\cdots\right\}\\
&= \prod_{\wp}\left\{1+\frac{(\wp-1)^p}{\wp^{2rp}}\frac{1}{1-\frac{1}{\wp^{(2rp-1)p}}}\right\}\\
&= \prod_{\wp}\left\{1+\frac{(\wp-1)^p}{\wp^{2rp}-\wp^p}\right\}.
\end{aligned}$$

This completes the proof of the theorem. □

THEOREM 3. *Let $\alpha>1$, $rp>1$, and let σ_α be the divisor functions defined in (2.1). Then we have*

$$\lim_{n\to\infty} \left\|\left(S_{\sigma_\alpha}\right)\right\|_p = \frac{\zeta(rp)^{2/p}\ \zeta(\alpha)}{\zeta(2rp)^{1/p}}\left\{\prod_{\wp}\sum_{m=0}^{\infty}\left(\frac{\wp^{\alpha m}-\wp^{-\alpha}}{\wp^{2rm}}\right)^p\right\}^{1/p}$$

where the product is over all the primes $\wp$.

Proof. Take $f = \sigma_\alpha$ in Theorem 1. Recall that σ_α is multiplicative and $\sigma_\alpha(\wp^m) = \sum_{k=0}^{m} \wp^{\alpha k}$ for all $\alpha > 1$. By the identity (2.4) we have

$$\begin{aligned}
\mathscr{L}_{\sigma_\alpha^p}(2rp) &= \sum_{i=1}^{\infty} \frac{\sigma_\alpha^p(i)}{i^{2rp}} \\
&= \prod_{\wp} \sum_{m=0}^{\infty} \left(\frac{1-\wp^{\alpha(m+1)}}{1-\wp^{\alpha}} \right)^p \frac{1}{\wp^{2rpm}} \\
&= \left(\prod_{\wp} \frac{1}{1-\frac{1}{\wp^{\alpha}}} \right)^p \cdot \left\{ \prod_{\wp} \frac{1}{\wp^{\alpha p}} \sum_{m=0}^{\infty} \left(\frac{\wp^{\alpha(m+1)}-1}{\wp^{2rm}} \right)^p \right\} \\
&= \zeta(\alpha)^p \cdot \prod_{\wp} \sum_{m=0}^{\infty} \left(\frac{\wp^{\alpha m}-\wp^{-\alpha}}{\wp^{2rm}} \right)^p .
\end{aligned}$$

Since $\alpha > 1$ and $rp > 1$ $\mathscr{L}_{\sigma_\alpha^p}(2rp)$ is convergent. Then by Theorem 1, the proof is complete. □

THEOREM 4. *Let ψ be Dedekind's totient defined in (2.2) and $rp > 1$. Then we have*

$$\lim_{n\to\infty} \left\| (S_\psi) \right\|_p = \frac{\zeta(rp)^{2/p}}{\zeta(2rp)^{1/p}} \prod_{\wp} \left\{ \frac{\wp^{2rp} - \wp^p + (\wp+1)^p}{\wp^{2rp} - \wp^p} \right\}^{1/p}$$

where the product is over all the primes $\wp$.

Proof. Take $f = \psi$ in Theorem 1. Since ψ is multiplicative we have by the identity (2.4)

$$\begin{aligned}
\mathscr{L}_{\psi^p}(2rp) &= \sum_{i=1}^{\infty} \frac{\psi^p(i)}{i^{2rp}} \\
&= \prod_{\wp} \left\{ 1 + \frac{(\wp+1)^p}{\wp^{2rp}} + \frac{\wp^p(\wp+1)^p}{\wp^{4rp}} + \frac{\wp^{2p}(\wp+1)^p}{\wp^{6rp}} + \cdots \right\} \\
&= \prod_{\wp} \left\{ 1 + \frac{(\wp+1)^p}{\wp^{2rp}} \frac{1}{1-\frac{\wp^p}{\wp^{2rp}}} \right\} \\
&= \prod_{\wp} \left\{ 1 + \frac{(\wp+1)^p}{\wp^{2rp}-\wp^p} \right\}.
\end{aligned}$$

By Theorem 1, the proof is complete. □

Define the class of multiplicative arithmetical functions

$$\mathscr{C} = \{ f : f(\wp) \geqslant \wp^{2r} - \wp^{r} + 1 \text{ for every prime } \wp \text{ and } r > 1 \}.$$

The following theorem gives the maximum row sum matrix norm of (S_f) when $f \in \mathscr{C}$.

THEOREM 5. *Let $f \in \mathscr{C}$ and $\left(S_f\right)$ be the $n \times n$ matrix given in (1.3). Then we have*

$$\left\|\left|\left(S_f\right)\right|\right\|_\infty = \sum_{j=1}^{n} \frac{1}{j^r}.$$

Proof. Let R_i be the $i-$th row sum of $\left(S_f\right)$. Then

$$\begin{aligned} R_i &= \sum_{j=1}^{n} \frac{f(i,j)}{i^r j^r} \\ &= \frac{1}{i^r} \sum_{j=1}^{n} \frac{1}{j^r} \sum_{\substack{d|i \\ d|j}} (f*\mu)(d) \\ &= \frac{1}{i^r} \sum_{d|i} \frac{(f*\mu)(d)}{d^r} \sum_{j=1}^{n} \frac{1}{j^r} \\ &= \frac{1}{i^{2r}} \left((f*\mu)*N^r\right)(i) \sum_{j=1}^{n} \frac{1}{j^r} \\ &= \frac{1}{i^{2r}} (f*J_r)(i) \sum_{j=1}^{n} \frac{1}{j^r}. \end{aligned}$$

f and J_r is multiplicative so is $f * J_r$. Thus it is sufficient to consider the difference between R_1 and $R_\wp$ for a prime $\wp$. Then we have

$$\begin{aligned} R_1 - R_\wp &= \frac{1}{i^{2r}} \left(1 - \frac{1}{\wp^{2r}} (f*J_r)(\wp)\right) \\ &= \frac{1}{i^{2r}} \frac{f(\wp) + \wp^{2r} - \wp^r + 1}{\wp^{2r}}. \end{aligned}$$

Since $f \in \mathscr{C}$ it is clear that $R_1 - R_\wp \geqslant 0$. Thus the maximum row sum matrix norm of $\left(S_f\right)$ is R_1, namely

$$\left\|\left|\left(S_f\right)\right|\right\|_\infty = R_1 = \sum_{j=1}^{n} \frac{1}{j^r}.$$

□

It should be noted that the multiplicative arithmetical functions φ, σ_α and ψ are in $\mathscr{C}$. Thus Theorem 5 holds for φ, σ_α and ψ. Moreover, we have obtained a sharp upper bound for the maximum row sum matrix norm of $\left(S_f\right)$, that is,

$$\lim_{n\to\infty} \left\|\left|\left(S_f\right)\right|\right\|_\infty = \zeta(r) \tag{3.1}$$

for a $f \in \mathscr{C}$ and $r > 1$.

It may be said that our results are direct consequences of known facts. On the other hand, it should be said that we have taken a few steps in the study of matrix norms of

GCD matrices. We will also present a different viewpoint for bounds of infinite prime products we have obtained in Theorem 2-4 by a new method. In the following section, we will discuss this viewpoint.

4. Inequalities For Some Infinite Prime Products

In this section we present some inequalities for some infinite prime products which we have obtained in the previous section.

Let $\left(S_f\right)$ be the matrix defined in (1.3). and let $f=\varphi$. Consider the double sum for the ℓ_p norm of $\left(S_f\right)$

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\varphi^p(i,j)}{i^{rp}j^{rp}}. \tag{4.1}$$

The $p-$th root of the double sum in (4.1) is a sharp upper bound for the ℓ_p norm of $\left(S_\varphi\right)$. On the other hand, the double sum is a sharp upper bound for ℓ_1 norm of the $n\times n$ matrix

$$\left(S_f^p\right)=\left(\frac{\varphi^p(i,j)}{i^{rp}j^{rp}}\right).$$

By (2.5) and Theorem 5 we have

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\varphi^p(i,j)}{i^{rp}j^{rp}}\geqslant\zeta(rp)^2.$$

Then by Theorem 2 we have the following inequality

$$\prod_{\wp}\left(1+\frac{(\wp-1)^p}{\wp^{2rp}-\wp^p}\right)\geqslant\zeta(2rp). \tag{4.2}$$

On the other hand, from the proof of Theorem 2 we have

$$\begin{aligned}\mathscr{L}_{\varphi^p}(2rp) &= \prod_{\wp}\left\{1+\frac{(\wp-1)^p}{\wp^{2rp}-\wp^p}\right\}\\ &= \sum_{i=1}^{\infty}\frac{\varphi(i)^p}{i^{2rp}}\\ &\leqslant \sum_{i=1}^{\infty}\frac{\varphi(i)}{i^{2r}}\\ &= \frac{\zeta(2r-1)}{\zeta(2r)}.\end{aligned}$$

Recall that if $r>1$ then $\sum_{i=1}^{\infty}\frac{\varphi(i)}{i^{2r}}$ converges. From the last two inequalities we have

$$\zeta(2rp)\leqslant\prod_{\wp}\left(1+\frac{(\wp-1)^p}{\wp^{2rp}-\wp^p}\right)\leqslant\frac{\zeta(2r-1)}{\zeta(2r)}. \tag{4.3}$$

The lower and upper bounds given in (4.3) for the infinite prime product in (4.2) are not the best ones. It should be noted that these bounds can be improved. To check these bounds consider the following computer calculations by the aid of Maple 9. For $p=r=2$ we have

$$1.004077357 \leqslant \prod_{\wp}\left(1+\frac{(\wp-1)^p}{\wp^{2rp}-\wp^p}\right) \leqslant 1.110626535.$$

For example, for the first 1000 primes the value of the infinite prime product in (4.2) is 1.004629313. It can be easily seen that the inequalities in (4.3) will be sharper for larger value of r and p.

We now consider the following infinite prime product in Theorem 3

$$\prod_{\wp}\sum_{m=0}^{\infty}\left(\frac{\wp^{\alpha m}-\wp^{-\alpha}}{\wp^{2rm}}\right)^p\Bigg\}.$$

Briefly, using similar procedure above we have the following bounds

$$\frac{\zeta(r)^p\ \zeta(2rp)}{\zeta(rp)^2\ \zeta(\alpha)^p} \leqslant \prod_{\wp}\sum_{m=0}^{\infty}\left(\frac{\wp^{\alpha m}-1}{\wp^{2rm}}\right)^p \leqslant \frac{\zeta(2r)\ \zeta(2r-\alpha)}{\zeta(\alpha)^p}. \tag{4.4}$$

Similarly, we obtain the following inequalities for the infinite prime product in Theorem 4

$$\frac{\zeta(r)^p\ \zeta(2rp)}{\zeta(rp)^2} \leqslant \prod_{\wp}\frac{\wp^{2rp}-\wp^p+(\wp+1)^p}{\wp^{2rp}-\wp^p} \leqslant \frac{\zeta(2rp)\ \zeta(2r)\ \zeta(2r-1)}{\zeta(rp)^2\ \zeta(4r)}. \tag{4.5}$$

5. Discussion and Further Studies

While studying matrix norms of GCD and related matrices, we have tried to tackle the problem about factorization of the Dirichlet series

$$\mathscr{L}_{f^p}(2rp)=\sum_{i=1}^{\infty}\frac{f^p(i)}{i^{2rp}}.$$

in terms of the Riemann zeta function. In the literature, there are some remarkable examples of such a factorization for particular arithmetical functions f and particular exponents p. One of them is due to Ramanujan [5, 7]

$$\sum_{i=1}^{\infty}\frac{\sigma_\alpha(i)\ \sigma_\beta(i)}{i^s}=\frac{\zeta(s)\ \zeta(s-\alpha)\ \zeta(s-\beta)\ \zeta(s-\alpha-\beta)}{\zeta(2s-\alpha-\beta)}$$

for $\operatorname{Re} s > \max\{1, \alpha+1, \beta+1, \alpha+\beta+1\}$. Borwein and Choi [5] generalized the above result. Indeed, they proved that if f_i and g_i are completely multiplicative then

$$\sum_{i=1}^{\infty}\frac{(f_1*g_1)(i)\cdot(f_2*g_2)(i)}{i^s}=\frac{\mathscr{L}_{f_1f_2}(s)\mathscr{L}_{g_1g_2}(s)\mathscr{L}_{f_1g_2}(s)\mathscr{L}_{g_1f_2}(s)}{\mathscr{L}_{f_1f_2g_1g_2}(2s)}.$$

If we focus on our problem in the above sense we can say that we have taken a few steps about such a factorization. Indeed, we only give the Dirichlet series $\mathcal{L}_{f^p}(s)$ for a multiplicative function f as a product of primes. On the other hand, for example, taking $p=2$ and $r=s/4$ in the proof of Theorem 3 we obtain a particular version of Ramanujan's beautiful result, that is,

$$\begin{aligned}\mathcal{L}_{f^p}(s) &= \sum_{i=1}^{\infty} \frac{\sigma_\alpha^2(i)}{i^s} \\ &= \zeta(\alpha)^2 \prod_{\wp} \sum_{m=0}^{\infty} \left(\frac{\wp^{\alpha m} - \wp^{-\alpha}}{\wp^{sm/2}} \right)^2 \\ &= \frac{\zeta(s)\ \zeta(s-\alpha)^2\ \zeta(s-2\alpha)}{\zeta(2s-2\alpha)}.\end{aligned}$$

Moreover, we present some upper and lower bounds for some particular infinite prime products. In further studies these bounds can be improved.

Another interesting viewpoint of the study of GCD matrices is the eigenvalues of these matrices. Lindqvist and Seip [12] obtained the best lower bound and upper bound of an $n \times n$ GCD related matrix $\left(S_{N^2}\right) = \left((i,j)^{2s}/(i^s j^s)\right)$. Indeed, they proved that if $\lambda\left(S_{N^2}\right)$ is an eigenvalue of $\left(S_{N^2}\right)$ then

$$\frac{\zeta(2s)}{\zeta(s)^2} \leqslant \lambda\left(S_{N^2}\right) \leqslant \frac{\zeta(s)^2}{\zeta(2s)}.$$

We are able to obtain upper bounds of the matrices we studied in this paper by using the inequality in (2.6). Since the matrices $\left(S_\varphi\right)$, $\left(S_{\sigma_\alpha}\right)$, and $\left(S_\psi\right)$ are real, symmetric, and positive definite all their eigenvalues are positive reals. Thus, the spectral radius, the spectral norm, and the maximum eigenvalue of $\left(S_f\right)$ coincide for each $f=\varphi$, σ_α, and ψ. Then, by (4.3) we have

$$\lambda\left(S_\varphi\right) \leqslant \frac{\zeta(2r-1)}{\zeta(2r)},$$

where $\lambda\left(S_\varphi\right)$ is an eigenvalue of the matrix $\left(S_\varphi\right)$. By a similar reasoning and the inequalities in (4.4) and (4.5), respectively, we have

$$\lambda\left(S_{\sigma_\alpha}\right) \leqslant \frac{\zeta(2r)\ \zeta(2r-\alpha)}{\zeta(\alpha)^p},$$

and

$$\lambda\left(S_\psi\right) \leqslant \frac{\zeta(2rp)\ \zeta(2r)\ \zeta(2r-1)}{\zeta(rp)^2\ \zeta(4r)}.$$

These upper bounds also hold for spectral norm of these matrices.

Our technique is completely different from Lindqvist and Seip's. They use the Riesz bases in the Hilbert space of Dirichlet series. Additionally, our method does not work for $\left(S_{N^2}\right)$ since the ℓ_p norms of the matrix $\left(S_{N^2}\right)$ are not convergent. On the other hand, their technique does not seem to work for our matrices. Briefly, one will be able to obtain interesting results in some consolidation of mathematical tools aforesaid above.

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Ercan Altınışık
Gazi University, Faculty of Arts and Sciences
Department of Mathematics
06500 Teknikokullar-Ankara
Turkey
e-mail: ealtinisik@gazi.edu.tr