A GENERAL FORM OF JORDAN'S
INEQUALITIES AND ITS APPLICATIONS

LING ZHU

(*communicated by P. Bullen*)

Abstract. In this work, the general form of Jordan's double inequalities:

$$P_{2N}(x) + \alpha(r^2 - x^2)^{N+1} \leqslant \frac{\sin x}{x} \leqslant P_{2N}(x) + \beta(r^2 - x^2)^{N+1}$$

are established, where $x \in (0, r], r \leqslant \frac{\pi}{2}, P_{2N}(x) = \sum_{n=0}^{N} a_n(r^2 - x^2)^n, a_0 = \frac{\sin r}{r}, a_1 = \frac{\sin r - r \cos r}{2r^3}, a_{n+1} = \frac{2n+1}{2(n+1)r^2}a_n - \frac{1}{4n(n+1)r^2}a_{n-1}$, $N \geqslant 0$ is a natural number, $\alpha = a_{N+1}$ and $\beta = \frac{1 - \sum_{n=0}^{N} a_n r^{2n}}{r^{2(N+1)}}$ are the best constants in inequalities above. The application of the result above give a new infinite series $(\sin x)/x = \sum_{n=0}^{\infty} a_n(r^2 - x^2)^n$ for $0 < |x| \leqslant r \leqslant \pi/2$, the general improvement of Yang Le inequality, and a general form of Kober's double inequality.

1. Introduction

The following result is known as Jordan's inequality [1]:

THEOREM 1. *If* $0 < x \leqslant \pi/2$, *then*

$$\frac{2}{\pi} \leqslant \frac{\sin x}{x} < 1 \tag{1}$$

with equality if and only if $x = \pi/2$.

Qi, Cui and Xu [2] showed a new lower and upper bounds for the function $\frac{\sin x}{x}$, and obtained the following results.

THEOREM 2. *If* $0 < x \leqslant \pi/2$, *then*

$$\frac{3}{\pi} - \frac{4}{\pi^3}x^2 \leqslant \frac{\sin x}{x} \leqslant 1 - \frac{4(\pi - 2)}{\pi^3}x^2. \tag{2}$$

Theorem 2 is equivalent to the following

THEOREM 3. *If $0 < x \leqslant \pi/2$, then*

$$\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) \leqslant \frac{\sin x}{x} \leqslant \frac{2}{\pi} + \frac{\pi - 2}{\pi^3}(\pi^2 - 4x^2). \tag{3}$$

Debnath and Zhao [3] proved the left inequality in (3) using other method. In fact, let $g(x) = \frac{\frac{\sin x}{x} - \frac{2}{\pi}}{\pi^2 - 4x^2}$, then $\lim_{x \to 0^+} g(x) = \frac{\pi - 2}{\pi^3}$ and $\lim_{x \to \frac{\pi}{2}^-} g(x) = \frac{1}{\pi^3}$, we find that $\frac{1}{\pi^3}$ and $\frac{\pi - 2}{\pi^3}$ are best constants in (3).

Now, let $0 < x \leqslant r \leqslant \frac{\pi}{2}$, then $(\frac{\sin x}{x})' = \frac{\cos x(x - \tan x)}{x^2} < 0$, so $\frac{\sin x}{x}$ is decreasing on $(0, r]$, we obtain

THEOREM 4. *If $0 < x \leqslant r \leqslant \pi/2$, then*

$$\frac{\sin r}{r} \leqslant \frac{\sin x}{x} < 1. \tag{4}$$

The equality in (4) holds if and only if $x = r$.

Put $r = \pi/2$ in (4), then (1) holds by (4).

The author of this paper [4] obtained a further result:

THEOREM 5. *If $0 < x \leqslant r \leqslant \pi/2$, then*

$$\frac{\sin r}{r} + \frac{\sin r - r\cos r}{2r^3}(r^2 - x^2) \leqslant \frac{\sin x}{x} \leqslant \frac{\sin r}{r} + \frac{r - \sin r}{r^3}(r^2 - x^2). \tag{5}$$

Furthermore, $\frac{\sin r - r\cos r}{2r^3}$ and $\frac{r - \sin r}{r^3}$ are best constants in (5).

Clearly, Theorem 5 is the generalization of Theorem 2 or Theorem 3.

In the form of (4) and (5), we finally obtain the general Jordan's double inequalities as follows

THEOREM 6 (ZHU-JORDAN INEQUALITY). *Let $0 < x \leqslant r \leqslant \pi/2$ and $N \geqslant 0$ be a natural number, then*

$$P_{2N}(x) + \alpha(r^2 - x^2)^{N+1} \leqslant \frac{\sin x}{x} \leqslant P_{2N}(x) + \beta(r^2 - x^2)^{N+1} \tag{6}$$

with the equalities if and only if $x = r$, where, $P_{2N}(x) = \sum_{n=0}^{N} a_n(r^2 - x^2)^n$ and

$$a_0 = \frac{\sin r}{r}, a_1 = \frac{\sin r - r\cos r}{2r^3}, a_{n+1} = \frac{2n+1}{2(n+1)r^2}a_n - \frac{1}{4n(n+1)r^2}a_{n-1}, n = 1, 2, \cdots.$$

Furthermore, $\alpha = a_{N+1}$ and $\beta = \frac{1 - \sum_{n=0}^{N} a_n r^{2n}}{r^{2(N+1)}}$ are the best constants in (6).

Letting $r = \pi/2$ in Theorem 6, we obtain the following results.

THEOREM 7. *Let $0 < x \leqslant \pi/2$ and $N \geqslant 0$ be a natural number, then*

$$Q_{2N}(x) + b_{N+1}(\pi^2 - 4x^2)^{N+1} \leqslant \frac{\sin x}{x} \leqslant Q_{2N}(x) + \frac{1 - \sum_{n=0}^{N} b_n \pi^{2n}}{\pi^{2(N+1)}}(\pi^2 - 4x^2)^{N+1} \quad (7)$$

with the equalities if and only if $x = \pi/2$, where, $Q_{2N}(x) = \sum_{n=0}^{N} b_n(\pi^2 - 4x^2)^n$ and

$$b_0 = \frac{2}{\pi}, b_1 = \frac{1}{\pi^3}, b_{n+1} = \frac{2n+1}{2(n+1)\pi^2} b_n - \frac{1}{16n(n+1)\pi^2} b_{n-1}, n = 1, 2, \cdots.$$

Furthermore, b_{N+1} and $\frac{1 - \sum_{n=0}^{N} b_n \pi^{2n}}{\pi^{2(N+1)}}$ are the best constants in (7).

2. Six lemmas

LEMMA 1 ([5,6]). *Let $f, g : [a, b] \to \mathbb{R}$ be two continuous functions which are differentiable on (a, b). Further, let $g' \neq 0$ on (a, b). If f'/g' is increasing (or decreasing) on (a, b), then the functions*

$$\frac{f(x) - f(b)}{g(x) - g(b)}$$

and

$$\frac{f(x) - f(a)}{g(x) - g(a)}$$

are also increasing (or decreasing) on (a, b).

LEMMA 2 ([7,8]). *Let $j_n(x)$ be the Spherical Bessel Functions of the first kind, $j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x)$, then*

$$j_n(x) = (-x)^n \left[\frac{1}{x} \frac{d}{dx} \right]^n \frac{\sin x}{x}, n = 0, 1, 2, \cdots. \quad (8)$$

LEMMA 3 ([9,10]). *Let $j_n(x)$ be the SBFs of the first kind, $j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x)$, then*

$$j_{n+1}(x) = \frac{2n+1}{x} j_n(x) - j_{n-1}(x) \quad (9)$$

or

$$(2n+1)j_n(x) = x[j_{n+1}(x) + j_{n-1}(x)]. \quad (10)$$

LEMMA 4 ([9]). *Let $j_n(x)$ be the SBFs of the first kind, $j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x)$, then*

$$x j_n'(x) = n j_n(x) - x j_{n+1}(x). \quad (11)$$

In the following, we show some new results about the properties of a function involving $j_n(x)$.

LEMMA 5. *Let $j_n(x)$ be the SBFs of the first kind, $j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x)$, $h_{n+1}(x) = x^{n+2} j_{n+1}(x)$, then*

$$h'_{n+1}(x) = x h_n(x)$$

or

$$\left[\frac{1}{x}\frac{d}{dx}\right] h_{n+1}(x) = h_n(x), n = 0, 1, 2, \cdots . \tag{12}$$

Proof. We know that $h'_{n+1}(x) = x^{n+1}[(n+2)j_{n+1}(x) + x j'_{n+1}(x)]$, and $x h_n(x) = x(x^{n+1} j_n(x)) = x^{n+1}[x j_n(x)]$, so (12) holds when we prove

$$(n+2)j_{n+1}(x) + x j'_{n+1}(x) = x j_n(x) \tag{13}$$

or

$$(n+1)j_n(x) + x j'_n(x) = x j_{n-1}(x). \tag{14}$$

In fact, (14) comes from (10) and (11).

LEMMA 6. *Let $j_n(x)$ be the SBFs of the first kind, $j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x)$, $h_{n+1}(x) = x^{n+2} j_{n+1}(x)$, then*

$$\left[\frac{1}{x}\frac{d}{dx}\right]^n h_n(x) = x j_0(x) = \sin x, n = 0, 1, 2, \cdots . \tag{15}$$

Proof. we finish proof of (15) by mathematical induction.
(i) When $n = 1$, $\frac{1}{x}\frac{d}{dx} h_1(x) = \frac{1}{x}\frac{d}{dx}[x^2 j_1(x)] = \frac{1}{x}\frac{d}{dx}[x^2(-\frac{\cos x}{x} + \frac{\sin x}{x^2})] = \sin x$.
(ii) Supposing (15) holds for $n = N$ or $\left[\frac{1}{x}\frac{d}{dx}\right]^N h_N(x) = \sin x$.
By (12) we have

$$\left[\frac{1}{x}\frac{d}{dx}\right]^{N+1} h_{N+1}(x) = \left[\frac{1}{x}\frac{d}{dx}\right]^N \left[\frac{1}{x}\frac{d}{dx}\right] h_{N+1}(x) = \left[\frac{1}{x}\frac{d}{dx}\right]^N h_N(x) = \sin x.$$

That is, (15) holds for $n = N + 1$.

3. A concise proof of Theorem 5

Let $F(x) = \frac{\frac{\sin x}{x} - P_{2N}(x)}{(r^2 - x^2)^{N+1}} = \frac{f_1(x)}{g_1(x)} = \frac{f_1(x) - f_1(r)}{g_1(x) - g_1(r)}$, where $f_1(x) = \frac{\sin x}{x} - P_{2N}(x)$, and $g_1(x) = (r^2 - x^2)^{N+1}$;

Let $\lim\limits_{x \to r^-} f_1(x) = 0$, we have $a_0 = \frac{\sin r}{r}$ and

$$\begin{aligned}
\frac{f'_1(x)}{g'_1(x)} &= \frac{\frac{d}{dx}\frac{\sin x}{x} - \sum_{n=1}^{N} n a_n (r^2 - x^2)^{n-1}(-2x)}{(N+1)(r^2 - x^2)^N(-2x)} \\
&= \frac{\frac{1}{-2}\left[\frac{1}{x}\frac{d}{dx}\right]\frac{\sin x}{x} - \sum_{n=1}^{N} n a_n (r^2 - x^2)^{n-1}}{(N+1)(r^2 - x^2)^N} \\
&= \frac{f_2(x)}{g_2(x)} = \frac{f_2(x) - f_2(r)}{g_2(x) - g_2(r)},
\end{aligned}$$

where $f_2(x) = \frac{1}{-2}\left[\frac{1}{x}\frac{d}{dx}\right]\frac{\sin x}{x} - \sum_{n=1}^{N} na_n(r^2 - x^2)^{n-1}$, and $g_2(x) = (N+1)(r^2 - x^2)^N$;

Let $\lim_{x \to r^-} f_2(x) = 0$, we have $a_1 = \frac{\sin r - r\cos r}{2r^3}$ and

$$\frac{f_2'(x)}{g_2'(x)} = \frac{\frac{1}{(-2)^2}\left[\frac{1}{x}\frac{d}{dx}\right]^2 \frac{\sin x}{x} - \sum_{n=2}^{N} n(n-1)a_n(r^2 - x^2)^{n-2}}{(N+1)N(r^2 - x^2)^{N-1}}$$

$$= \frac{f_3(x)}{g_3(x)} = \frac{f_3(x) - f_3(r)}{g_3(x) - g_3(r)},$$

where $f_3(x) = \frac{1}{(-2)^2}\left[\frac{1}{x}\frac{d}{dx}\right]^2 \frac{\sin x}{x} - \sum_{n=2}^{N} n(n-1)a_n(r^2 - x^2)^{n-2}$, and $g_3(x) = (N+1)N(r^2 - x^2)^{N-1}$;

$$\cdots\cdots;$$

$$\frac{f_N'(x)}{g_N'(x)} = \frac{\frac{1}{(-2)^N}\left[\frac{1}{x}\frac{d}{dx}\right]^N \frac{\sin x}{x} - \sum_{n=N}^{N} n(n-1)\cdots 2a_n}{(N+1)N\cdots 2(r^2 - x^2)}$$

$$= \frac{f_{N+1}(x)}{g_{N+1}(x)} = \frac{f_{N+1}(x) - f_{N+1}(r)}{g_{N+1}(x) - g_{N+1}(r)},$$

where $f_{N+1}(x) = \frac{1}{(-2)^N}\left[\frac{1}{x}\frac{d}{dx}\right]^N \frac{\sin x}{x} - \sum_{n=N}^{N} n(n-1)\cdots 2a_n$, and $g_{N+1}(x) = (N+1)N\cdots 2(r^2 - x^2)$.

Let $\lim_{x \to r^-} f_{N+1}(x) = 0$, we have $a_N = \frac{\frac{1}{(-2)^N}\left[\frac{1}{x}\frac{d}{dx}\right]^N \frac{\sin x}{x}|_{x=r}}{N!} = \frac{\frac{j_N(r)}{(-r)^N}}{(-2)^N N!}$ by Lemma 2. That is,

$$a_N = \frac{1}{(2r)^N N!} j_N(r). \tag{16}$$

From (9) and (16), we obtain

$$a_N = \frac{1}{(2r)^N N!}\left[\frac{2N-1}{r} j_{N-1}(r) - j_{N-2}(r)\right]$$

$$= \frac{2N-1}{2Nr^2} a_{N-1} - \frac{1}{4N(N-1)r^2} a_{N-2}. \tag{17}$$

At the same time,

$$\frac{f_{N+1}'(x)}{g_{N+1}'(x)} = \frac{\frac{1}{(-2)^{N+1}}\left[\frac{1}{x}\frac{d}{dx}\right]^{N+1}\frac{\sin x}{x}}{(N+1)!}$$

$$= \frac{1}{2^{N+1}(N+1)!}\frac{j_{N+1}(x)}{x^{N+1}} = \frac{1}{2^{N+1}(N+1)!}\frac{h_{N+1}(x)}{x^{2N+3}}$$

$$= \frac{1}{2^{N+1}(N+1)!}\frac{p_1(x)}{q_1(x)} = \frac{1}{2^{N+1}(N+1)!}\frac{p_1(x) - p_1(0)}{q_1(x) - q_1(0)},$$

where $p_1(x) = h_{N+1}(x)$, and $q_1(x) = x^{2N+3}$;

Then

$$\frac{p_1'(x)}{q_1'(x)} = \frac{\frac{d}{dx}h_{N+1}(x)}{(2N+3)x^{2N+2}} = \frac{\left[\frac{1}{x}\frac{d}{dx}\right]h_{N+1}(x)}{(2N+3)x^{2N+1}}$$

$$= \frac{p_2(x)}{q_2(x)} = \frac{p_2(x) - p_2(0)}{q_2(x) - q_2(0)},$$

where $p_2(x) = \left[\frac{1}{x}\frac{d}{dx}\right]h_{N+1}(x)$, and $q_2(x) = (2N+3)x^{2N+1}$;

$$\frac{p_2'(x)}{q_2'(x)} = \frac{\left[\frac{1}{x}\frac{d}{dx}\right]^2 h_{N+1}(x)}{(2N+3)(2N+1)x^{2N-1}} = \frac{p_3(x)}{q_3(x)} = \frac{p_3(x) - p_3(0)}{q_3(x) - q_3(0)},$$

where $p_3(x) = \left[\frac{1}{x}\frac{d}{dx}\right]^2 h_{N+1}(x)$, and $q_3(x) = (2N+3)(2N+1)x^{2N-1}$;

$$\cdots\cdots;$$

$$\frac{p_{N+1}'(x)}{q_{N+1}'(x)} = \frac{\left[\frac{1}{x}\frac{d}{dx}\right]^{N+1} h_{N+1}(x)}{[(2N+3)!!]x}.$$

By Lemma 6 we know that $\frac{p_{N+1}'(x)}{q_{N+1}'(x)} = \frac{\frac{\sin x}{x}}{(2N+3)!!}$, which is decreasing on $(0, r)$. Therefore, we obtain that $F(x)$ is decreasing on $(0, r)$ by using Lemma 1 repeatedly.

Furthermore,

$$\lim_{x \to 0^+} F(x) = \frac{1 - \sum_{n=0}^{N} a_n r^{2n}}{r^{2(N+1)}},$$

$$\lim_{x \to r^-} F(x) = \lim_{x \to r^-} \frac{\frac{\sin x}{x} - P_{2N}(x)}{(r^2 - x^2)^{N+1}} = \lim_{x \to r^-} \frac{f_{N+1}'(x)}{g_{N+1}'(x)}$$

$$= \lim_{x \to r^-} \frac{\frac{1}{(-2)^{N+1}}\left[\frac{1}{x}\frac{d}{dx}\right]^{N+1}\frac{\sin x}{x}}{(N+1)!} = \frac{1}{2^{N+1}(N+1)!}\frac{j_{N+1}(r)}{r^{N+1}}$$

$$= a_{N+1}.$$

Thus $\frac{1 - \sum_{n=0}^{N} a_n r^{2n}}{r^{2(N+1)}}$ and a_{N+1} are the best constants in (6).

4. New infinite series for $\frac{\sin x}{x}$

In this section, we consider to show a new infinite series of $(\sin x)/x$ and a property of the constants in series.

THEOREM 8. *Let $0 < x \leqslant r \leqslant \pi/2$ and $N \geqslant 0$ be a natural number, then*

$$\frac{\sin x}{x} = P_{2N}(x) + R_{2N+2}, \tag{18}$$

where $P_{2N}(x) = \sum_{n=0}^{N} a_n(r^2 - x^2)^n$ and

$$a_0 = \frac{\sin r}{r}, a_1 = \frac{\sin r - r\cos r}{2r^3}, a_{n+1} = \frac{2n+1}{2(n+1)r^2}a_n - \frac{1}{4n(n+1)r^2}a_{n-1}, n = 1, 2, \cdots.$$

The reminder term is

$$R_{2N+2} = \frac{j_{N+1}(\xi)}{(N+1)!(2\xi)^{N+1}}(r^2 - x^2)^{N+1}, 0 < \xi < r \leqslant \frac{\pi}{2} \tag{19}$$

or

$$R_{2N+2} = \frac{1}{2^{N+1}(N+1)!(2N+3)!!}\frac{\sin \eta}{\eta}(r^2 - x^2)^{N+1}, 0 < \eta < r \leqslant \frac{\pi}{2}. \tag{20}$$

Proof. Let $\frac{\frac{\sin x}{x} - P_{2N}(x)}{(r^2 - x^2)^{N+1}} = \frac{f_1(x)}{g_1(x)} = \frac{f_1(x) - f_1(r)}{g_1(x) - g_1(r)}$, where $f_1(x) = \frac{\sin x}{x} - P_{2N}(x)$, and $g_1(x) = (r^2 - x^2)^{N+1}$. By using Cauchy mean value theorem repeatedly we obtain

$$\begin{aligned}
\frac{\frac{\sin x}{x} - P_{2N}(x)}{(r^2 - x^2)^{N+1}} &= \frac{f_1'(\xi_1)}{g_1'(\xi_1)} = \frac{f_1'(\xi_1) - f_1'(r)}{g_1'(\xi_1) - g_1'(r)} \\
&= \frac{f_1''(\xi_2)}{g_1''(\xi_2)} = \frac{f_1''(\xi_2) - f_1''(r)}{g_1''(\xi_2) - g_1''(r)} \\
&\vdots \\
&= \frac{1}{2^{N+1}(N+1)!}\frac{j_{N+1}(\xi)}{\xi^{N+1}}, 0 < \xi < r \leqslant \frac{\pi}{2} \\
&= \frac{1}{2^{N+1}(N+1)!}\frac{h_{N+1}(\xi)}{\xi^{2N+3}} \\
&= \frac{1}{2^{N+1}(N+1)!}\frac{p_1(\xi)}{q_1(\xi)} = \frac{1}{2^{N+1}(N+1)!}\frac{p_1(\xi) - p_1(0)}{q_1(\xi) - q_1(0)},
\end{aligned}$$

where $p_1(\xi) = h_{N+1}(\xi)$, and $q_1(\xi) = \xi^{2N+3}$.

Then

$$\begin{aligned}
\frac{\frac{\sin x}{x} - P_{2N}(x)}{(r^2 - x^2)^{N+1}} &= \frac{1}{2^{N+1}(N+1)!}\frac{p_1'(\eta_1)}{q_1'(\eta_1)} = \frac{1}{2^{N+1}(N+1)!}\frac{p_1'(\eta_1) - p_1'(0)}{q_1'(\eta_1) - q_1'(0)} \\
&= \frac{1}{2^{N+1}(N+1)!}\frac{p_1''(\eta_2)}{q_1''(\eta_2)} = \frac{1}{2^{N+1}(N+1)!}\frac{p_1''(\eta_2) - p_1''(0)}{q_1''(\eta_2) - q_1''(0)} \\
&\vdots \\
&= \frac{1}{2^{N+1}(N+1)!}\frac{\left[\frac{1}{x}\frac{d}{dx}\right]^{N+1}h_{N+1}(x)|_{x=\eta}}{(2N+3)!!\eta} \\
&= \frac{1}{2^{N+1}(N+1)!(2N+3)!!}\frac{\sin \eta}{\eta}, \quad 0 < \eta < r \leqslant \frac{\pi}{2}.
\end{aligned}$$

So

$$Q_{2N+2} = \frac{j_{N+1}(\xi)}{(N+1)!(2\xi)^{N+1}}(r^2 - x^2)^{N+1}, 0 < \xi < r \leqslant \frac{\pi}{2}$$

or

$$Q_{2N+2} = \frac{1}{2^{N+1}(N+1)!(2N+3)!!} \frac{\sin \eta}{\eta} (r^2 - x^2)^{N+1}, \quad 0 < \eta < r \leqslant \frac{\pi}{2}.$$

Since $\lim\limits_{N \to +\infty} Q_{2N+2} = 0$ from (20), we can obtain the result about a new expansion for $(\sin x)/x$ as follows

THEOREM 9. *Let* $0 < |x| \leqslant r \leqslant \pi/2$, *then*

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} a_n (r^2 - x^2)^n, \tag{21}$$

where

$$a_0 = \frac{\sin r}{r}, \quad a_1 = \frac{\sin r - r \cos r}{2r^3},$$

$$a_{n+1} = \frac{2n+1}{2(n+1)r^2} a_n - \frac{1}{4n(n+1)r^2} a_{n-1}, n = 1, 2, \cdots.$$

At the same time, the constants in series (21) have the following property

$$\sum_{n=0}^{\infty} a_n r^{2n} = \lim_{x \to 0} \frac{\sin x}{x} = 1. \tag{22}$$

Particularly, let $r = \pi/2$ in Theorem 9, we have

THEOREM 10. *Let* $0 < |x| \leqslant \pi/2$, *then*

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} b_n (\pi^2 - 4x^2)^n, \tag{23}$$

where

$$b_0 = \frac{2}{\pi}, \quad b_1 = \frac{1}{\pi^3}, \quad b_{n+1} = \frac{2n+1}{2(n+1)\pi^2} b_n - \frac{1}{16n(n+1)\pi^2} b_{n-1}, n = 1, 2, \cdots.$$

And

$$\sum_{n=0}^{\infty} b_n \pi^{2n} = \lim_{x \to 0} \frac{\sin x}{x} = 1. \tag{24}$$

5. The application of general Jordan's inequality

Let $H_{ij} = \cos^2 \lambda A_i + \cos^2 \lambda A_j - 2 \cos \lambda \pi \cos \lambda A_i \cos \lambda A_j$. It follows from [11] that

$$\sin^2 \lambda \pi \leqslant H_{ij} \leqslant 4 \sin^2 \frac{\lambda}{2} \pi. \tag{25}$$

Let $1 \leqslant i < j \leqslant n$. Taking the sum for all inequalities in (25), we obtain

$$\sum_{1 \leqslant i < j \leqslant n} \sin^2 \lambda \pi \leqslant \sum_{1 \leqslant i < j \leqslant n} H_{ij} \leqslant \sum_{1 \leqslant i < j \leqslant n} 4 \sin^2 \frac{\lambda}{2} \pi. \tag{26}$$

It follows from the definition of H_{ij} that

$$\sum_{1 \leqslant i < j \leqslant n} H_{ij} = \sum_{1 \leqslant i < j \leqslant n} (\cos^2 \lambda A_i + \cos^2 \lambda A_j - 2 \cos \lambda \pi \cos \lambda A_i \cos \lambda A_j)$$

$$= (n-1) \sum_{k=1}^{n} \cos^2 \lambda A_k - 2 \cos \lambda \pi \sum_{1 \leqslant i < j \leqslant n} \cos \lambda A_i \cos \lambda A_j. \tag{27}$$

Making use of the inequalities in (6) we obtain

$$\sum_{1 \leqslant i < j \leqslant n} 4 \sin^2 \frac{\lambda}{2} \pi \leqslant \sum_{1 \leqslant i < j \leqslant n} 4 (\frac{\lambda}{2} \pi)^2 [P_{2N}(\frac{\lambda}{2} \pi) + \frac{1 - \sum_{n=0}^{N} a_n r^{2n}}{r^{2(N+1)}} (r^2 - \frac{1}{4} \lambda^2 \pi^2)^{N+1}]^2$$

$$= \binom{n}{2} (\lambda \pi)^2 [P_{2N}(\frac{\lambda}{2} \pi) + \frac{1 - \sum_{n=0}^{N} a_n r^{2n}}{r^{2(N+1)}} (r^2 - \frac{1}{4} \lambda^2 \pi^2)^{N+1}]^2 \tag{28}$$

and

$$\sum_{1 \leqslant i < j \leqslant n} \sin^2 \lambda \pi = 4 \cos^2 \frac{\lambda}{2} \pi \sum_{1 \leqslant i < j \leqslant n} \sin^2 \frac{\lambda}{2} \pi$$

$$\geqslant 4 \cos^2 \frac{\lambda}{2} \pi \sum_{1 \leqslant i < j \leqslant n} (\frac{\lambda}{2} \pi)^2 [P_{2N}(\frac{\lambda}{2} \pi) + a_{N+1}(r^2 - \frac{1}{4} \lambda^2 \pi^2)^{N+1}]^2$$

$$= \binom{n}{2} (\lambda \pi)^2 [P_{2N}(\frac{\lambda}{2} \pi) + a_{N+1}(r^2 - \frac{1}{4} \lambda^2 \pi^2)^{N+1}]^2 \cos^2 \frac{\lambda}{2} \pi, \tag{29}$$

where $0 < r \leqslant \pi/2$.

Substituting (28) and (29) into (26), we obtain the general improvement of Yang Le inequality as follows

THEOREM 11. *Let $A_i > 0$ $(i = 1, 2, \cdots, n)$ with $\sum_{i=1}^{n} A_i \leqslant \pi$, let $0 \leqslant \lambda \leqslant 1$, $0 < r \leqslant \pi/2$, and let $n \geqslant 2$ be a natural number. Then*

$$L(\lambda) \leqslant (n-1) \sum_{k=1}^{n} \cos^2 \lambda A_k - 2 \cos \lambda \pi \sum_{1 \leqslant i < j \leqslant n} \cos \lambda A_i \cos \lambda A_j \leqslant R(\lambda), \tag{30}$$

where,

$$L(\lambda) = \binom{n}{2} (\lambda \pi)^2 [P_{2N}(\frac{\lambda}{2} \pi) + a_{N+1}(r^2 - \frac{1}{4} \lambda^2 \pi^2)^{N+1}]^2 \cos^2 \frac{\lambda}{2} \pi,$$

$$R(\lambda) = \binom{n}{2} (\lambda \pi)^2 [P_{2N}(\frac{\lambda}{2} \pi) + \frac{1 - \sum_{n=0}^{N} a_n r^{2n}}{r^{2(N+1)}} (r^2 - \frac{1}{4} \lambda^2 \pi^2)^{N+1}]^2.$$

6. A general form of Kober's inequality

The following result is known as Kober's inequality [1,12]:

THEOREM 12. *(1) If $0 < u < 1$, then $\cos \frac{\pi}{2}u > 1 - u$;*
(2) If $1 < u < 2$, then $\cos \frac{\pi}{2}u < 1 - u$.

Qi et al.[13,14] had done some refinement work on Kober's inequality. In this section, we obtain a general Kober's double inequality as follows

THEOREM 13 (ZHU-KOBER INEQUALITY). *Let $u \in [0,2]$, $R(u) = \sum_{n=0}^{N} \frac{b_n}{2}\pi^{2n+1}(1 - u)u^n(2 - u)^n$, $S(u) = \frac{1}{2}\pi^{2N+3}(1 - u)u^{N+1}(2 - u)^{N+1}$, $b_0 = \frac{2}{\pi}$, $b_1 = \frac{1}{\pi^3}$, $b_{n+1} = \frac{2n+1}{2(n+1)\pi^2}b_n - \frac{1}{16n(n+1)\pi^2}b_{n-1}$, $n = 1, 2, \cdots$, and $N \geqslant 0$ be a natural number, then*

$$R(u) + \lambda S(u) \leqslant \cos \frac{\pi}{2}u \leqslant R(u) + \mu S(u) \tag{31}$$

holds in cases:

(a) When $0 \leqslant u \leqslant 1$, we have $\lambda = b_{N+1}$, $\mu = \frac{1 - \sum_{n=0}^{N} b_n \pi^{2n}}{\pi^{2(N+1)}}$;

(b) When $1 \leqslant u \leqslant 2$, we have $\lambda = \frac{1 - \sum_{n=0}^{N} b_n \pi^{2n}}{\pi^{2(N+1)}}$, $\mu = b_{N+1}$.
Furthermore, these paired numbers λ and μ are the best constants in (31).

Proof of Theorem 13. We have the following result from (7):

$$\sum_{n=0}^{N} b_n x(\pi^2 - 4x^2)^n + \alpha x(\pi^2 - 4x^2)^{N+1} \leqslant \sin x \leqslant \sum_{n=0}^{N} b_n x(\pi^2 - 4x^2)^n + \beta x(\pi^2 - 4x^2)^{N+1}, \tag{32}$$

where $0 \leqslant x \leqslant \frac{\pi}{2}$, $\alpha = b_{N+1}$, and $\beta = \frac{1 - \sum_{n=0}^{N} b_n \pi^{2n}}{\pi^{2(N+1)}}$.
We deal with (32) in two cases.
(a) Let $\frac{\pi}{2}(1 - u) = x$ in (32), then $0 \leqslant u \leqslant 1$ and

$$\sum_{n=0}^{N} b_n \frac{\pi}{2}(1 - u)\pi^{2n}u^n(2 - u)^n + \alpha \frac{\pi}{2}(1 - u)\pi^{2(N+1)}u^{N+1}(2 - u)^{N+1} \leqslant \cos \frac{\pi}{2}u$$

$$\leqslant \sum_{n=0}^{N} b_n \frac{\pi}{2}(1 - u)\pi^{2n}u^n(2 - u)^n + \beta \frac{\pi}{2}(1 - u)\pi^{2(N+1)}u^{N+1}(2 - u)^{N+1},$$

that is,

$$R(u) + \lambda S(u) \leqslant \cos \frac{\pi}{2}u \leqslant R(u) + \mu S(u),$$

where $\lambda = \alpha = b_{N+1}$, and $\mu = \beta = \frac{1 - \sum_{n=0}^{N} b_n \pi^{2n}}{\pi^{2(N+1)}}$.
(b) Let $\frac{\pi}{2}(u - 1) = x$ in (32), then $1 \leqslant u \leqslant 2$ and

$$\sum_{n=0}^{N} b_n \frac{\pi}{2}(u - 1)\pi^{2n}u^n(2 - u)^n + \alpha \frac{\pi}{2}(u - 1)\pi^{2(N+1)}u^{N+1}(2 - u)^{N+1} \leqslant -\cos \frac{\pi}{2}u$$

$$\leqslant \sum_{n=0}^{N} b_n \frac{\pi}{2}(u-1)\pi^{2n}u^n(2-u)^n + \beta\frac{\pi}{2}(u-1)\pi^{2(N+1)}u^{N+1}(2-u)^{N+1},$$

that is,

$$R(u) + \lambda S(u) \leqslant \cos\frac{\pi}{2}u \leqslant R(u) + \mu S(u),$$

where $\lambda = \beta = \frac{1-\sum_{n=0}^{N}b_n\pi^{2n}}{\pi^{2(N+1)}}$, and $\mu = \alpha = b_{N+1}$.

REFERENCES

[1] D. S. MITRINOVIC, *Analytic inequalities*, Springer-Verlag, 1970.

[2] F. QI, L.-H. CUI AND S.-L. XU, *Some Inequalities Constructed by Tchebysheff's Integral Inequality*, Mathematical Inequalities and Applications **4** (1999), 517–528.

[3] L. DEBNATH AND C. J. ZHAO, *New Strengthened Jordan's Inequality and Its Applications*, Mathematical Inequalities and Applications **16** (4) (2003), 557–560.

[4] L. ZHU, *Sharpening of Jordan's Inequalities and Its Applications*, Mathematical Inequalities and Applications, **9** (1) (2006), 103–106.

[5] G. D. ANDERSON, M. K. VAMANAMURTHY, AND M. VUORINEN, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, New York, 1997.

[6] G. D. ANDERSON, S.-L. QIU, M. K. VAMANAMURTHY, AND M. VUORINEN, *Generalized Elliptic Integral and Modular Equations*, Pacific J.Math., **192** (2000), 1–37.

[7] M. ABRAMOWITZ, I. STEGUN, *Handbook of Mathematical Functions, U.S. National Bureau of Standards, Washington, DC*, 1964.

[8] D. ZWILLINGER, *CRC Standard Mathematical Tables and Formulae*, CRC Press, 1996.

[9] F. E. HARRIS, *Spherical Bessel Expansions of Sine, Cosine, and Exponential Integrals*, Applied Numerical Mathematics, **34** (2000), 95–98.

[10] J. L. BASTARDO, S. ABRAHAM IBRAHIM, P. FERNFINDEZ DE CORDOBA, J. F. URCHUEGUFA SCHOLZEL, YU. L. RATIS, *Evaluation of Fresnel Integrals Based on the Continued Fractions Method*, Applied Mathematics Letters, **18** (2005), 23–28.

[11] C. J. ZHAO, *Generalization and Strengthen of Yang Le Inequality*, Math. Practice Theory (in Chinese), **4** (2000), 493–497.

[12] H. KOBER, *Approximation by integral functions in the complex domain*, Trans. Amer. Math. Soc. **56** (1944), 22.

[13] F. QI, *Refinements and extensions of Jordan's and Kober's inequalities*, Gongke Shuxue (Journal of Mathematics for Technology) **12** (4) (1996), 98–102. (Chinese).

[14] F. QI AND Q.-D. HAO, *Refinements and sharpenings of Jordan's and Kober's inequality*, Mathematics and Informatics Quarterly **8** (3) (1998), 116–120.

Ling Zhu
Department of Mathematics
Zhejiang Gongshang University
Hangzhou
Zhejiang 310035
P. R. of China
e-mail: zhuling0571@163.com

Mathematical Inequalities & Applications
www.ele-math.com
mia@ele-math.com