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AN APPROACH TO KY FAN TYPE INEQUALITIES FROM BINOMIAL EXPANSIONS

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Abstract. In this article, using binomial expansions, we get some interesting recursive identities concerning arithmetic, geometric and harmonic means of positive numbers, from which, the most important Ky Fan type inequalities are handled by induction at once.

1. Introduction

Throughout this article, let $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$ with $\sum_{i=1}^n \lambda_i = 1$, and A_n, G_n and H_n be the arithmetic, geometric and harmonic means of $x_1, \ldots, x_n > 0$ respectively, i.e.

$$A_n = \sum_{i=1}^n \lambda_i x_i, \qquad G_n = \prod_{i=1}^n x_i^{\lambda_i}, \qquad H_n = \frac{1}{\sum_{i=1}^n \lambda_i \frac{1}{x_i}}. \tag{1}$$

Also, if $x_i \in \left(0, \frac{1}{2}\right]$, we denote by A'_n, G'_n, and H'_n the arithmetic, geometric and harmonic means of $1-x_1, \ldots, 1-x_n$ respectively, i.e.

$$A'_n = \sum_{i=1}^n \lambda_i (1-x_i), \qquad G'_n = \prod_{i=1}^n (1-x_i)^{\lambda_i}, \qquad H'_n = \frac{1}{\sum_{i=1}^n \lambda_i \frac{1}{1-x_i}}. \tag{2}$$

When emphasizing, we write $A_n(x_1, \ldots, x_n)$ instead of A_n, and so on.

The following inequalities are the most important inequalities concerning these means:

$$G_n \leqslant A_n, \tag{3}$$

$$\frac{A'_n}{G'_n} \leqslant \frac{A_n}{G_n}, \tag{4}$$

$$A'_n - G'_n \leqslant A_n - G_n, \tag{5}$$

$$\frac{G'_n}{H'_n} \leqslant \frac{G_n}{H_n}, \tag{6}$$

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and

$$\frac{1}{H_n'} - \frac{1}{G_n'} \leqslant \frac{1}{H_n} - \frac{1}{G_n}. \tag{7}$$

In (3), all x_i's are positive, and in the others all belong to $\left(0, \frac{1}{2}\right]$. Moreover, equality holds in each of them if and only if $x_1 = \ldots = x_n$.
In literature, (3) and (4) are known as AGM and Ky Fan inequalities respectively, and all inequalities above and from the above kinds are referred as Ky Fan type inequalities. There are several interesting proofs for the AGM inequality (3) and more than fifty of them have been mentioned in [5] in order of their appearances.

The Ky Fan inequality (4), was published for the first time in the well-known book *Inequalities* by Beckenbach and Bellman [4, p. 5], and from then it has evoked the interest of several mathematicians and in numerous articles new proofs, extensions, refinements and various related results have been published; see the survey paper [2] and the references therein. Among these remarkable results, the additive analogue of Ky Fan's inequality (5) and also the inequality (7) are due to H. Alzer, and the inequality (6) is due to Wang-Wang; see [3], [1] and [9] respectively, and see also [2].

The aim of this paper is to establish the above inequalities via binomial expansions. This shows the power of binomial expansions on the one hand and the close relations between these inequalities on the other hand. In the following sections, we introduce two different binomial methods, which work in the cases of equal and arbitrary weights. Although the methods are similar to each other, but they are quite independent.

2. The Case of Equal Weights

In this section, we consider only the case of equal weights $\lambda_1 = \lambda_2 = \ldots = \lambda_n = \frac{1}{n}$. In [7] and [8], using the binomial theorem, we proved AGM and Ky Fan inequalities respectively. Now, we are going to establish the other inequalities by the same method. Unfortunately, this method is not so straight forward for inequality (5) which remains a challenging problem. The following trivial lemma is the heart of this method. With the aid of this lemma, first we get some recursive identities concerning arithmetic, geometric and harmonic means of positive numbers in the case of equal weights, and then using these identities, we prove (6) and (7) by induction on n.

LEMMA 2.1. *If $a, b \geqslant 0$, then for each $n = 1, 2, \ldots$, we have*

$$\frac{(n-1)a+b}{n} = a^{\frac{n-1}{n}} b^{\frac{1}{n}} + \frac{1}{n}\sum_{k=2}^{n}\binom{n}{k} a^{\frac{n-k}{n}}\left(b^{\frac{1}{n}} - a^{\frac{1}{n}}\right)^k. \tag{8}$$

Proof. We can write

$$b = \left(b^{\frac{1}{n}} - a^{\frac{1}{n}} + a^{\frac{1}{n}}\right)^n = a + na^{\frac{n-1}{n}}\left(b^{\frac{1}{n}} - a^{\frac{1}{n}}\right) + \sum_{k=2}^{n}\binom{n}{k} a^{\frac{n-k}{n}}\left(b^{\frac{1}{n}} - a^{\frac{1}{n}}\right)^k,$$

and (8) is obtained. □

COROLLARY 2.2.

$$\frac{A_n}{G_n} = \left(\frac{A_{n-1}}{G_{n-1}}\right)^{\frac{n-1}{n}} + \frac{1}{n}\sum_{k=2}^{n}\binom{n}{k}\left(\frac{A_{n-1}}{G_{n-1}}\right)^{\frac{n-k}{n}} \frac{\left(x_n^{\frac{1}{n}} - A_{n-1}^{\frac{1}{n}}\right)^k}{G_{n-1}^{\frac{k-1}{n}} x_n^{\frac{1}{n}}}, \tag{9}$$

and

$$A_n - G_n = \left(A_{n-1}^{\frac{n-1}{n}} - G_{n-1}^{\frac{n-1}{n}}\right) x_n^{\frac{1}{n}} + \frac{1}{n}\sum_{k=2}^{n}\binom{n}{k} A_{n-1}^{\frac{n-k}{n}}\left(x_n^{\frac{1}{n}} - A_{n-1}^{\frac{1}{n}}\right)^k. \tag{10}$$

Proof. The identities (9) and (10) follow from (8) by taking $a = A_{n-1}, b = x_n$, and considering $A_n = \frac{n-1}{n}A_{n-1} + \frac{1}{n}x_n$ and $G_n = G_{n-1}^{\frac{n-1}{n}} x_n^{\frac{1}{n}}$. □

COROLLARY 2.3.

$$\frac{G_n}{H_n} = \left(\frac{G_{n-1}}{H_{n-1}}\right)^{\frac{n-1}{n}} + \frac{1}{n}\sum_{k=2}^{n}\binom{n}{k}\left(\frac{G_{n-1}}{H_{n-1}}\right)^{\frac{n-k}{n}}\left(\frac{G_{n-1}}{x_n}\right)^{\frac{k-1}{n}}\left[1 - \left(\frac{x_n}{H_{n-1}}\right)^{\frac{1}{n}}\right]^k, \tag{11}$$

and

$$\frac{1}{H_n} - \frac{1}{G_n} = \left(\frac{1}{H_{n-1}^{\frac{n-1}{n}}} - \frac{1}{G_{n-1}^{\frac{n-1}{n}}}\right)\frac{1}{x_n^{\frac{1}{n}}} + \frac{1}{n}\sum_{k=2}^{n}\binom{n}{k}\frac{1}{H_{n-1}^{\frac{n-k}{n}} x_n^{\frac{k}{n}}}\left[1 - \left(\frac{x_n}{H_{n-1}}\right)^{\frac{1}{n}}\right]^k. \tag{12}$$

Proof. Clearly,

$$A_n\left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right) = \frac{1}{H_n(x_1, \ldots, x_n)},$$

and

$$G_n\left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right) = \frac{1}{G_n(x_1, \ldots, x_n)}.$$

Now, changing the roles of x_i's by $\frac{1}{x_i}$'s, the identities (11) and (12) follow from Corollary 2.2 by replacing A_n, G_n, A_{n-1}, G_{n-1} and x_n, by $\frac{1}{H_n}, \frac{1}{G_n}, \frac{1}{H_{n-1}}, \frac{1}{G_{n-1}}$ and $\frac{1}{x_n}$ respectively. □

In order to use only discrete methods and so avoid the mean value theorem, the following lemma and its consequences, are useful in the proof of (7).

LEMMA 2.4. *For each $x \geqslant 0$ and $n = 2, 3, \ldots$, we have*

$$\frac{1 - x^{n-1}}{n-1} \geqslant \frac{1 - x^n}{n}, \tag{13}$$

with equality holding if and only if $x = 1$.

Consequently, if $a, b > 0$, then for each $n = 2, 3, \ldots$,

$$\frac{n-1}{n}\frac{a-b}{a^{\frac{1}{n}}} \leqslant a^{\frac{n-1}{n}} - b^{\frac{n-1}{n}} \leqslant \frac{n-1}{n}\frac{a-b}{b^{\frac{1}{n}}}, \tag{14}$$

and equality holds in each of them if and only if $a=b$.

Proof. For (13), it is sufficient to consider

$$n\left(1-x^{n-1}\right)-(n-1)\left(1-x^{n}\right)=(1-x)\left[\sum_{k=0}^{n-2}x^{k}-(n-1)x^{n-1}\right].$$

Now, using (13) with $x=\left(\frac{b}{a}\right)^{\frac{1}{n}}$ and $x=\left(\frac{a}{b}\right)^{\frac{1}{n}}$, we have

$$a^{\frac{n-1}{n}}-b^{\frac{n-1}{n}}=a^{\frac{n-1}{n}}\left[1-\left(\frac{b}{a}\right)^{\frac{n-1}{n}}\right]\geqslant\frac{n-1}{n}a^{\frac{n-1}{n}}\left(1-\frac{b}{a}\right)=\frac{n-1}{n}\frac{a-b}{a^{\frac{1}{n}}},$$

and

$$a^{\frac{n-1}{n}}-b^{\frac{n-1}{n}}=b^{\frac{n-1}{n}}\left[\left(\frac{a}{b}\right)^{\frac{n-1}{n}}-1\right]\leqslant\frac{n-1}{n}b^{\frac{n-1}{n}}\left(\frac{a}{b}-1\right)=\frac{n-1}{n}\frac{a-b}{b^{\frac{1}{n}}}.$$

Evidently, equality holds in each of inequalities in (14), if and only if $a=b$. $\square$

Proof of inequalities (6) and (7). We prove (6) and (7) by induction on n. If $n=1$, there is nothing to prove. Suppose $n\geqslant 2$ and the assertions hold for $n-1$. If $x_1=\ldots=x_n$, then obviously equality holds in each of (6) and (7). Let not all x_i's be equal. Arrange x_i's so that $x_n=\min\limits_{1\leqslant i\leqslant n}x_i$. Applying Corollary 2.3 for $(1-x_i)$'s instead of x_i's, the following similar identities hold:

$$\frac{G'_n}{H'_n}=\left(\frac{G'_{n-1}}{H'_{n-1}}\right)^{\frac{n-1}{n}}+\frac{1}{n}\sum_{k=2}^{n}\binom{n}{k}\left(\frac{G'_{n-1}}{H'_{n-1}}\right)^{\frac{n-k}{n}}\left(\frac{G'_{n-1}}{1-x_n}\right)^{\frac{k-1}{n}}\left[1-\left(\frac{1-x_n}{H'_{n-1}}\right)^{\frac{1}{n}}\right]^{k}, \tag{15}$$

and

$$\begin{aligned}\frac{1}{H'_n}-\frac{1}{G'_n}=&\left(\frac{1}{{H'_{n-1}}^{\frac{n-1}{n}}}-\frac{1}{{G'_{n-1}}^{\frac{n-1}{n}}}\right)\frac{1}{(1-x_n)^{\frac{1}{n}}}\\&+\frac{1}{n}\sum_{k=2}^{n}\binom{n}{k}\frac{1}{{H'_{n-1}}^{\frac{n-k}{n}}(1-x_n)^{\frac{k}{n}}}\left[1-\left(\frac{1-x_n}{H'_{n-1}}\right)^{\frac{1}{n}}\right]^{k}.\end{aligned} \tag{16}$$

Clearly,

$$1-x_n>x_n,\qquad H'_{n-1}\geqslant H_{n-1},\qquad \frac{G_{n-1}}{x_n}>\frac{G'_{n-1}}{1-x_n},\qquad \frac{x_n}{H_{n-1}}<\frac{1-x_n}{H'_{n-1}},$$

and

$$\begin{aligned}1-\left(\frac{x_n}{H_{n-1}}\right)^{\frac{1}{n}} &= \left[\sum_{i=0}^{n-1}\left(\frac{x_n}{H_{n-1}}\right)^{\frac{i}{n}}\right]^{-1}\left(1-\frac{x_n}{H_{n-1}}\right)\\ &= \left[\sum_{i=0}^{n-1}\left(\frac{x_n}{H_{n-1}}\right)^{\frac{i}{n}}\right]^{-1}\frac{1}{n-1}\sum_{i=1}^{n-1}\frac{x_i-x_n}{x_i}\\ &> \left[\sum_{i=0}^{n-1}\left(\frac{1-x_n}{H'_{n-1}}\right)^{\frac{i}{n}}\right]^{-1}\frac{1}{n-1}\sum_{i=1}^{n-1}\frac{x_i-x_n}{1-x_i}\\ &= \left[\sum_{i=0}^{n-1}\left(\frac{1-x_n}{H'_{n-1}}\right)^{\frac{i}{n}}\right]^{-1}\left(\frac{1-x_n}{H'_{n-1}}-1\right)\\ &= \left(\frac{1-x_n}{H'_{n-1}}\right)^{\frac{1}{n}}-1>0.\end{aligned}$$

Now, using the the induction hypothesis

$$\frac{G_{n-1}}{H_{n-1}} \geqslant \frac{G'_{n-1}}{H'_{n-1}},$$

the second terms in the right hand sides of (11) and (12) are strictly greater than the second terms in the right hand sides of (15) and (16), respectively. Now comparing (11) with (15), evidently, we have strict inequality in (6).

Finally, using (14) and the induction hypothesis

$$\frac{1}{H_{n-1}}-\frac{1}{G_{n-1}} \geqslant \frac{1}{H'_{n-1}}-\frac{1}{G'_{n-1}} \geqslant 0,$$

we have

$$\left(\frac{1}{H_{n-1}^{\frac{n-1}{n}}}-\frac{1}{G_{n-1}^{\frac{n-1}{n}}}\right)\frac{1}{x_n^{\frac{1}{n}}} \geqslant \frac{n-1}{n}\left(\frac{1}{H_{n-1}}-\frac{1}{G_{n-1}}\right)\left(\frac{H_{n-1}}{x_n}\right)^{\frac{1}{n}}$$

$$\geqslant \frac{n-1}{n}\left(\frac{1}{H'_{n-1}}-\frac{1}{G'_{n-1}}\right)\left(\frac{G'_{n-1}}{1-x_n}\right)^{\frac{1}{n}} \geqslant \left(\frac{1}{{H'_{n-1}}^{\frac{n-1}{n}}}-\frac{1}{{G'_{n-1}}^{\frac{n-1}{n}}}\right)\frac{1}{(1-x_n)^{\frac{1}{n}}},$$

since

$$\frac{H_{n-1}}{x_n} > 1 > \frac{G'_{n-1}}{1-x_n}.$$

So, comparing (12) with (16), we have strict inequality in (7), and the proof is complete. □

3. The General Case of Arbitrary weights

In this section, we consider the general case of arbitrary weights $\lambda_1, \lambda_2, \ldots, \lambda_n$. Here, using binomial series, first we establish a trivial lemma, from which is similarly obtained some new recursive identities among arithmetic, geometric and harmonic means of positive numbers in the general case. Then using these identities, we prove (3-7) by induction on n. Fortunately, this method is stronger here than in the previous case and gives the inequality (5) directly.

LEMMA 3.1. *If $0 < a < 2b$, then for each real λ,*

$$(1-\lambda)a + \lambda b = a^{1-\lambda} b^{\lambda} + \sum_{k=2}^{\infty} (-1)^{k-1} \binom{1-\lambda}{k} (b-a)^k b^{1-k}. \tag{17}$$

Thus, in the case of $0 < \lambda < 1$, considering

$$(-1)^{k-1} \binom{1-\lambda}{k} > 0 \qquad (k \geqslant 2), \tag{18}$$

and arranging a and b so that $0 < a \leqslant b$, we get the difference of arithmetic and geometric means of a and b as a series of nonnegative terms.

Proof. Since $|\frac{a}{b} - 1| < 1$, using binomial series [6, p. 90], we have

$$\left(\frac{a}{b}\right)^{1-\lambda} = 1 + \sum_{k=1}^{\infty} \binom{1-\lambda}{k} \left(\frac{a}{b} - 1\right)^k.$$

Now, multiplying each side by b we get (17). □

COROLLARY 3.2. *If $A_{n-1} < 2x_n$, then*

$$\frac{A_n}{G_n} = \left(\frac{A_{n-1}}{G_{n-1}}\right)^{1-\lambda_n} + \frac{1}{G_n} \sum_{k=2}^{\infty} (-1)^{k-1} \binom{1-\lambda_n}{k} (x_n - A_{n-1})^k x_n^{1-k}, \tag{19}$$

and

$$A_n - G_n = \left(A_{n-1}^{1-\lambda_n} - G_{n-1}^{1-\lambda_n}\right) x_n^{\lambda_n} + \sum_{k=2}^{\infty} (-1)^{k-1} \binom{1-\lambda_n}{k} (x_n - A_{n-1})^k x_n^{1-k}, \tag{20}$$

where $A_{n-1} = \sum_{i=1}^{n-1} \frac{\lambda_i}{1-\lambda_n} x_i$ and $G_{n-1} = \prod_{i=1}^{n-1} x_i^{\frac{\lambda_i}{1-\lambda_n}}$.

Proof. The identities (19) and (20) follow from (17) by taking $a = A_{n-1}, b = x_n$ and $\lambda = \lambda_n$ in Lemma 3.1, and considering $A_n = (1-\lambda_n)A_{n-1} + \lambda_n x_n$ and $G_n = G_{n-1}^{1-\lambda_n} x_n^{\lambda_n}$. □

COROLLARY 3.3. *If* $x_n < 2H_{n-1}$, *then*

$$\frac{G_n}{H_n} = \left(\frac{G_{n-1}}{H_{n-1}}\right)^{1-\lambda_n} + \frac{G_n}{x_n}\sum_{k=2}^{\infty}(-1)^{k-1}\binom{1-\lambda_n}{k}\left(1-\frac{x_n}{H_{n-1}}\right)^k, \tag{21}$$

and

$$\frac{1}{H_n} - \frac{1}{G_n} = \left(\frac{1}{H_{n-1}^{1-\lambda_n}} - \frac{1}{G_{n-1}^{1-\lambda_n}}\right)\frac{1}{x_n^{\lambda_n}} + \frac{1}{x_n}\sum_{k=2}^{\infty}(-1)^{k-1}\binom{1-\lambda_n}{k}\left(1-\frac{x_n}{H_{n-1}}\right)^k \tag{22}$$

where $H_{n-1} = 1/\left(\sum_{i=1}^{n-1}\frac{\lambda_i}{1-\lambda_n}\frac{1}{x_i}\right)$ *and* $G_{n-1} = \prod_{i=1}^{n-1} x_i^{\frac{\lambda_i}{1-\lambda_n}}$.

Proof. Clearly,

$$A_n\left(\frac{1}{x_1},\ldots,\frac{1}{x_n}\right) = \frac{1}{H_n\left(x_1,\ldots,x_n\right)},$$

and

$$G_n\left(\frac{1}{x_1},\ldots,\frac{1}{x_n}\right) = \frac{1}{G_n\left(x_1,\ldots,x_n\right)}.$$

Now since $\frac{1}{H_{n-1}} < \frac{2}{x_n}$, changing the roles of x_i 's by $\frac{1}{x_i}$'s, the identities (21) and (22) follow from Corollary 3.2 by replacing A_n, G_n, A_{n-1}, G_{n-1} and x_n, by $\frac{1}{H_n}, \frac{1}{G_n}, \frac{1}{H_{n-1}}, \frac{1}{G_{n-1}}$ and $\frac{1}{x_n}$ respectively. □

Proof of inequalities (3-7). We prove (3-7) by induction on n. If $n=1$, there is nothing to prove. Suppose $n \geqslant 2$ and the assertions hold for $n-1$. If $x_1 = \ldots = x_n$, then obviously equality holds in each inequality of (3-7). Let not all x_i 's be equal. For proving (3), (4) and (5), arrange x_i 's so that $x_n = \max_{1\leqslant i\leqslant n} x_i$. Since $A_{n-1} < x_n < 2x_n$, the identities (19) and (20) hold. Now, using the induction hypothesis

$$A_{n-1} \geqslant G_{n-1},$$

and considering (18), the AGM inequality (3) follows from (20) with strict inequality.

Since $A'_{n-1} < 2(1-x_n)$, using Corollary 3.2, we have

$$\frac{A'_n}{G'_n} = \left(\frac{A'_{n-1}}{G'_{n-1}}\right)^{1-\lambda_n} + \frac{1}{G'_n}\sum_{k=2}^{\infty}(-1)^{k-1}\binom{1-\lambda_n}{k}(A_{n-1}-x_n)^k(1-x_n)^{1-k}, \tag{23}$$

and

$$\begin{aligned} A'_n - G'_n =&(A'^{1-\lambda_n}_{n-1} - G'^{1-\lambda_n}_{n-1})(1-x_n)^{\lambda_n} \\ &+ \sum_{k=2}^{\infty}(-1)^{k-1}\binom{1-\lambda_n}{k}(A_{n-1}-x_n)^k(1-x_n)^{1-k}, \end{aligned} \tag{24}$$

where $A'_{n-1} = \sum_{i=1}^{n-1}\frac{\lambda_i}{1-\lambda_n}(1-x_i)$ and $G'_{n-1} = \prod_{i=1}^{n-1}(1-x_i)^{\frac{\lambda_i}{1-\lambda_n}}$. But,

$$G_n < G'_n, \qquad x_n > A_{n-1} \qquad \text{and} \qquad x_n \leqslant 1-x_n.$$

Therefore, considering (18), the second terms in the right hand sides of (19) and (20) are strictly greater than the second terms in the right hand sides of (23) and (24) respectively. Now, considering the induction hypothesis

$$\frac{A_{n-1}}{G_{n-1}} \geqslant \frac{A'_{n-1}}{G'_{n-1}},$$

the Ky Fan's inequality (4) follows with strict inequality by comparing (19) with (23).

Moreover, using the mean value theorem and the induction hypothesis

$$A_{n-1} - G_{n-1} \geqslant A'_{n-1} - G'_{n-1} \geqslant 0,$$

we have

$$\begin{aligned}\left(A_{n-1}^{1-\lambda_n} - G_{n-1}^{1-\lambda_n}\right) x_n^{\lambda_n} &= (1-\lambda_n)(A_{n-1} - G_{n-1}) \left(\frac{x_n}{\theta_{n-1}}\right)^{\lambda_n} \\ &\geqslant (1-\lambda_n)(A'_{n-1} - G'_{n-1}) \left(\frac{1-x_n}{\theta'_{n-1}}\right)^{\lambda_n} \\ &= \left({A'}_{n-1}^{1-\lambda_n} - {G'}_{n-1}^{1-\lambda_n}\right)(1-x_n)^{\lambda_n},\end{aligned}$$

where $\theta_{n-1} \in [G_{n-1}, A_{n-1}]$ and $\theta'_{n-1} \in [G'_{n-1}, A'_{n-1}]$, since

$$\frac{x_n}{\theta_{n-1}} > \frac{A_{n-1}}{\theta_{n-1}} \geqslant 1 \geqslant \frac{G'_{n-1}}{\theta'_{n-1}} > \frac{1-x_n}{\theta'_{n-1}}.$$

Thus, comparing (20) with (24), we get strict inequality in (5).

For the proof of (6) and (7), arrange x_i's so that $x_n = \min_{1 \leqslant i \leqslant n} x_i$. Since $x_n < H_{n-1} < 2H_{n-1}$ and $1 - x_n < 2H'_{n-1}$, using Corollary 3.3, the identities (21), (22), and also the following similar identities hold:

$$\frac{G'_n}{H'_n} = \left(\frac{G'_{n-1}}{H'_{n-1}}\right)^{1-\lambda_n} + \frac{G'_n}{1-x_n} \sum_{k=2}^{\infty} (-1)^{k-1} \binom{1-\lambda_n}{k} \left(1 - \frac{1-x_n}{H'_{n-1}}\right)^k, \tag{25}$$

and

$$\begin{aligned}\frac{1}{H'_n} - \frac{1}{G'_n} = &\left(\frac{1}{{H'}_{n-1}^{1-\lambda_n}} - \frac{1}{{G'}_{n-1}^{1-\lambda_n}}\right) \frac{1}{(1-x_n)^{\lambda_n}} \\ &+ \frac{1}{1-x_n} \sum_{k=2}^{\infty} (-1)^{k-1} \binom{1-\lambda_n}{k} \left(1 - \frac{1-x_n}{H'_{n-1}}\right)^k.\end{aligned} \tag{26}$$

Now, since

$$1 - \frac{x_n}{H_{n-1}} = \sum_{i=1}^{n-1} \frac{\lambda_i}{1-\lambda_n} \frac{x_i - x_n}{x_i} \geqslant \sum_{i=1}^{n-1} \frac{\lambda_i}{1-\lambda_n} \frac{x_i - x_n}{1-x_i} = \frac{1-x_n}{H'_{n-1}} - 1 > 0,$$

$$\frac{G_n}{x_n} > \frac{G'_n}{1-x_n}, \qquad \text{and} \qquad \frac{1}{x_n} > \frac{1}{1-x_n},$$

considering (18), the second terms in the right hand sides of (21) and (22) are strictly greater than the second terms in the right hand sides of (25) and (26), respectively. Now, by the induction hypothesis

$$\frac{G_{n-1}}{H_{n-1}} \geqslant \frac{G'_{n-1}}{H'_{n-1}},$$

comparing (21) with (25), we have strict inequality in (6).

Finally, by the mean value theorem and the induction hypothesis

$$\frac{1}{H_{n-1}} - \frac{1}{G_{n-1}} \geqslant \frac{1}{H'_{n-1}} - \frac{1}{G'_{n-1}} \geqslant 0,$$

we have

$$\begin{aligned}
\left(\frac{1}{H_{n-1}^{1-\lambda_n}} - \frac{1}{G_{n-1}^{1-\lambda_n}}\right)\frac{1}{x_n^{\lambda_n}} &= (1-\lambda_n)\left(\frac{1}{H_{n-1}} - \frac{1}{G_{n-1}}\right)\left(\frac{1}{x_n\eta_{n-1}}\right)^{\lambda_n} \\
&\geqslant (1-\lambda_n)\left(\frac{1}{H'_{n-1}} - \frac{1}{G'_{n-1}}\right)\left(\frac{1}{(1-x_n)\eta'_{n-1}}\right)^{\lambda_n} \\
&= \left(\frac{1}{{H'_{n-1}}^{1-\lambda_n}} - \frac{1}{{G'_{n-1}}^{1-\lambda_n}}\right)\frac{1}{(1-x_n)^{\lambda_n}},
\end{aligned}$$

where $\eta_{n-1} \in [\frac{1}{G_{n-1}}, \frac{1}{H_{n-1}}]$ and $\eta'_{n-1} \in [\frac{1}{G'_{n-1}}, \frac{1}{H'_{n-1}}]$, since

$$x_n\eta_{n-1} \leqslant \frac{x_n}{H_{n-1}} < 1 < \frac{1-x_n}{G'_{n-1}} \leqslant (1-x_n)\eta'_{n-1}.$$

So, comparing (22) with (26), we have strict inequality in (7), and the proof is complete. □

REMARK 3.4. It is noted that if $x_n \geqslant A_{n-1}$, each of identities (9) and (19) gives a refinement for the Popoviciu inequality

$$\frac{A_n}{G_n} \geqslant \left(\frac{A_{n-1}}{G_{n-1}}\right)^{1-\lambda_n}, \tag{27}$$

in the case of equal and arbitrary weights respectively, where $A_{n-1} = \sum_{i=1}^{n-1} \frac{\lambda_i}{1-\lambda_n} x_i$ and $G_{n-1} = \prod_{i=1}^{n-1} x_i^{\frac{\lambda_i}{1-\lambda_n}}$; see [5] for details.

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REFERENCES

[1] H. ALZER, *Inequalities for arithmetic, geometric and harmonic means*, Bull. London Math. Soc., **22** (1990), 362-366.
[2] H. ALZER, *The inequality of Ky Fan and related results*, Acta Appl. Math., **38** (1995), 305-354.
[3] H. ALZER, *Ungleichungen für geometrische und arithmetische Mittelwerte*, Proc. Kon. Nederl. Akad. Wetensch., **91** (1988), 365-374.
[4] E.F. BECKENBACH AND R. BELLMAN, *Inequalities*, Springer-Verlag, Berlin, 1961.
[5] P.S. BULLEN, D.S. MITRINOVIĆ AND P.M. VASIĆ, *Means and their inequalities*, Reidel, Dordrecht, 1988.
[6] E. HEWITT AND K. STROMBERG, *Real and abstract analysis*, Springer-Verlag, New York, 1975.
[7] J. ROOIN, *AGM inequality with binomial expansion*, Elemente Der Mathematik, **58** (2003), 115-117.
[8] J. ROOIN, *Ky Fan's inequality with binomial expansion*, Elemente Der Mathematik, **60** (2005), 171-173.
[9] W.L. WANG AND P.F. WANG, *A class of inequalities for the symmetric functions* (in chinese), Acta Math. Sinica, **27** (1984), 485-497.

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