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PARAMETRIZED KLAMKIN'S INEQUALITY AND IMPROVED EULER'S INEQUALITY

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Abstract. In this paper, the authors present a generalization of Klamkin's inequality by introducing a parameter, which relaxes the conditions of Klamkin's inequality. As applications, some improved versions of Euler's inequality are obtained.

1. Introduction

We begin by recalling here the following well-known inequality related to the angles A, B, C of a triangle

$$\sin A + \sin B + \sin C \leqslant \frac{3\sqrt{3}}{2}. \tag{1}$$

In 1969, P. M. Vasić [1] presented a weighted generalization of the inequality (1), as follows

THEOREM A. *Let x, y, z be positive numbers, and let A, B, C be real numbers with $A+B+C=\pi$. Then*

$$x\sin A + y\sin B + z\sin C \leqslant \frac{\sqrt{3}}{2}\left(\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z}\right). \tag{2}$$

In 1984, M. S. Klamkin [2] sharpened Vasić's inequality (2) in the from asserted by Theorem B below.

THEOREM B. *Let x, y, z be positive numbers, and let A, B, C be real numbers with $A+B+C=\pi$. Then*

$$x\sin A + y\sin B + z\sin C \leqslant \frac{1}{2}(xy+yz+zx)\sqrt{\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}}. \tag{3}$$

It is well-known that Klamkin's inequality (3) plays an important role in the study of geometric inequalities. A number of geometric inequalities can be obtained from

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the Klamkin's inequality (3) by assigning appropriate values to the parameters. Due to the importance of Klamkin's inequality, this inequality has been given considerable attention by mathematicians. A comprehensive survey on this inequality can be found in [3], where a large number of references connected to this subject are listed.

The aim of this paper is to give a generalization of Klamkin's inequality by introducing a parameter. Moreover, we provide an application of the obtained result to the improvement of the classical Euler's inequality and Băndilă's inequality.

2. Lemmas

In order to prove the main result in Section 3, we need the following lemmas.

LEMMA 1. *Let x, y, z, A, B, C be real numbers with $A+B+C=\theta$, $-\pi \leqslant \theta \leqslant \pi$. Then*

$$x^2+y^2+z^2 \geqslant \sec\frac{\theta}{3}\left(yz\cos A+zx\cos B+xy\cos C\right). \tag{4}$$

Proof. We rewrite the inequality (4) as

$$x^2-\sec\frac{\theta}{3}(y\cos C+z\cos B)x+y^2-yz\sec\frac{\theta}{3}\cos A+z^2 \geqslant 0. \tag{5}$$

To prove the inequality (5), it is enough to prove that

$$\Delta_1=\sec^2\frac{\theta}{3}(y\cos C+z\cos B)^2-4\left(y^2-yz\sec\frac{\theta}{3}\cos A+z^2\right)\leqslant 0,$$

i.e.,

$$\begin{aligned}\Delta_1=&\left(\sec^2\frac{\theta}{3}\cos^2 C-4\right)y^2+\left(2z\cos C\cos B\sec^2\frac{\theta}{3}+4z\sec\frac{\theta}{3}\cos A\right)y\\&+z^2\left(\sec^2\frac{\theta}{3}\cos^2 B-4\right)\leqslant 0.\end{aligned}\tag{6}$$

When $C=0$ and $\theta=\pi$, we deduce directly that $\Delta_1=4z^2(\cos^2 B-1)\leqslant 0$. When $C\neq 0$ or $\theta\neq\pi$, it implies that $\sec^2\frac{\theta}{3}\cos^2 C-4<0$. Thus, in order to prove the inequality (6), it is enough to prove that

$$\begin{aligned}\Delta_2=&\left(2z\cos C\cos B\sec^2\frac{\theta}{3}+4z\sec\frac{\theta}{3}\cos A\right)^2\\&-4\left(\sec^2\frac{\theta}{3}\cos^2 C-4\right)\left(\sec^2\frac{\theta}{3}\cos^2 B-4\right)\leqslant 0.\end{aligned}\tag{7}$$

Direct computation gives

$$\Delta_2=16z^2\sec^3\frac{\theta}{3}\left[\cos A\cos B\cos C+\cos\frac{\theta}{3}\left(\cos^2 A+\cos^2 B+\cos^2 C\right)-4\cos^3\frac{\theta}{3}\right],$$

and

$$
\begin{aligned}
&\cos A\cos B\cos C+\cos\frac{\theta}{3}\left(\cos^2 A+\cos^2 B+\cos^2 C\right)-4\cos^3\frac{\theta}{3}\\
&\quad=\frac{1}{2}\left[\cos(\theta-C)+\cos(A-B)\right]\cos C\\
&\qquad+\cos\frac{\theta}{3}\left[1+\cos(\theta-C)\cos(A-B)+\cos^2 C\right]-4\cos^3\frac{\theta}{3}\\
&\quad=\frac{1}{2}\cos(A-B)\left[\cos C+2\cos\frac{\theta}{3}\cos(\theta-C)\right]\\
&\qquad+\frac{1}{2}\cos C\left[\cos(\theta-C)+2\cos\frac{\theta}{3}\cos C\right]+\cos\frac{\theta}{3}-4\cos^3\frac{\theta}{3}\\
&\quad=\frac{1}{2}\cos(A-B)\left[\cos(\frac{4}{3}\theta-C)+\cos(\frac{2}{3}\theta-C)+\cos C\right]\\
&\qquad+\frac{1}{2}\cos C\left[\cos(\frac{\theta}{3}+C)+\cos(\frac{\theta}{3}-C)+\cos(\theta-C)\right]\\
&\qquad-\cos\frac{\theta}{3}\left(1+2\cos\frac{2}{3}\theta\right)\\
&\quad=\frac{1}{2}\cos(A-B)\cos(\frac{2}{3}\theta-C)\left(1+2\cos\frac{2}{3}\theta\right)\\
&\qquad+\frac{1}{2}\cos C\cos(\frac{\theta}{3}-C)\left(1+2\cos\frac{2}{3}\theta\right)-\cos\frac{\theta}{3}\left(1+2\cos\frac{2}{3}\theta\right)\\
&\quad=\left(\cos\frac{2}{3}\theta+\frac{1}{2}\right)\left[\cos(\frac{2}{3}\theta-C)\cos(A-B)+n\cos C\cos(\frac{\theta}{3}-C)-2\cos\frac{\theta}{3}\right].
\end{aligned}\tag{8}
$$

By the assumption of Lemma 1 $-\pi\leqslant\theta\leqslant\pi$, we conclude that $\cos\frac{2}{3}\theta\geqslant-\frac{1}{2}$, $\cos\frac{\theta}{3}\geqslant\frac{1}{2}$. In order to prove that $\Delta_2\leqslant 0$, we consider the following two cases.

Case (I). When $\cos(\frac{2}{3}\theta-C)\geqslant 0$, we have

$$
\begin{aligned}
&\cos(\frac{2}{3}\theta-C)\cos(A-B)+\cos C\cos(\frac{\theta}{3}-C)-2\cos\frac{\theta}{3}\\
&\qquad\leqslant\cos(\frac{2}{3}\theta-C)+\cos C\cos(\frac{\theta}{3}-C)-2\cos\frac{\theta}{3}\\
&\qquad=2\cos\frac{\theta}{3}\cos(\frac{\theta}{3}-C)-\cos C+\cos C\cos(\frac{\theta}{3}-C)-2\cos\frac{\theta}{3}\\
&\qquad=-\left[1-\cos(\frac{\theta}{3}-C)\right]\left[\cos C+2\cos\frac{\theta}{3}\right]\\
&\qquad=-2\left[1-\cos(\frac{\theta}{3}-C)\right]\left[\cos^2\frac{C}{2}+\cos\frac{\theta}{3}-\frac{1}{2}\right]\\
&\qquad\leqslant 0.
\end{aligned}
$$

Case (II). When $\cos(\frac{2}{3}\theta - C) < 0$, we have

$$\cos(\frac{2}{3}\theta - C)\cos(A-B) + \cos C\cos(\frac{\theta}{3} - C) - 2\cos\frac{\theta}{3}$$

$$\leqslant -\cos(\frac{2}{3}\theta - C) + \cos C\cos(\frac{\theta}{3} - C) - 2\cos\frac{\theta}{3}$$

$$= -2\cos\frac{\theta}{3}\cos(\frac{\theta}{3} - C) + \cos C + \cos C\cos(\frac{\theta}{3} - C) - 2\cos\frac{\theta}{3}$$

$$= -\left[1 + \cos(\frac{\theta}{3} - C)\right]\left[-\cos C + 2\cos\frac{\theta}{3}\right]$$

$$= -2\left[1 + \cos(\frac{\theta}{3} - C)\right]\left[\sin^2\frac{C}{2} + \cos\frac{\theta}{3} - \frac{1}{2}\right] \leqslant 0.$$

Combining the identity (8) and the above inequalities, we deduce that $\Delta_2 \leqslant 0$, which implies the validity of inequality (4). The Lemma 1 is proved. □

LEMMA 2. *Let* $x,\ y,\ z,\ A,\ B,\ C$ *be real numbers with* $A+B+C=\theta,\ (6k-1)\pi \leqslant \theta \leqslant (6k+1)\pi,\ (k=0,\ \pm1,\ \pm2,\ \ldots)$. *Then*

$$x^2+y^2+z^2 \geqslant \sec\frac{\theta}{3}\left(yz\cos A + zx\cos B + xy\cos C\right). \tag{9}$$

Proof. Based on the assumption of Lemma 2 $(6k-1)\pi \leqslant \theta \leqslant (6k+1)\pi$, we have

$$(A-2k\pi)+(B-2k\pi)+(C-2k\pi) = \theta - 6k\pi \quad \text{and} \quad -\pi \leqslant \theta - 6k\pi \leqslant \pi.$$

It thus follows from Lemma 1 that

$$x^2+y^2+z^2 \geqslant \sec\left(\frac{\theta}{3} - 2k\pi\right)\left[yz\cos(A-2k\pi)+zx\cos(B-2k\pi)+xy\cos(C-2k\pi)\right]. \tag{10}$$

this yields

$$x^2+y^2+z^2 \geqslant \sec\frac{\theta}{3}\left(yz\cos A + zx\cos B + xy\cos C\right),$$

which is the required inequality in Lemma 2. The proof of Lemma 2 is complete. □

3. Main result

Our main result is stated in the following theorem.

THEOREM 1. *Let* $x,\ y,\ z$ *be positive numbers, and let* $A,\ B,\ C$ *be real numbers with* $A+B+C=\theta,\ (3k+1)\pi \leqslant \theta \leqslant (3k+2)\pi\ \ (k=0,\ \pm1,\ \pm2,\ \ldots)$. *Then*

$$x\sin A+y\sin B+z\sin C \leqslant \tfrac{1}{2}\sqrt{(xy+yz+zx)^2-\left(2\cos\tfrac{2}{3}\theta+1\right)(x^2y^2+y^2z^2+z^2x^2)}\cdot\sqrt{\tfrac{1}{xy}+\tfrac{1}{yz}+\tfrac{1}{zx}}\,. \tag{11}$$

Proof. Using a substitution in (11) by

$$A\longmapsto\frac{\pi-\varphi_1}{2},\quad B\longmapsto\frac{\pi-\varphi_2}{2},\quad C\longmapsto\frac{\pi-\varphi_3}{2},$$

we find that the inequality (11) is equivalent to the following inequality:

$$x\cos\tfrac{\varphi_1}{2}+y\cos\tfrac{\varphi_2}{2}+z\cos\tfrac{\varphi_3}{2} \leqslant \tfrac{1}{2}\sqrt{(xy+yz+zx)^2-\left(1-2\cos\tfrac{\phi}{3}\right)(x^2y^2+y^2z^2+z^2x^2)}\cdot\sqrt{\tfrac{1}{xy}+\tfrac{1}{yz}+\tfrac{1}{zx}}, \tag{12}$$

where $\phi=\varphi_1+\varphi_2+\varphi_3,\ (6k-1)\pi\leqslant\phi\leqslant(6k+1)\pi\quad(k=0,\ \pm1,\ \pm2,\ \ldots)$.

In order to prove the inequality (11), it is enough to prove that the inequality (12) is valid.

Applying the Cauchy-Schwarz's inequality [4, p.30], one obtain

$$\begin{aligned} &x\cos\frac{\varphi_1}{2}+y\cos\frac{\varphi_2}{2}+z\cos\frac{\varphi_3}{2}\\ &\leqslant\sqrt{\left(zxxy\cos^2\frac{\varphi_1}{2}+xyyz\cos^2\frac{\varphi_2}{2}+yzzx\cos^2\frac{\varphi_3}{2}\right)\left(\frac{1}{yz}+\frac{1}{zx}+\frac{1}{xy}\right)}\\ &=\frac{1}{2}\sqrt{2\,(zxxy+xyyz+yzzx)+2\,(zxxy\cos\varphi_1+xyyz\cos\varphi_2+yzzx\cos\varphi_3)}\\ &\quad\cdot\sqrt{\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}}\,. \end{aligned}\tag{13}$$

On the other hand, in view of the hypothesis that

$$\phi=\varphi_1+\varphi_2+\varphi_3,\ (6k-1)\pi\leqslant\phi\leqslant(6k+1)\pi\qquad(k=0,\ \pm1,\ \pm2,\ \ldots).$$

We thus deduce from Lemma 2 that

$$zxxy\cos\varphi_1+xyyz\cos\varphi_2+yzzx\cos\varphi_3\leqslant\cos\frac{\phi}{3}\left(z^2x^2+x^2y^2+y^2z^2\right). \tag{14}$$

Combining inequalities (13) and (14) gives

$$\begin{aligned} &x\cos\frac{\varphi_1}{2}+y\cos\frac{\varphi_2}{2}+z\cos\frac{\varphi_3}{2}\\ &\leqslant\frac{1}{2}\sqrt{2\,(zxxy+xyyz+yzzx)+2\cos\frac{\phi}{3}\,(z^2x^2+x^2y^2+y^2z^2)}\sqrt{\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}}\\ &=\frac{1}{2}\sqrt{(xy+yz+zx)^2-\left(1-2\cos\frac{\phi}{3}\right)(x^2y^2+y^2z^2+z^2x^2)}\sqrt{\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}}\,. \end{aligned}$$

Hence the inequality (12) is proved, which leads us to the desired inequality (11). This completes the proof of Theorem 1. □

REMARK 1. It is obvious that the Klamkin's inequality (3) would follow as a special case of Theorem 1 when $\theta = \pi$.

4. Application to the improvement of Euler's inequality

In what follows, we state that A, B, C denote the angles of triangle ABC, a, b, c denote the lengths of the corresponding sides. Let R, r be the radii of the circumscribed and inscribed circles of triangle ABC respectively. Similarly define the triangle $A'B'C'$.

The inequality

$$R \geqslant 2r \tag{15}$$

is called Euler's inequality. This inequality was proved by L. Euler in 1765, it is one of the oldest geometric inequalities, As is well-known, Euler's inequality (15) is an important tool in the study of geometric inequalities, see [5] for history and more details, refer to [6–13] for some results concerning improvements and applications of this inequality .

In 1985, an interesting sharpened version of Euler's inequality was presented by V. Băndilă [14], as follows

$$\frac{R}{r} \geqslant \frac{b}{c} + \frac{c}{b}. \tag{16}$$

In 2003, Zh.-H. Zhang and Q. Song et al [15] established an analogue of Băndilă's inequality (16) as a sharpening of Euler's inequality, i.e.,

$$\frac{R}{r} \geqslant \frac{2}{3}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right). \tag{17}$$

As application of Theorem 1, we give here a new improvement of Euler's inequality, it will be shown that the present result is a unified improved version of Băndilă's inequality (16) and Zhang-Song's inequality (17).

THEOREM 2. *Suppose that A, B, C are the real numbers such that $A+B+C = \theta$ $(\pi \leqslant \theta \leqslant 2\pi)$, A', B', C' are the angles of triangle $A'B'C'$, and R', r' are the radii of the circumscribed and inscribed circles respectively. Then the following inequality holds true*

$$\frac{\sin A}{\sin A'}+\frac{\sin B}{\sin B'}+\frac{\sin C}{\sin C'} \leqslant \sqrt{3\left(2\cos\frac{2}{3}\theta+1\right)\frac{R'}{r'}-2\left(\cos\frac{2}{3}\theta\right)\left(1+\frac{R'}{r'}\right)^2}. \tag{18}$$

Proof. Direct computation gives

$$\begin{aligned}&\left(x^2y^2+y^2z^2+z^2x^2\right)\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)\\&\quad=\left[(xy+yz+zx)^2-2xyz(x+y+z)\right]\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)\end{aligned}$$

$$= (xy+yz+zx)^2\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)-2(x+y+z)^2$$
$$= (xy+yz+zx)^2\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)-6(xy+yz+zx)-(x-y)^2-(y-z)^2-(z-x)^2$$
$$\leqslant (xy+yz+zx)^2\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)-6(xy+yz+zx).$$

By using Theorem 1 with $k=1$, and appealing to the above inequality, we get that

$$x\sin A+y\sin B+z\sin C$$
$$\leqslant \frac{1}{2}\sqrt{6\left(2\cos\frac{2}{3}\theta+1\right)(xy+yz+zx)-\left(2\cos\frac{2}{3}\theta\right)(xy+yz+zx)^2\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)}. \tag{19}$$

Now, substituting

$$x=\frac{1}{\sin A'}\ ,\quad y=\frac{1}{\sin B'}\ ,\quad z=\frac{1}{\sin C'}$$

into (19), and making use of the known results (see [3, p.180]):

$$\frac{1}{\sin A'\sin B'}+\frac{1}{\sin B'\sin C'}+\frac{1}{\sin C'\sin A'}=\frac{2R'}{r'},$$

$$\sin A'\sin B'+\sin B'\sin C'+\sin C'\sin A'\leqslant\left(1+\frac{r'}{R'}\right)^2,$$

we obtain immediately the desired inequality (18). The proof of Theorem 2 is complete. □

As a consequence of Theorem 2, letting A, B, C be the angles of a triangle and using the law of sine, it yields immediately the following interesting and valuable inequality between two triangles.

COROLLARY 1. *For any triangle ABC and triangle $A'B'C'$, the following inequality holds true*

$$R\left(\frac{1}{r'}+\frac{1}{R'}\right)\geqslant\frac{a}{a'}+\frac{b}{b'}+\frac{c}{c'}\ . \tag{20}$$

REMARK 2. In particular, the Băndilă's inequality (16) would follow from the inequality (20) by setting $a'=a,\ b'=c,\ c'=b$ (where, it evidently implies that $r'=r$ and $R'=R$).

Moreover, putting $a'=b,\ b'=c,\ c'=a$ in inequality (20), a new sharpened version of Euler's inequality is derived as follows

COROLLARY 2. *For any triangle ABC, the following inequality holds true*

$$\frac{R}{r}\geqslant\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-1. \tag{21}$$

REMARK 3. It is clear that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-1 \geqslant \frac{2}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right). \tag{22}$$

Thus, the inequality (21) is stronger than the inequality (17) given by Zhang and Song et al in [15].
In fact, we can now prove that the inequality (21) is the strongest possible inequalities of the form

$$\frac{R}{r} \geqslant \mu\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+2-3\mu. \tag{23}$$

By using the identity

$$\frac{R}{r}=\frac{2abc}{(a+b-c)(b+c-a)(c+a-b)},$$

we conclude that the inequality (23) is equivalent to the inequality:

$$\frac{2abc}{(a+b-c)(b+c-a)(c+a-b)} \geqslant \mu\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+2-3\mu. \tag{24}$$

Putting $a=b=1$, $c=\varepsilon$ in (24) and then taking limits as $\varepsilon\to 0$, we get that $\mu\leqslant 1$. Consequently, the coefficient $\mu=1$ is best possible in the sense that it cannot be replaced by a larger constant.

REMARK 4. In a recent paper [16], D. Svrtan and I. Urbiha proved a sharp inequality:

$$\frac{2abc}{(a+b-c)(b+c-a)(c+a-b)} \geqslant \frac{1}{2}\left(1+\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\right). \tag{25}$$

By direct computation, we find that

$$\begin{aligned}\frac{1}{2}\left(1+\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\right) &= \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-1\right)\\ &\quad+\frac{(b+c-a)(c+a-b)(a+b-c)}{8abc}\left(\frac{b+c-a}{c+a-b}+\frac{c+a-b}{a+b-c}+\frac{a+b-c}{b+c-a}-3\right).\end{aligned} \tag{26}$$

Combining inequalities (25) and (26), a refinement of inequality (21) is derived as follows

$$\frac{R}{r} \geqslant \frac{1}{2}\left(1+\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\right) \geqslant \frac{a}{b}+\frac{b}{c}+\frac{c}{a}-1. \tag{27}$$

In addition, applying inequalities (21) and (25) to identity (26), we get immediately the following sharpened version of inequality (21):

$$\left(\frac{R}{r}\right)^2 \geqslant \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-1\right)^2+\frac{1}{4}\left(\frac{b+c-a}{c+a-b}+\frac{c+a-b}{a+b-c}+\frac{a+b-c}{b+c-a}-3\right). \tag{28}$$

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