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A NOTE ON STRONG APPROXIMATION OF FOURIER SERIES AND EMBEDDING THEOREMS

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Abstract. The present paper proves embedding results which arise from strong approximation by Fourier series and generalizes a theorem of Tikhonov under the ultimate MVBV condition.

§1. Introduction

Let $f(x)$ be an odd continuous function of period 2π and let

$$\sum_{n=1}^{\infty} b_n \sin nx \tag{1}$$

its Fourier series. As usual, denote $S_n(f,x)$ the nth partial sum of series (1). Write $\|\cdot\|$ as the usual supremum norm.

The modulus of smoothness of order β, $\beta > 0$ is defined by

$$\omega_\beta(f,t) = \sup_{|h|\leqslant t}\left\|\sum_{v=0}^{\infty}(-1)^v\binom{\beta}{v} f(x+(\beta-v)h)\right\|,$$

where

$$\binom{\beta}{v} = \begin{cases} \frac{\beta(\beta-1)\cdots(\beta-v+1)}{v!}, & v \geqslant 1; \\ 1, & v = 0. \end{cases}$$

When β is an inter k, $\omega_k(f,t)$ then reduces to the ordinary modulus of smoothness of order k.

Let $\omega(\delta)$ be a non-decreasing continuous function on $[0,2\pi]$ having the following properties:

$$\omega(0) = 0, \quad \omega(\delta_1+\delta_2) \leqslant \omega(\delta_1)+\omega(\delta_2)$$

for any $0 \leqslant \delta_1 \leqslant \delta_2 \leqslant \delta_1+\delta_2 \leqslant 2\pi$, then we write $\omega \in \Omega$.

A nonnegative sequence $\{b_n\}_{n=0}^{\infty}$ is said to be *quasimonotone* sequence (in symbol, $\{b_n\}_{n=0}^{\infty} \in QMS$) if, for some $\alpha \geqslant 0$, the sequence $\{b_n/n^\alpha\}$ is non-increasing for $n \geqslant 1$. This is a classical generalization to (nonnegative) non-increasing condition.

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For another direction, Leindler [2] raised the rest bounded variation condition. A nonnegative sequence $\mathbf{A} = \{a_n\}$ with $\lim_{n\to\infty} a_n = 0$ is said to be a *rest bounded variation sequence* (in symbol, $\{a_n\} \in \mathrm{RBVS}$) if

$$\sum_{k=n}^{\infty} |a_k - a_{k+1}| \leqslant C(\mathbf{A})a_n$$

holds for all $n = 1, 2, \ldots$ and some constant $C(\mathbf{A})$ depending only upon the sequence A. Surely, it is also a generalization to (nonnegative) non-increasing condition.

Leindler [3] proved that *quasimonotonicity* and *the rest bounded variation condition* are not comparable.

A unified condition called *Group Bounded Variation condition* (GBV condition) was introduced by our work [7] to contain both RBV condition and various quasimonotone conditions and to generalize the classical result of Chaundy and Jollife [1] which states that if $\{ b_n \}$ is a non-increasing sequence with $\lim_{n\to\infty} b_n = 0$, then a necessary and sufficient condition for the uniform convergence of series (1) is $\lim_{n\to\infty} nb_n = 0$.

Very recently, S.P. Zhou, P. Zhou and Yu [4] proposed a new condition, Mean Value Bounded Variation condition (MVBV condition), which contains all previous condition including GBV condition. Furthermore, the Chaundy-Jollife's theorem still holds true in MVBV condition, and the condition cannot be further weakened for uniform convergence case.

In every sense, we believe that this MVBV condition could be also an ultimate generalization to monotonicity in most classical important convergence problems of Fourier analysis.

The definition can be stated as follows:

A nonnegative sequence $\mathbf{A} = \{a_n\}_{n=0}^{\infty}$ is said to be a mean value bounded variation sequence ($\{a_n\} \in \mathrm{MVBVS}$) *if there is a $\lambda \geqslant 2$ such that*

$$\sum_{k=n}^{2n} |a_k - a_{k+1}| \leqslant \frac{C(\mathbf{A})}{n} \sum_{k=[\lambda^{-1}n]}^{[\lambda n]} a_k$$

holds for all $n = 1, 2, \ldots$ and some constant $C(\mathbf{A})$ depending only upon the sequence $\mathbf{A}$.

In the present note, we will consider the strong approximation problem and related embedding theorems in MVBV condition. We need the following definitions.

A nonnegative sequence $\mathbf{b} = \{b_n\}$ is said to be an *almost monotone (increaing) sequence* ($\{b_n\} \in$ AMS) if there is a positive constant $C(\mathbf{b})$ such that

$$b_k \geqslant C(\mathbf{b})b_n \text{ for all } k \geqslant n.$$

Let $\{\lambda_n\}$ be a sequence of positive numbers and set

$$\Lambda_n = \sum_{v=1}^{n} \lambda_v.$$

Denote

$$h_n(f,\lambda,p) := \left\| \left(\frac{1}{\Lambda_n} \sum_{v=1}^{n} \lambda_v |f(x) - S_v(x)|^p \right)^{1/p} \right\|.$$

Define the following classes of functions:

$$H(\lambda,p,r,\omega) = \{f \in C_{2\pi} : h_n(f,\lambda,p) = O(n^{-r}\omega(n^{-1}))\},$$

$$W^r H_\beta^\omega = \{f \in C_{2\pi} : \omega_\beta(f^{(r)},\delta) = O(\omega(\delta))\},$$

$$C_1 = \{f \in C_{2\pi} : f(x) = \sum_{n=1}^{\infty} b_n \sin nx, b_n \in QMS\},$$

$$C_2 = \{f \in C_{2\pi} : f(x) = \sum_{n=1}^{\infty} b_n \sin nx, b_n \in RBVS\},$$

$$C_3 = \{f \in C_{2\pi} : f(x) = \sum_{n=1}^{\infty} b_n \sin nx, b_n \in MVBVS\},$$

where $C_{2\pi}$ denotes the class of continuous functions of period 2π.

Set $I_1 \ll I_2$ if there exists a positive constant C such that $I_1 \leqslant CI_2$. If both $I_1 \ll I_2$ and $I_2 \ll I_1$ hold, then we write $I_1 \simeq I_2$.

Tikhonov [6] proved the following theorem for the classes C_1 and C_2:

THEOREM T. *Let $\beta, p > 0$, $r \geqslant 0$, $\omega \in \Omega$, λ_n be a positive sequence satisfying*

$$\Lambda_{2n} \ll \Lambda_n \ll n\lambda_n. \tag{2}$$

If

$$\lambda_n \omega^p(1/n) n^{1-rp} \in \text{AMS}, \tag{3}$$

then

$$W^r H_\beta^\omega \bigcap C_j \in H(\lambda,p,r,\omega)$$

for $j = 1,2$.

In this note, we generalized the above theorem to MVBVS:

THEOREM. *Let $\beta, p > 0$, $r \geqslant 0$, $\omega \in \Omega$, λ_n be a positive sequence satisfying* (2). *If* (3) *holds, then*

$$W^r H_\beta^\omega \bigcap C_3 \in H(\lambda,p,r,\omega).$$

§2. Lemmas

LEMMA 2.1. ([5]) *Let $a_n \geqslant 0$, $\lambda_n > 0$. If $p \geqslant 1$, then*

$$\sum_{n=1}^{\infty} \lambda_n \left(\sum_{v=n}^{\infty} a_v \right)^p \ll \sum_{n=1}^{\infty} \lambda_n^{1-p} a_n^p \left(\sum_{v=1}^{n} \lambda_v \right)^p ; \tag{4}$$

If $0 < p < 1$, $a_n \downarrow$, then

$$\sum_{n=1}^{\infty} \lambda_n \left(\sum_{v=n}^{\infty} a_v \right)^p \ll \sum_{n=1}^{\infty} n^{p-1} a_n^p \left(n\lambda_n + \sum_{v=1}^{n-1} \lambda_v \right). \tag{5}$$

LEMMA 2.2. *Let $p > 0$, $r \geqslant 0$, $\{\lambda_n\}$ be a positive sequence satisfying*

$$\Lambda_{2n} \ll \Lambda_n \ll n\lambda_n.$$

Let $b_n \in$ MVBVS and $b_n \ll n^{-r-1}\omega(\frac{1}{n})$, and $\omega \in \Omega$, then the Fourier series (1) converges to $f(x)$ uniformly, i. e.,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

Furthermore, if $\{\lambda_n \omega^p(\frac{1}{n}) n^{1-rp}\} \in$ AMS, then

$$f(x) \in H(\lambda, p, r, \omega).$$

Proof. The first part of Lemma 2.2 is clear by the generalization of Chaundy-Jollife's theorem in MVBV condition (see [4: Theorem 5]) and the condition $b_n \ll n^{-r-1}\omega(\frac{1}{n})$. In view of $S_k(f,0) = S_k(f,\pi) = 0$, we may restrict $x \in (0,\pi)$, say, $\frac{\pi}{m+1} < x \leqslant \frac{\pi}{m}$ for $m = 1, 2, \cdots$. By Abel's transformation, for $k \leqslant m$, we have

$$\begin{aligned} |f(x) - S_k(f,x)| &\leqslant \left| \sum_{j=k+1}^{m} b_j \sin jx \right| + \left| \sum_{j=m+1}^{\infty} b_j \sin jx \right| \\ &\ll \left| x \sum_{j=k+1}^{m} jb_j \right| + \sum_{j=m+1}^{\infty} |b_j - b_{j+1}| |\bar{D}_j(x)| \\ &\ll \frac{1}{m} \sum_{j=k+1}^{m} jb_j + m \sum_{j=m+1}^{\infty} |b_j - b_{j+1}|, \end{aligned} \tag{6}$$

where $\bar{D}_j(x) := \sum_{k=1}^{j} \sin kx$, and the estimate $|\bar{D}_j(x)| = O(\frac{1}{x})$ clearly holds. Similarly, for $k \geqslant m$, we obtain that

$$|f(x) - S_k(x)| \ll m \sum_{j=k+1}^{\infty} |b_j - b_{j+1}|.$$

Therefore

$$\begin{aligned}\sum_{k=1}^{n}\lambda_k|f(x)-S_k(x)|^p &= \sum_{k=1}^{m}\lambda_k|f(x)-S_k(x)|^p+\sum_{k=m+1}^{n}\lambda_k|f(x)-S_k(x)|^p\\ &=: I_1+I_2.\end{aligned}$$

From (6) we see that

$$I_1 \ll \sum_{k=1}^{m}\lambda_k\left(\frac{1}{m}\sum_{j=k+1}^{m} jb_j\right)^p+\sum_{k=1}^{m}\lambda_k m^p\left(\sum_{j=m+1}^{\infty}|b_j-b_{j+1}|\right)^p =: I_{11}+I_{12}.$$

First, we estimate I_{11}. If $p \geqslant 1$, by using (4), (2), (3) and $b_k \ll k^{-r-1}\omega(\frac{1}{k})$, we obtain that (Setting $a_v = vb_v$ for $v \leqslant m$ and $a_v = 0$ for $v > m$ we apply (4))

$$\begin{aligned}I_{11} &\leqslant \frac{1}{m^p}\sum_{k=1}^{m}\lambda_k\left(\sum_{j=k}^{m} jb_j\right)^p \ll \frac{1}{m^p}\sum_{k=1}^{m}\lambda_k^{1-p}\left(\sum_{j=1}^{k}\lambda_j\right)^p(kb_k)^p\\ &\ll \frac{1}{m^p}\sum_{k=1}^{m}\lambda_k^{1-p}k^p k^{-(r+1)p}\omega^p(1/k)\Lambda_k^p \ll \frac{1}{m^p}\sum_{k=1}^{m}\lambda_k k^{p-rp}\omega^p(1/k)\\ &\ll m^{1-rp}\lambda_m\omega^p(1/m).\end{aligned}$$

If $0<p<1$, by using (2), (3), (5) and $b_k \ll k^{-r-1}\omega(\frac{1}{k})$, we have

$$\begin{aligned}I_{11} &\ll \frac{1}{m^p}\sum_{k=1}^{m}\lambda_k\left(\sum_{j=k}^{m} j^{-r}\omega(1/j)\right)^p\\ &\ll \frac{1}{m^p}\sum_{k=1}^{m}k^{p-1}k^{-rp}\omega^p(1/k)\left(k\lambda_k+\sum_{j=1}^{k-1}\lambda_j\right)\\ &\ll \frac{1}{m^p}\sum_{k=1}^{m}k^{p-1}k^{1-rp}\omega^p(1/k)\lambda_k\\ &\ll \frac{1}{m^p}\lambda_m\omega^p(1/m)m^{1-rp}m^p\\ &\ll m^{1-rp}\lambda_m\omega^p(1/m).\end{aligned}$$

Therefore, in any case,

$$I_{11} \ll m^{1-rp}\lambda_m\omega^p(1/m). \tag{7}$$

On the other hand, by Abel's transformation and that $\{b_n\} \in \mathrm{MVBVS}$, we check that

$$\begin{aligned}\sum_{j=m+1}^{\infty}|b_j-b_{j+1}| &\leqslant \sum_{i=0}^{\infty}\sum_{j=2^i m}^{2^{i+1}m}|\triangle b_j| \ll \sum_{i=0}^{\infty}\frac{1}{2^i m}\sum_{j=[\frac{2^i m}{\lambda}]}^{[\lambda 2^i m]} b_j\\ &\ll \sum_{i=0}^{\infty}\frac{1}{2^i m}\sum_{j=[\frac{2^i m}{\lambda}]}^{[\lambda 2^i m]} j^{-r-1}\omega(1/j)\\ &\ll \sum_{i=0}^{\infty}\frac{1}{2^i m}\omega\left(\frac{1}{2^i m}\right)\sum_{j=[\frac{2^i m}{\lambda}]}^{[\lambda 2^i m]} j^{-r-1}\\ &\ll \sum_{i=0}^{\infty}\frac{1}{2^i m}\omega\left(\frac{1}{2^i m}\right)\frac{1}{(2^i m)^{r+1}}2^i m\\ &\ll \omega(1/m)\frac{1}{m^{r+1}},\end{aligned}$$

with $\Lambda_m \ll m\lambda_m$, we get

$$I_{12} \ll \Lambda_m m^p \omega^p(1/m)\frac{1}{m^{(r+1)p}} \ll \lambda_m m^{1-rp}\omega^p(1/m). \tag{8}$$

At the same time,

$$I_2 = \sum_{k=m+1}^{n} \lambda_k |f(x) - S_k(x)|^p \ll \sum_{k=m+1}^{n} \lambda_k m^p \left(\sum_{l=k}^{\infty} |\triangle b_l|\right)^p,$$

while

$$\begin{aligned}\sum_{l=k}^{\infty} |\triangle b_l| &= \sum_{j=0}^{\infty}\sum_{l=2^j k}^{2^{j+1}k} |\triangle b_l| \ll \sum_{j=0}^{\infty} \frac{1}{2^j k} \sum_{l=[\frac{2^j k}{\lambda}]}^{[\lambda 2^j k]} b_l \\ &\ll \sum_{j=0}^{\infty} \frac{1}{2^j k} \sum_{l=[\frac{2^j k}{\lambda}]}^{[\lambda 2^j k]} l^{-r-1}\omega(1/l) \ll \sum_{j=0}^{\infty} \frac{1}{2^j k}\omega\left(\frac{1}{2^j k}\right)\left(\frac{2^j k}{\lambda}\right)^{-r-1} 2^j k \\ &\ll \sum_{j=0}^{\infty} \omega\left(\frac{1}{2^j k}\right)(2^j k)^{-r-1} \ll \omega(1/k)k^{-r-1},\end{aligned}$$

then

$$\begin{aligned} I_2 &\ll m^p \sum_{k=m+1}^{n} \lambda_k \omega^p(1/k)k^{-(r+1)p} = m^p \sum_{k=m+1}^{n} \lambda_k \omega^p(1/k) k^{1-rp}k^{-1-p} \\ &\ll m^p \lambda_n \omega^p(1/n) n^{1-rp} \sum_{k=m+1}^{n} k^{-1-p} \ll n^{1-rp}\lambda_n\omega^p(1/n). \end{aligned} \tag{9}$$

From the above estimates (7)-(9) we achieve that

$$\sum_{k=1}^{n} \lambda_k |f(x) - S_k(f,x)|^p \ll n^{1-rp}\lambda_n\omega^p(1/n).$$

It is easy to prove that $\Lambda_{2n} \ll \Lambda_n \ll n\lambda_n$ and $\Lambda_{2n} \simeq \Lambda_n \simeq n\lambda_n$ is equivalent (in the same manner as the proof of [6]). With all the above discussions, we have proved

$$f(x) \in H(\lambda, p, r, \omega).$$

Lemma 2.2 is completed. □

§3. Proof of the Theorem

Proof. Following the technique of Tikhonov [6], we get

$$\omega(n^{-1}) \gg n^{-(\beta+1)} \sum_{k=1}^{n} k^{r+\beta+1} b_k.$$

Let $m = [\lambda n] + 1$, $\{b_k\} \in \text{MVBVS}$, we have

$$\begin{aligned}\omega(n^{-1}) &\geqslant \omega(m^{-1}) \gg m^{-(\beta+1)} \sum_{k=1}^{m} k^{r+\beta+1} b_k \\ &\gg m^{-(\beta+1)} \sum_{k=1}^{m} b_k \sum_{j=1}^{k} j^{r+\beta} = m^{-(\beta+1)} \sum_{k=1}^{m} k^{r+\beta} \sum_{j=k}^{m} b_j \\ &\gg (\lambda n)^{-(\beta+1)} \sum_{k=1}^{[\frac{n}{\lambda}]} k^{r+\beta} \sum_{j=[\frac{n}{\lambda}]}^{[\lambda n]} b_j.\end{aligned}$$

Let $n \leqslant k \leqslant 2n$,

$$b_n \leqslant \sum_{j=n}^{k-1} |\triangle b_j| + b_k \leqslant \sum_{j=n}^{2n} |\triangle b_j| + b_k \leqslant M(b) n^{-1} \sum_{j=[\frac{n}{\lambda}]}^{[\lambda n]} b_j + b_k,$$

so that

$$n b_n = \sum_{k=n}^{2n} b_n \leqslant M(b) n^{-1} \sum_{k=n}^{2n} \sum_{j=[\frac{n}{\lambda}]}^{[\lambda n]} b_j + \sum_{k=n}^{2n} b_k \leqslant M(b) \sum_{j=[\frac{n}{\lambda}]}^{[\lambda n]} b_j,$$

and it yields that

$$\omega(n^{-1}) \gg n^{-(\beta+1)} \sum_{k=1}^{[\frac{n}{\lambda}]} k^{r+\beta} n b_n \gg n^{-(\beta+1)} n b_n n^{r+\beta+1} = n^{r+1} b_n,$$

i. e.,

$$b_n \ll n^{-r-1} \omega(n^{-1}).$$

From Lemma 2.2 we know that

$$f(x) \in H(\lambda, p, r, \omega).$$

The proof of the Theorem is completed. □

FINAL REMARK. During improving the monotonicity conditions, before appearing the above mentioned ultimate class MVBVS due to Zhou-Zhou-Yu [4], some other classes as independent antecedents have also been defined, see e.g. Leindler [8-10].

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