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ON SOME FURTHER SECOND ORDER INTEGRAL INEQUALITIES

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Abstract. Some weighted quadratic integral inequalities of the second order involving a function, its first and second derivative has been derived using the uniform method of obtaining integral inequalities. An example in which some new integral inequalities with Chebyshev weight functions appeared has been considered.

1. Introduction

In this paper we would like to derive the integral inequality of the form

$$\int_I uh'^2 dt \leqslant \int_I \left(sh^2 + rh''^2\right) dt, \qquad h \in \hat{H}, \tag{1}$$

where $I = (\alpha, \beta)$, $-\infty \leqslant \alpha < \beta \leqslant \infty$, r, u and s are real functions of the variable t, $\hat{H}$ is a class of functions defined later on.

We will use the uniform method for obtaining integral inequalities first introduced by Florkiewicz and Rybarski in [4] for first order integral inequalities and then used to obtain different types of integral inequalities involving a function and its first derivative (for references see [3] , [7]) and then, in [7] applied to obtain second order integral inequalities of Hardy type of the form

$$\int_I sh^2 dt \leqslant \int_I rh''^2 dt, \qquad h \in H, \tag{2}$$

In [6] the inequality (2) has been obtained in a different class of functions h.

Further studies made it possible to extend the method which allowed to obtain in [5] an inequality of the form (1).

In [5] the uniform method of obtaining integral inequalities is as follows. Given positive and absolutely continuous weight functions r, u and an auxiliary function $\varphi > 0$, where $2r\varphi''\varphi^{-1} + u \leqslant 0$, there is determined directly the weight function s and auxiliary functions v_0 and v_1. Then using these functions the class of functions for which the inequality (1) holds is constructed.

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Now we extend the results obtained in [7], [6] and [5] and we modify the method which let us obtain the integral inequalities of the form (1) in the new class of functions $\hat{H}$.

Our modification of the method let us for given positive weight functions r and u and auxiliary function $\varphi > 0$, $(r\varphi')'\varphi + 2r\varphi\varphi'' + u\varphi^2 - 2r\varphi'^2 \leqslant 0$, determine directly the weight function s such that a suitable differential identity is satisfied. Next, we use the obtained differential identity to construct the class $\hat{H}$ of functions h, depending on the auxiliary functions w_0, w_1 and w_2 for which the considered integral inequality holds.

Then we derive some new integral inequalities of the form (1) with the Chebyshev weight functions $\big($i.e. $(1-t^2)^a$, $a = const\big)$ on $I = (-1,1)$ and show that for such a weight functions boundary conditions for the class $\hat{H}$ can be simplified. We also illustrate on the cosidered example that the classes H in [5] and $\hat{H}$ determined in this paper are different because this extension allows to obtaine the cases that cannot be obtained in [5].

Integral inequalities is the branch of mathematics which developes rapidly during last years. It has applications both to other branches of mathematics to other areas (see: [10]).

Many authors considered the inequality of the form (2) e.g. Nasyrova and Stepanov [16], Kufner and Sinnamon [11], Kufner and Wannebo [12]. Further references can be found in the fairly new book [10] by Kufner and Persson and also in Kufner and Opic book [9] and in [7]. In fact, (2) is a special case of the higher order Hardy type inequalities studied in Chapter 4 of the book [10] but the results in this Chapter do not cover the results in this paper.

However the method used by others was different. Namely they considered the boundary conditions of the form that functions h and /or its derivatives vanish at the endpoints and found the conditions for weights such that the inequality is valid. In the uniform method given positive functions r and auxiliary φ satisfying some additional conditions, e.g. $\varphi'' \leqslant 0$ a.e. on I, the weight function $s = (r\varphi'')''\varphi^{-1}$ is determined directly and class of functions satisfying some integral and limit conditions for which the inequality (2) holds is built.

Also the inequalities of the form (1) have been considered by others (see e.g. Benson [1], Leighton [14], Talenti [17]) but different approach was used. Further detailed studies can be found in the monographs [8], [13] and [15].

2. Main result

Let $I = (\alpha, \beta), -\infty \leqslant \alpha < \beta \leqslant \infty,$ be an arbitrary open interval. We denote by $AC(I)$ the class of real functions absolutely continuous on the interval I, and by $AC^1(I)$ the class of functions $f \in AC(I)$ such that $f' \in AC(I)$. Let $r \in AC(I)$, $u \in AC(I)$ and $\varphi \in AC^1(I)$ be given functions such that $r > 0$, $u > 0$ and $\varphi > 0$ on the interval I and $r\varphi'' \in AC^1(I)$.

Put

$$s = -\left[\left(r\varphi''\right)'' + \left(u\varphi'\right)'\right]\varphi^{-1}. \tag{3}$$

Denote by $\hat{H}$ the class of functions $h \in AC^1(I)$ satisfying the following integrability conditions

$$\int_I rh''^2 dt < \infty, \qquad \int_I sh^2 dt < \infty \tag{4}$$

and the limit conditions

$$\begin{aligned} \liminf_{t\to\alpha} \left(w_0h^2 + 2w_1hh' + w_2h'^2\right) &< \infty, \\ \limsup_{t\to\beta} \left(w_0h^2 + 2w_1hh' + w_2h'^2\right) &> -\infty \end{aligned} \tag{5}$$

and

$$\liminf_{t\to\alpha} \left(w_0h^2 + 2w_1hh' + w_2h'^2\right) \leqslant \limsup_{t\to\beta} \left(w_0h^2 + 2w_1hh' + w_2h'^2\right), \tag{6}$$

where

$$w_0 = r(\varphi^{-1}\varphi')^3 + r\varphi''(\varphi^{-1})' - (r\varphi'')'\varphi^{-1} - u\varphi'\varphi^{-1} \tag{7}$$
$$w_1 = r(\varphi^{-1}\varphi')' \tag{8}$$
$$w_2 = r\varphi^{-1}\varphi'. \tag{9}$$

THEOREM 1. *Let* $w = (r\varphi')'\varphi + 2r\varphi\varphi'' + u\varphi^2 - 2r\varphi'^2 \leqslant 0$ *almost everywhere on the interval I.*

Then for every function $h \in \hat{H}$ *the inequality*

$$\int_I uh'^2 dt \leqslant \int_I \left(sh^2 + rh''^2\right) dt \tag{10}$$

holds.

If $w \not\equiv 0$ *and* $h \not\equiv 0$ *then the inequality* (10) *becomes an equality if and only if* $h = c\varphi$ *with* $c =$ const $\neq 0$ *and the additional conditions*

$$\varphi \in \hat{H}, \quad \lim_{t\to\alpha} \left(w_0h^2 + 2w_1hh' + w_2h'^2\right) = \lim_{t\to\beta} \left(w_0h^2 + 2w_1hh' + w_2h'^2\right) \tag{11}$$

are satisfied.

Proof. This proof is a modification of the proof of Theorem 1 in [5], [6]. It has been shown in [5] that the identity

$$rh''^2 - uh'^2 + sh^2 = \left(v_0h^2 + 2v_1hh'\right)' + g \tag{12}$$

holds, where

$$v_0 = r\varphi''(\varphi^{-1})' - (r\varphi'')'\varphi^{-1} - u\varphi'\varphi^{-1}, \tag{13}$$
$$v_1 = r\varphi''\varphi^{-1} \tag{14}$$

and

$$g_1 = r\left[\varphi\left(\varphi^{-1}h\right)'' + 2\varphi'\left(\varphi^{-1}h\right)'\right]^2 - \left(2r\varphi\varphi'' + u\varphi^2\right)\left[\left(\varphi^{-1}h\right)'\right]^2 \tag{15}$$

is valid almost everywhere on I. Now we transform the right hand side of the identity (12).

Let $h \in AC^1(I)$. By virtue of (7),(8) and (9) and from assumptions we have $\varphi^{-1}h \in AC^1(I)$ and $w_0h^2+2w_1hh'+w_2h'^2 \in AC(I)$. If we substitute $h=\varphi f$, where $f \in AC^1(I)$, in the expression $rh''^2-uh'^2+sh^2$, then we have

$$rh''^2 = r\varphi''\left(\varphi f^2\right)''+r\left[\left(\varphi f''+\varphi' f'\right)+\varphi' f'\right]^2-2r\varphi\varphi'' f'^2.$$

Moreover, by using the identities

$$r\varphi''\left(\varphi f^2\right)''-\left(r\varphi''\right)''\varphi f^2=\left[r\varphi''\left(\varphi f^2\right)'-\left(r\varphi''\right)'\varphi f^2\right]'$$

and

$$\begin{aligned} r\left[\varphi' f'+(\varphi f''+\varphi' f')\right]^2 &= (r\varphi')\varphi' f'^2+(r\varphi')\varphi(f'^2)'+(r\varphi')'\varphi f'^2-(r\varphi')'\varphi f'^2 \\ &\quad +2r\varphi'^2 f'^2+r(\varphi f''+\varphi' f')^2 \\ &= \left(r\varphi\varphi' f'^2\right)'+\left(2r\varphi'^2-(r\varphi')'\varphi\right)f'^2+r(\varphi f''+\varphi' f')^2 \end{aligned}$$

we obtain

$$\begin{aligned} rh''^2 &= (r\varphi'')''\varphi f^2+\left[r\varphi''\left(\varphi f^2\right)'-(r\varphi'')'\varphi f^2+r\varphi\varphi' f'^2\right]' \\ &\quad +r(\varphi f''+\varphi' f')^2-\left(2r\varphi\varphi''-2r\varphi'^2+(r\varphi')'\varphi\right)f'^2. \end{aligned} \tag{16}$$

Similarly

$$uh'^2=u\left(\varphi'^2 f^2+2\varphi\varphi' ff'+\varphi^2 f'^2\right),$$

and, due to the fact that

$$u\varphi'\left(\varphi f^2\right)'=u\varphi'^2 f^2+2u\varphi\varphi' ff',$$

we obtain

$$uh'^2=u\varphi'\left(\varphi f^2\right)'+u\varphi^2 f'^2. \tag{17}$$

Moreover

$$sh^2=-\left[(r\varphi'')''+(u\varphi')'\right]\varphi f^2$$

and using

$$(u\varphi')'\varphi f^2=\left(u\varphi\varphi' f^2\right)'-u\varphi'\left(\varphi f^2\right)'$$

we get

$$sh^2=-(r\varphi'')''\varphi f^2-(u\varphi\varphi' f^2)'+u\varphi'(\varphi f^2)' \tag{18}$$

From (16), (17) and (18) substituting $f = \varphi^{-1}h,\ \varphi f^2 = \varphi^{-1}h^2,$ in the expression

$$r\varphi''(\varphi f^2)' - (r\varphi'')'\varphi f^2 + r\varphi\varphi' f'^2 - u\varphi\varphi' f^2,$$

we have

$$\left(r\left(\varphi^{-1}\varphi'\right)^3 + r\varphi''\left(\varphi^{-1}\right)' - (r\varphi'')'\varphi^{-1} - u\varphi'\varphi^{-1}\right)h^2 + 2r\left(\varphi^{-1}\varphi'\right)' hh' + r\varphi^{-1}\varphi' h'^2$$
$$= w_0h^2 + 2w_1hh' + w_2h'^2.$$

Therefore we get the following identity

$$rh''^2 - uh'^2 + sh^2 = \left(w_0h^2 + 2w_1hh' + w_2h'^2\right)' + g, \tag{19}$$

where

$$\begin{aligned} g &= r\left[\varphi\left(\varphi^{-1}h\right)'' + \varphi'\left(\varphi^{-1}h\right)'\right]^2 - \left((r\varphi')'\varphi + 2r\varphi\varphi'' + u\varphi^2 - 2r\varphi'^2\right)\left[\left(\varphi^{-1}h\right)'\right]^2 \\ &= r\left[\varphi\left(\varphi^{-1}h\right)'' + \varphi'\left(\varphi^{-1}h\right)'\right]^2 - w\left[\left(\varphi^{-1}h\right)'\right]^2. \end{aligned} \tag{20}$$

The identity (19) is valid almost everywhere on I and, according to assumptions, we have $g \geqslant 0$ a.e. on I.

Now let $h \in \hat{H}$. The first condition of (4) implies that the function rh''^2 is summable on I since $rh''^2 \geqslant 0$ on I. It follows from the assumptions that the functions uh'^2, sh^2 and $\left(w_0h^2 + 2w_1hh' + w_2h'^2\right)'$ are summable on each compact interval $[a,b] \subset I$.

Thus by (19) we get the summability of the function g on $[a,b] \subset I$ and we obtain the equality

$$\int_a^b \left(rh''^2 + sh^2\right)dt = \int_a^b uh'^2 dt + \left(w_0h^2 + 2w_1hh' + w_2h'^2\right)\Big|_a^b + \int_a^b g\,dt. \tag{21}$$

for arbitrary $\alpha < a < b < \beta$. In view of (5) there exist two sequences $\{a_n\}$ and $\{b_n\}$ such that $\alpha < a_n < b_n < \beta$, $a_n \to \alpha$, $b_n \to \beta$ and

$$\lim_{n\to\infty}\left[-\left(w_0h^2 + 2w_1hh' + w_2h'^2\right)\right]\Big|_{a_n} > -\infty$$

$$\lim_{n\to\infty}\left(w_0h^2 + 2w_1hh' + w_2h'^2\right)\Big|_{b_n} > -\infty.$$

Thus there is a constant C such that

$$\left(w_0h^2 + 2w_1hh' + w_2h'^2\right)\Big|_{a_n}^{b_n} \geqslant C > -\infty.$$

By virtue of the condition $g \geqslant 0$ a.e. on I, the assumption that $u > 0$ on I and from the equality (21) we infer that

$$\int_{a_n}^{b_n}\left(rh''^2 + sh^2\right)dt \geqslant C$$

and from this by letting $n \to \infty$ we obtain

$$\int_I \left(rh''^2 + sh^2\right) dt \geqslant C > -\infty.$$

By this estimate and by (4) we conclude that $rh''^2 + sh^2$ is summable on I. Next, in a similar way using (21) and the summability of the function $rh''^2 + sh^2$ on I we prove, in turn, the summability of the functions uh'^2 and g on I. Thus all the integrals in the equality (21) have finite limits as $a \to \alpha$ or $b \to \beta$, and hence both of the limits in (5) are proper and finite. Therefore the conditions (5) and (6) may be written in the equivalent form

$$-\infty < \lim_{t\to\alpha}\left(w_0h^2 + 2w_1hh' + w_2h'^2\right) \leqslant \lim_{t\to\beta}\left(w_0h^2 + 2w_1hh' + w_2h'^2\right) < \infty. \quad (22)$$

Now by (21) as $a \to \alpha$ and $b \to \beta$ we obtain the equality

$$\int_I \left(rh''^2 dt + sh^2\right) dt = \int_I uh'^2 + \int_I gdt$$
$$+\lim_{t\to\beta}\left(w_0h^2 + 2w_1hh' + w_2h'^2\right) - \lim_{t\to\alpha}\left(w_0h^2 + 2w_1hh' + w_2h'^2\right), \quad (23)$$

whence, in view of (22), the inequality (10) follows since $g \geqslant 0$ a.e. on I.

The proof of equality condition follows as the proof of equality condition in [6] however in [6] conditions for w were different.

If the inequality (10) becomes an equality for a non-vanishing function $h \in \hat{H}$, then by (22) and (23) we have

$$\int_I gdt = 0$$
$$\lim_{t\to\alpha}\left(w_0h^2 + 2w_1hh' + w_2h'^2\right) = \lim_{t\to\beta}\left(w_0h^2 + 2w_1hh' + w_2h'^2\right). \quad (24)$$

As $g \geqslant 0$ a.e. on I we obtain $g = 0$ a.e. on I. Hence

$$\varphi\left(\varphi^{-1}h\right)'' + 2\varphi'\left(\varphi^{-1}h\right)' = 0 \quad (25)$$

a.e. on I and

$$w\left[\left(\varphi^{-1}h\right)'\right]^2 = 0$$

a.e. on I. According to the equality assumptions $w = (r\varphi')'\varphi + 2r\varphi\varphi'' + u\varphi^2 - 2r\varphi'^2 \not\equiv 0$. Moreover it follows from the assumptions that $w \leqslant 0$ and $w \in AC(I)$. Therefore $w < 0$ on some $(a,b) \in I$. Whence $(\varphi^{-1}h)'(t_0) = 0$ for some $t_0 \in I$ and the function $(\varphi^{-1}h)' \in AC(I)$ satisfies the homogeneous linear differential equation (25) with the initial value $(\varphi^{-1}h)'(t_0) = 0$, and whence $(\varphi^{-1}h)' = 0$ on I. This implies that $h = c\varphi$, where $c = \text{const} \neq 0$, since $\varphi^{-1}h \in AC^1(I)$. Thus $\varphi \in \hat{H}$ so that we obtain the first condition of (11) and from the second condition of (24) we get the second condition of (11).

Now let (11) be satisfied and let $h = c\varphi$, where $c = \text{const} \neq 0$. That implies $g = 0$ a.e. on I so that $\int_I g dt = 0$ and, in view of (23), the inequality (10) becomes an equality which completes the proof.

3. Example

Now we derive some new integral inequalities with the Chebyshev weight functions and show, by using [6], that in some cases the limit conditions can be simplified.

EXAMPLE. Let us take $I = (-1,1)$ and the functions $r = (1-t^2)^a$, $u = A(1-t^2)^{a-1}$ and $\varphi = (1-t^2)^{2-a}$ on I, where a and A are arbitrary constants such that $a < 1$ and $A > 6-4a$ on I. Then

$$s = (2-a)(A+4a-6)\left(1-t^2\right)^{a-2} > 0,$$

on I. Moreover

$$w = \left(6a-12+A+(4-2a-A)t^2\right)\left(1-t^2\right)^{2-a} < 0$$

on I, since

$$4-2a-A < 4-2a-6+4a = 2(a-1) < 0$$

and

$$6a-12+A = (4a-6+A)+2(a-1)-10 < 0.$$

Therefore from (10) we obtain the inequality

$$A\int_{-1}^{1}\left(1-t^2\right)^{a-1} h'^2 dt \leqslant (2-a)(A+4a-6)\int_{-1}^{1}\left(1-t^2\right)^{a-2} h^2 dt + \int_{-1}^{1}\left(1-t^2\right)^{a} h''^2 dt, \quad (26)$$

which holds for $h \in AC^1(-1,1)$ satisfying the integral conditions

$$\int_{-1}^{1}\left(1-t^2\right)^{a} h''^2 dt < \infty, \ \int_{-1}^{1}\left(1-t^2\right)^{a-2} h^2 dt < \infty \quad (27)$$

and the limit condition (22) with w_0, w_1 and w_2 equal to:

$$w_0(t) = 2(2-a)\,t\,\left[A+6a-10-(2a+A-2)t^2\right]\left(1-t^2\right)^{a-3} = \hat{\omega}_0(t)\left(1-t^2\right)^{a-3},$$

$$w_1(t) = 4(a-2)\left(1+t^2\right)\left(1-t^2\right)^{a-2} = \hat{\omega}_1(t)\left(1-t^2\right)^{a-2},$$

$$w_2(t) = 2(a-2)t\left(1-t^2\right)^{a-1} = \hat{\omega}_2(t)\left(1-t^2\right)^{a-1},$$

respectively.

Now we show that a function $h \in AC^1((-1,1))$ satisfying the integral conditions (27) and the limit conditions

$$h(-1) = h'(-1) = h(1) = h'(1) = 0 \quad (28)$$

belongs to the class $\hat{H}$ using the method introduced in [6].

It was shown in [6] that if $h(1) = h'(1) = 0$ and $\int_{-1}^{1} \left(1-t^2\right)^a h''^2 dt < \infty$, then the limit condition

$$\lim_{t \to 1} S(t, h, h') = 0$$

is satisfied, where

$$S(t,h,h') = \omega_0(t)\left(1-t^2\right)^{a-3} h^2 + \omega_1(t)\left(1-t^2\right)^{a-2} hh' + \omega_2(t)\left(1-t^2\right)^{a-1} h'^2 \qquad (29)$$

with

$$\begin{aligned} \omega_0(t) &= 4(a-2)\, t\, \left[(5-3a)+(a-1)t^2\right], \\ \omega_1(t) &= 4(a-2)\left(1+t^2\right), \\ \omega_2(t) &= 2(a-2)t. \end{aligned}$$

It is easy to see that if we put $\hat{\omega}_0$ instead if ω_0, $\hat{\omega}_1$ instead of ω_1 and $\hat{\omega}_2$ instead of ω_2 than we obtain that

$$\begin{aligned} \hat{S}(t,h,h') &= \lim_{t \to 1} \Big(\hat{\omega}_0(t)\left(1-t^2\right)^{a-3} h^2 + \hat{\omega}_1(t)\left(1-t^2\right)^{a-2} hh' \\ &\qquad + \hat{\omega}_2(t)\left(1-t^2\right)^{a-1} h'^2 \Big) = 0 \qquad (30) \end{aligned}$$

provided that integral conditions (27) are fulfilled.

In analogous way we show that if $h(-1) = h'(-1) = 0$ and (27) holds, then

$$\lim_{t \to -1} \hat{S}(t, h, h') = 0.$$

Based on the above considerations we obtain that from the conditions (27) and (28) follows the limit condition (22).

Therefore we get the following:

LEMMA. *If a function $h \in AC^1((-1,1))$ satisfies the integral condition*

$$\int_{-1}^{1} \left(1-t^2\right)^a h''^2 dt < \infty, \qquad \int_{-1}^{1} \left(1-t^2\right)^{a-2} h^2 dt < \infty$$

and the limit conditions

$$h(-1) = h'(-1) = h(1) = h'(1) = 0,$$

then the inequality

$$A \int_{-1}^{1} (1-t^2)^{a-1} h'^2 dt \leqslant (2-a)(A+4a-6) \int_{-1}^{1} (1-t^2)^{a-2} h^2 dt + \int_{-1}^{1} (1-t^2)^a h''^2 dt,$$

holds on $I = (-1,1)$, where $a < 1$, $A > 6-4a$ on I.

The inequality (26) becomes an equality if and only if $h = c\left(1-t^2\right)^{2-a}$, where $c = \text{const}$.

REMARK. In the paper [5] the inequality (26) was derived under the assumption that $1 < a < \frac{3}{2}$ on I. This assumption appeared as a result of the condition $w = 2r\varphi''\varphi^{-1} + u \leqslant 0$. Indeed since r, u and φ are positive it is necessary that $\varphi'' < 0$ and the inequality (26) with $a < 1$ cannot be obtained from Theorem 1 in [5], as for such a the function $\varphi = \left(1 - t^2\right)^{2-a}$ doesn't satisfy the condition $\varphi'' \leqslant 0$ on $(-1, 1)$. In this paper for $a < 1$ we have $w = (r\varphi')'\varphi + 2r\varphi\varphi'' + u\varphi^2 - 2r\varphi'^2 \leqslant 0$ so the inequality (26) follows from Theorem 1. Therefore the classes of functions h for which the inequality (1) holds obtained in [5] and in this paper are not equal.

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