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GENERALIZED HYERS–ULAM STABILITY OF RICCATI DIFFERENTIAL EQUATION

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Abstract. In this paper, we will prove the generalized Hyers-Ulam stability of the Riccati differential equation of the form $y'(t)+g(t)y(t)+h(t)y(t)^2=k(t)$ under some additional conditions. Some concrete examples will be introduced.

1. Introduction

Let X be a normed space over a scalar field $\mathbb{K}$ and let I be an open interval. By $P(D)=a_nD^n+a_{n-1}D^{n-1}+\cdots+a_1D+a_0$ we denote an n-th order differential operator, where D is the differentiation with respect to t and $a_i:I\to\mathbb{K}$ are continuous functions for $i\in\{0,1,\ldots,n\}$.

Assume that for a fixed $h:I\to X$ and for any n times strongly differentiable function $y:I\to X$ satisfying the inequality

$$\|P(D)y(t)+h(t)\|\leqslant\varepsilon$$

for all $t\in I$ and for some $\varepsilon\geqslant 0$, there exists a function $y_0:I\to X$ satisfying $P(D)y_0(t)+h(t)=0$ and $\|y(t)-y_0(t)\|\leqslant K(\varepsilon)$ for any $t\in I$, where $K(\varepsilon)$ is an expression for ε only. Then, we say that the above differential equation has the Hyers-Ulam stability.

If the above statement is also true when we replace ε and $K(\varepsilon)$ by $\varphi(t)$ and $\Phi(t)$, where $\varphi,\Phi:I\to[0,\infty)$ are functions not depending on y and y_0 explicitly, then we say that the corresponding differential equation has the generalized Hyers-Ulam stability.

We may apply these terminologies for other differential equations. For more detailed definitions of the Hyers-Ulam stability and the generalized Hyers-Ulam stability (or the Hyers-Ulam-Rassias stability), we refer the reader to [2, 3, 4, 5, 13, 15].

C. Alsina and R. Ger were the first authors who investigated the Hyers-Ulam stability of differential equations: They proved in [1] that if a differentiable function $y:I\to\mathbb{R}$ satisfies the differential inequality $|y'(t)-y(t)|\leqslant\varepsilon$, where I is an open

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subinterval of $\mathbb{R}$, then there exists a differentiable function $y_0 : I \to \mathbb{R}$ satisfying $y_0'(t) = y_0(t)$ and $|y(t) - y_0(t)| \leqslant 3\varepsilon$ for any $t \in I$.

This result of Alsina and Ger has been generalized by S.-E. Takahasi, T. Miura and S. Miyajima. They proved in [14] that the Hyers-Ulam stability holds true for the Banach space valued differential equation $y'(t) = \lambda y(t)$ (see also [11]).

Recently, the Hyers-Ulam stability problems for various linear differential equations have been investigated by the first author (see [6, 7, 8, 9]).

Assume that $g, h, k : I \to \mathbb{C}$ are continuous functions, where I is an open interval. The Riccati differential equation

$$y'(t) + g(t)y(t) + h(t)y(t)^2 = k(t) \tag{1}$$

is one of important differential equations.

If one solution of the Riccati's equation is known, then the other solutions can be represented in terms of the known solution: Assume that $f_0 : I \to \mathbb{C}$ is a known solution and $f : I \to \mathbb{C}$ is another solution of (1). If we define $y : I \to \mathbb{C}$ by $y(t) = f(t) - f_0(t)$, then y satisfies the Bernoulli differential equation

$$y'(t) + [g(t) + 2h(t)f_0(t)]y(t) + h(t)y(t)^2 = 0.$$

In [10, Theorem 1], the authors investigated the generalized Hyers-Ulam stability of a special form of the Bernoulli differential equation. In this paper, we will prove the generalized Hyers-Ulam stability of the Riccati differential equation by applying [10, Theorem 1]. Moreover, some concrete examples will be introduced.

2. Generalized Hyers-Ulam stability of Riccati's equation

Throughout this section, let $I = (a, b)$ be an open interval with $-\infty \leqslant a < b \leqslant \infty$. The following theorem dealing with a stability problem for the Bernoulli's equation was proved in [10].

THEOREM 1. *Let $y : I \to \mathbb{C}$ be a continuously differentiable function satisfying the differential inequality*

$$|y(t)^{-\alpha}y'(t) + g(t)y(t)^{1-\alpha} + h(t)| \leqslant \varphi(t) \tag{2}$$

for all $t \in I$, where $g, h : I \to \mathbb{C}$ are continuous functions, $\varphi : I \to [0, \infty)$ is a function, and where $\alpha \neq 1$ is a fixed real number satisfying $y(t)^{-\alpha} \in \mathbb{C}$ for all $t \in I$. Suppose $g(t)$ is integrable on (a, s) for any $s \in I$ and let $G(t) = \int_a^t g(v)\,dv$. Assume that

(a) *$h(t)e^{(1-\alpha)G(t)}$ is integrable on (a, s) for each $s \in I$;*

(b) *$\varphi(t)e^{(1-\alpha)\Re(G(t))}$ is integrable on I, where $\Re(G(t))$ denotes the real part of $G(t)$.*

Then, there exists a unique complex number c_0 *such that*

$$\left|y(t)^{1-\alpha} - (1-\alpha)e^{(\alpha-1)G(t)}\left[c_0 - \int_a^t e^{(1-\alpha)G(v)}h(v)\,dv\right]\right| \leqslant |1-\alpha|e^{(\alpha-1)\Re(G(t))}\int_t^b \varphi(v)e^{(1-\alpha)\Re(G(v))}dv$$

for all $t \in I$.

Using the above result, we can now prove the generalized Hyers-Ulam stability of the Riccati differential equation (1).

THEOREM 2. *Let* $f_0 : I \to \mathbb{C}$ *be a continuously differentiable function satisfying the differential inequality*

$$|f_0'(t) + g(t)f_0(t) + h(t)f_0(t)^2 - k(t)| \leqslant \varphi(t) \tag{3}$$

for all $t \in I$, *where* $g,h,k : I \to \mathbb{C}$ *are continuous functions and* $\varphi : I \to [0,\infty)$ *is a function. Moreover, suppose* $g(t)$ *and* $f_0(t)h(t)$ *are integrable on* (a,s) *for any* $s \in I$ *and let* $G(t) = \int_a^t [g(v) + 2f_0(v)h(v)]\,dv$. *Assume that*

(a) $h(t)e^{-G(t)}$ *is integrable on* (a,s) *for each* $s \in I$;

(b) $\varphi(t)e^{-\Re(G(t))}$ *is integrable on* I.

For any $\varepsilon > 0$, *define*

$$M_{f_0}^{\varepsilon}(I,\mathbb{C}) = \left\{f_1 : I \to \mathbb{C} \;:\; \inf_{t\in I} |f_1(t) - f_0(t)| \geqslant \varepsilon\right\}.$$

If $f \in M_{f_0}^{\varepsilon}(I,\mathbb{C})$ *is continuously differentiable and satisfies* (3) *instead of* f_0, *then there exists a unique complex number* c_0 *such that*

$$\left|\frac{1}{f(t)-f_0(t)} + e^{G(t)}\left[c_0 - \int_a^t e^{-G(v)}h(v)\,dv\right]\right| \leqslant \frac{2}{\varepsilon^2}e^{\Re(G(t))}\int_t^b \varphi(v)e^{-\Re(G(v))}\,dv$$

for all $t \in I$.

Proof. Let us define a function $y : I \to \mathbb{C}$ by $y(t) = f(t) - f_0(t)$. Since f_0 and f satisfy the inequality (3), if we subtract the inequality (3) for f_0 from that for f, then we get

$$\left|y'(t) + g(t)y(t) + h(t)[f(t)^2 - f_0(t)^2]\right| \leqslant 2\varphi(t)$$

or

$$|y'(t) + g(t)y(t) + h(t)y(t)[y(t) + 2f_0(t)]| \leqslant 2\varphi(t),$$

since $f(t) + f_0(t) = y(t) + 2f_0(t)$. Equivalently, we have

$$\left|y'(t) + [g(t) + 2f_0(t)h(t)]y(t) + h(t)y(t)^2\right| \leqslant 2\varphi(t)$$

for any $t \in I$.

Since $f \in M_{f_0}^{\varepsilon}(I, \mathbb{C})$, we see that $\inf_{t\in I} |y(t)| \geqslant \varepsilon$. Hence, if we divide both sides of the last inequality by $|y(t)|^2$, then

$$\left|y(t)^{-2}y'(t) + [g(t) + 2f_0(t)h(t)]y(t)^{-1} + h(t)\right| \leqslant \frac{2}{\varepsilon^2}\varphi(t) \tag{4}$$

for each $t \in I$.

If we assume $\alpha = 2$ in Theorem 1, we infer from (4) that there exists a unique complex number c_0 such that

$$\left|y(t)^{-1} + e^{G(t)}\left[c_0 - \int_a^t e^{-G(v)}h(v)\,dv\right]\right| \leqslant \frac{2}{\varepsilon^2}e^{\Re(G(t))}\int_t^b \varphi(v)e^{-\Re(G(v))}\,dv$$

for all $t \in I$. □

We can now raise a question about whether Theorem 2 can be proved without the condition $f \in M_{f_0}^{\varepsilon}(I, \mathbb{C})$. It is an open problem.

3. Examples

EXAMPLE 1. We know that the function $f_0(t) = -\frac{1}{t}$ is a solution of the Riccati's equation

$$y'(t) - 2ty(t) - y(t)^2 = 2. \tag{5}$$

Assume that a continuously differentiable function $f : (a,b) \to \mathbb{R}$ satisfies the inequality

$$\left|f'(t) - 2tf(t) - f(t)^2 - 2\right| \leqslant \varphi(t) \tag{6}$$

for all $t \in (a,b)$, where a and b satisfy $0 < a < b \leqslant \infty$ and $\varphi : (a,b) \to [0,\infty)$ is a function for which $t^{-2}\varphi(t)e^{t^2}$ is integrable on (a,b). According to Theorem 2, if there exists an $\varepsilon > 0$ such that $\left|f(t) + \frac{1}{t}\right| \geqslant \varepsilon$ for all $t \in (a,b)$, then there exists a unique real number c_0 such that

$$\left|\frac{1}{f(t) + \frac{1}{t}} + t^2e^{-t^2}\left(c_0 + \int_a^t \frac{1}{v^2}e^{v^2}\,dv\right)\right| \leqslant \frac{2}{\varepsilon^2}t^2e^{-t^2}\int_t^b \frac{\varphi(v)}{v^2}e^{v^2}\,dv$$

for all $t \in (a,b)$. Since if we set $E(t) = \int_a^t e^{v^2}dv$ then

$$\int_a^t \frac{1}{v^2}e^{v^2}\,dv = \frac{1}{a}e^{a^2} - \frac{1}{t}e^{t^2} + 2E(t),$$

we can equivalently conclude that there exists a unique real number c_1 such that

$$\left|\frac{1}{f(t) + \frac{1}{t}} - \left[t + t^2e^{-t^2}(c_1 - 2E(t))\right]\right| \leqslant \frac{2}{\varepsilon^2}t^2e^{-t^2}\int_t^b \frac{\varphi(v)}{v^2}e^{v^2}\,dv$$

for each $t \in (a,b)$. Indeed, it holds that $c_1 = -c_0 - \frac{1}{a}e^{a^2}$.

REMARK 1. The general solution of the Riccati differential equation (5) can be expressed by

$$y(t) = -\frac{1}{t} + \frac{1}{t + t^2 e^{-t^2}(C - 2E(t))}, \tag{7}$$

where C is an arbitrary (real) constant. Thus, if b is a fixed real number, there certainly exist an $\varepsilon > 0$ and a continuously differentiable function $f : (a,b) \to \mathbb{R}$ such that f satisfies both inequalities $\left|f(t) + \frac{1}{t}\right| \geqslant \varepsilon$ and (6) for all $t \in (a,b)$. (In the worst case, we can choose an f in the form of (7). Then f satisfies the Riccati's equation (5), and hence it satisfies the inequality (6).)

EXAMPLE 2. The function $f_0(t) = t$ is a solution of the Riccati differential equation

$$y'(t) + \left(2t^4 - \frac{1}{t}\right)y(t) - t^3 y(t)^2 = t^5. \tag{8}$$

Assume that a continuously differentiable function $f : (a,b) \to \mathbb{R}$ satisfies the inequality

$$\left|f'(t) + \left(2t^4 - \frac{1}{t}\right)f(t) - t^3 f(t)^2 - t^5\right| \leqslant \varphi(t) \tag{9}$$

for all $t \in (a,b)$, where a and b satisfy $0 < a < b \leqslant \infty$ and $\varphi : (a,b) \to [0,\infty)$ is a function for which $\varphi(t)t$ is integrable on (a,b). According to Theorem 2, if there exists an $\varepsilon > 0$ such that $|f(t) - t| \geqslant \varepsilon$ for all $t \in (a,b)$, then there exists a unique real number c_0 such that

$$\left|\frac{1}{f(t) - t} - \left(\frac{c_0}{t} - \frac{1}{5}t^4\right)\right| \leqslant \frac{2}{\varepsilon^2}\frac{1}{t}\int_t^b \varphi(v)v\,dv$$

for any $t \in (a,b)$.

REMARK 2. It is not difficult to see that the general solution of the Riccati's equation (8) is given by

$$y(t) = t + \frac{1}{\frac{C}{t} - \frac{1}{5}t^4}, \tag{10}$$

where C is an arbitrary (real) constant. Thus, if b is a fixed real number, there surely exist an $\varepsilon > 0$ and a continuously differentiable function $f : (a,b) \to \mathbb{R}$ such that f satisfies both inequalities $|f(t) - t| \geqslant \varepsilon$ and (9) for all $t \in (a,b)$. (As in Remark 1, in the worst case, we may choose an f in the form of (10). Then f satisfies the Riccati's equation (8), and hence it naturally satisfies the inequality (9).)

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