Mathematical
Inequalities
& Applications
Volume 11, Number 4 (2008), 783–794

A NEW SYSTEM OF GENERAL NONLINEAR VARIATIONAL INCLUSIONS INVOLVING (A,η)–ACCRETIVE MAPPINGS IN BANACH SPACES

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(*communicated by R. Verma*)

Abstract. In this paper, a new system of general nonlinear variational inclusions involving (A,η)-accretive mappings in Banach spaces is introduced and studied, which includes many variational inequality (inclusion) problems as special cases. By using the resolvent operator technique for (A,η)-accretive mapping due to Lan-Cho-Verma, an existence and uniqueness theorem of solutions for this system of variational inclusion is proved. A new iterative algorithm for finding approximate solution of this system variational inclusion is suggested and discussed, the convergence and stability of iterative sequence generated by new iterative algorithm is also given. The theorems presented in this paper improve and unify many known results variational inequalities and variational inclusions.

1. Introduction

Variational inequalities and variational inclusions are among the most interesting and important mathematical problems and have been studied intensively in the past years since they have wide applications in mechanics, physics, optimization and control, nonlinear programming, economics, and transportation equilibrium, and engineering sciences, etc. In the theory of variational inequalities and variational inclusions, the development of an efficient and implementable iterative algorithm is interesting and important. Various kinds of iterative algorithms to solve the variational inequalities and inclusions have been developed by many authors. For details, we refer the reader to [1]-[36] and the references therein. Among these methods, the resolvent operator techniques for solving variational inequalities and variational inclusions are interesting and important.

Zhu-Marcotte [36] introduced and investigated a class of system of variational inequalities in R^n. Afterwards, Ansari and Yao [3], Cho et al. [4], Fang and Huang [8], Fang et al. [9], He et al.[14], Jin [20], Kazmi and Bhat [22], Verma [27]-[29], [32]and others studied the approximation solvability of a few kinds of systems of variational inequalities (inclusions).

Mathematics subject classification (2000): 49J40, 47H17, 47H10.

Key words and phrases: (A,η)-accretive mapping; ralaxed cocoercive mapping; system of variational inclusion; iterative algorithm; convergence and stability.

This work was supported by the National Natural Science Foundation of China(10471151) and the Educational Science Foundation of Chongqing, Chongqing of China.

On the other hand, in 2001, Huang and Fang [37] were the first to introduce the generalized m-accretive mapping and give the definition of the resolvent operator for the generalized m-accretive mappings in Banach spaces. They also showed some properties of the resolvent operator for the generalized m-accretive mappings in Banach spaces. For further works, see Huang [15] and the references therein. Recently, inspired and motivated by the works of [7], [9]–[11], [15], [30], [31], [37]. Lan et al. [24] introduced a new concept of (A,η)-accretive mappings, which generalizes the existing monotone or accretive operators, and studied some properties of (A,η)-accretive mappings and defined resolvent operators associated with (A,η)-accretive mappings. They also studied a class of variational inclusions using the resolvent operator associated with (A,η)-accretive mappings.

Inspired and motivated by recent research works in this field, in this paper, we shall introduce and study a new system of general nonlinear variational inclusions involving (A,η)-accretive mappings in Banach spaces, which includes many variational inequality (inclusion) problems as special cases. By using the resolvent operator technique for (A,η)-accretive mapping due to Lan-Cho-Verma, an existence and uniqueness theorem of solutions for this system of variational inclusion is proved. A new iterative algorithm for finding approximate solution of this system variational inclusion is suggested and discussed, the convergence and stability of iterative sequence generated by new iterative algorithm is also given. The theorems presented in this paper improve and unify many known results variational inequalities and variational inclusions.

2. Preliminaries

Throughout this paper, we assume that X is a real Banach space with dual space X^*, $\langle \cdot,\cdot\rangle$ is the dual pair between X and X^*, and 2^X denote the family of all the nonempty subsets of X. The generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \{f^* \in X^* : \langle x, f^*\rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\}, \quad \forall x \in X,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that, in general, $J_q(x) = \|x\|^{q-2}J_2(x)$ for all $x \neq 0$ and J_q is single-valued if X^* is strictly convex, and if $X = H$, the Hilbert space,then J_2 becomes the identity mapping on H.

The modulus of smoothness of X is the function $\rho_X : [0,\infty) \to [0,\infty)$ defined by

$$\rho_X(t) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \leqslant 1, \|y\| \leqslant t\right\}.$$

A Banach space X is called uniformly smooth if

$$\lim_{t\to 0} \frac{\rho_X(t)}{t} = 0.$$

X is called q-uniformly smooth if there exists a constant $c > 0$, such that

$$\rho_X(t) \leqslant ct^q, \qquad q > 1.$$

Note that J_q is single -valued if X is uniformly smooth. In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [38] proved the following result:

LEMMA 2.1. ([38]) *Let X be a real uniformly smooth Banach space. Then X is q-uniformly smooth if and only if there exists a constant $C_q > 0$, such that for all $x, y \in X$,*

$$\|x+y\|^q \leqslant \|x\|^q + q\langle y, J_q(x)\rangle + C_q\|y\|^q.$$

DEFINITION 2.1. Let X_1, X_2 be real Banach spaces. Let Q be a mapping from $X_1 \times X_2 \to X_1 \times X_2$, $(x_0, y_0) \in X_1 \times X_2$ and $(x_{n+1}, y_{n+1}) = f(Q, x_n, y_n)$ define an iterative procedure which yields a sequence of points $\{(x_n, y_n)\}$ in $X_1 \times X_2$. Let $F(Q) = \{(x,y) \in X_1 \times X_2 : (x,y) = Q(x,y)\} \neq \phi$. Suppose that $\{(x_n, y_n)\}$ converges to $(x^*, y^*) \in F(Q)$. Let $\{(u_n, v_n)\}$ be an arbitrary sequence in $X_1 \times X_2$ and $\varepsilon_n = \|\{(u_{n+1}, v_{n+1})\} - f(Q, u_n, v_n)\|$ for each $n \geqslant 0$. If $\lim\limits_{n\to\infty} \varepsilon_n = 0$ implies that $\lim\limits_{n\to\infty} (u_n, v_n) = (x^*, y^*)$, then the iteration procedure defined by $(x_{n+1}, y_{n+1}) = f(Q, x_n, y_n)$ is said to be Q-stable or stable with respect to Q.

REMARK 2.1. Recently, some stability results of iteration procedures for variational inequalities (inclusions) have been established by various authors, see for example [2, 16, 21, 23, 26].

LEMMA 2.2. ([39]) *Let $\{a_n\}$ be a nonnegative real sequence and $\{b_n\}$ be a real sequence in $[0,1]$ such that $\sum\limits_{n=0}^{\infty} b_n = \infty$. If there exists a positive integer n_1 such that*

$$a_{n+1} \leqslant (1-b_n)a_n + b_n c_n, \qquad \forall n \geqslant n_1,$$

where $c_n \geqslant 0$ for all $n \geqslant 0$ and $c_n \to 0 (n \to \infty)$, then $\lim\limits_{n\to\infty} a_n = 0$.

DEFINITION 2.2. A single-valued mapping $\eta : X \times X \to X$ is said to be τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that $\|\eta(x,y)\| \leqslant \tau\|x-y\|, \forall x, y \in X$.

DEFINITION 2.3. Let $\eta : X \times X \to X$ and $A : X \to X$ be single-valued mappings. Then set-valued mapping $M : X \to 2^X$ is said to be

(i) accretive if

$$\langle u - v, J_q(x-y)\rangle \geqslant 0, \quad \forall x, y \in X, \quad u \in M(x), \quad v \in M(y);$$

(ii) η-accretive if

$$\langle u - v, J_q(\eta(x,y))\rangle \geqslant 0, \quad \forall x, y \in X, \quad u \in M(x), \quad v \in M(y);$$

(iii) strictly η-accretive if M is η-accretive and equality holds if and only if $x = y$;

(iv) r-strongly η-accretive if there exists a constant $r > 0$ such that

$$\langle u - v, J_q(\eta(x,y))\rangle \geqslant r\|x-y\|^q, \quad \forall x, y \in X, \quad u \in M(x), \quad v \in M(y);$$

(v) α-relaxed η-accretive if here exists a constant $m > 0$ such that

$$\langle u - v, J_q(\eta(x,y))\rangle \geqslant (-\alpha)\|x-y\|^q, \quad \forall x, y \in X, \quad u \in M(x), \quad v \in M(y).$$

In a similar way, we can define strictly η-accretivity and strongly η-accretivity of the single-valued mapping A.

DEFINITION 2.4. Let $A : X \to X, \eta : X \times X \to X$ is two single-valued mappings. Then a set-valued mapping $M : X \to 2^X$ is called (A,η)-accretive if M is m-relaxed η-accretive and $(A+\rho M)(X) = X$ for every $\rho > 0$.

REMARK 2.2. For appropriate and suitable choices of m, A, η and X, it is easy to see Definition 2.4 includes a number of definitions of monotone operators and accretive operators (see [24]).

In [24], Lan et al. showed that $(A+\rho M)^{-1}$ is a single-valued operator if $M : X \to 2^X$ be an (A,η)-accretive mapping and $A : X \to X$ be r-strongly η-accretive mapping. Based on this fact, we can define the resolvent operator $R_{\rho,A}^{\eta,M}$ associated with an (A,η)-accretive mapping M as follows:

DEFINITION 2.5. Let $A : X \to X$ be a strictly η-accretive mapping and $M : X \to 2^X$ be an (A,η)-accretive mapping. The resolvent operator $R_{\rho,A}^{\eta,M} : X \to X$ is defined by

$$R_{\rho,A}^{\eta,M}(x) = (A+\rho M)^{-1}(x), \qquad \forall x \in X.$$

LEMMA 2.3. *([24]) Let $\eta : X \times X \to X$ be τ-Lipschitz continuous, $A : X \to X$ be r-strongly η-accretive mapping and $M : X \to 2^X$ be an (A,η)-accretive mapping. Then the resolvent operator $R_{\rho,A}^{\eta,M} : X \to X$ is $\frac{\tau^{q-1}}{r-\rho m}$-Lipschitz continuous, i. e.,*

$$\|R_{\rho,A}^{\eta,M} - R_{\rho,A}^{\eta,M}(y)\| \leqslant \frac{\tau^{q-1}}{r-\rho m}\|x-y\|, \quad \forall x,y \in X,$$

where $\rho \in (0, \frac{r}{m})$ is a constant.

3. A system of general variational inclusions and iterative algorithm

In this section, we shall introduce a new system of general variational inclusions involving (A,η)-accretive mappings and construct a new iterative algorithm for solving this kind of system of general variational inclusions in Banach spaces. In what follows, unless other specifird, we assume that for $i = 1,2$, X_i be real q_i-uniformly smooth Banach spaces with norm $\|\cdot\|_i$.

For $i = 1,2$, let $\eta_i : X_i \times X_i \to X_i$, $A_i, g_i : X_i \to X_i$, $F : X_1 \times X_2 \to X_1$, $G : X_1 \times X_2 \to X_2$ be nonlinear mappings, and Let $M_i : X_i \to 2^{X_i}$ be (A_i,η_i)-accretive mappings. We consider the following problem of find $(x,y) \in X_1 \times X_2$ such that

$$\begin{cases} 0 \in F(x,y) + M_1(g_1(x)), \\ 0 \in G(x,y) + M_2(g_2(y)). \end{cases} \tag{3.1}$$

Problem (3.1) is called a system of general variational inclusions involving (A,η)-accretive mappings.

We remark that for suitable choices of the mappings $F, G, A_1, A_2, g_1, g_2, \eta_1, \eta_2, M_1, M_2$ and the spaces X_1, X_2, problem (3.1) includes many system of variational inequality (inclusion) problems as special cases, see for example, [1], [4], [6]-[9], [11], [13]–[15], [22], [27]–[29] and the references therein.

DEFINITION 3.1. A single-valued mappings $T : X_1 \to X_1$ is said to be
(i) accretive if

$$\langle T(x) - T(y), J_{q_1}(x-y)\rangle \geqslant 0, \quad \forall x, y \in X_1;$$

(ii) r-strongly accretive if there exists a constant $r > 0$ such that

$$\langle T(x) - T(y), J_{q_1}(x-y)\rangle \geqslant r\|x-y\|_1^{q_1}, \quad \forall x, y \in X_1;$$

(iii) s-relaxed cocoercive if there exists a constant $s > 0$ such that

$$\langle T(x) - T(y), J_{q_1}(x-y)\rangle \geqslant (-s)\|T(x) - T(y)\|_1^{q_1}, \qquad \forall x, y \in X_1;$$

(iv) (α, ξ)-relaxed cocoercive if there exist constants $\alpha, \xi > 0$ such that

$$\langle T(x) - T(y), J_{q_1}(x-y)\rangle \geqslant (-\alpha)\|T(x) - T(y)\|_1^{q_1} + \xi\|x-y\|_1^{q_1}, \qquad \forall x, y \in X_1;$$

(v) t-Lipschitz continuous if there exists a constant $t > 0$ such that

$$\|T(x) - T(y)\|_1 \leqslant t\|x-y\|_1, \quad \forall x, y \in X_1.$$

DEFINITION 3.2. Let $A : X_1 \to X_1$ and $F : X_1 \times X_2 \to X_1$ be single-valued mappings. F is said to be
(i) (α, β)-Lipschitz continuous, if there exist constants $\alpha > 0$, $\beta > 0$ such that

$$\|F(x_1, y_1) - F(x_2, y_2)\|_1 \leqslant \alpha\|x_1 - x_2\|_1 + \beta\|y_1 - y_2\|_2, \quad \forall x, x_2 \in X_1, y_1, y_2 \in X_2.$$

(ii) (a, b)-relaxed cocoercive with respect to A in first argument if there exist constants $a, b > 0$ such that

$$\langle F(x_1, y) - F(x_2, y), J_{q_1}(A(x_1) - A(x_2))\rangle \geqslant (-a)\|F(x_1, y) - F(x_2, y)\|_1^{q_1} + b\|x_1 - x_2\|_1^{q_1},$$

for all $x_1, x_2 \in X_1, y \in X_2$.

LEMMA 3.1. *For any given* $(x, y) \in X_1 \times X_2$, (x, y) *is a solution of problem (3.1) if and only if* (x, y) *satisfies*

$$\begin{cases} g_1(x) = R_{\rho_1, A_1}^{\eta_1, M_1}[A_1(g_1(x)) - \rho_1 F(x, y)], \\ g_2(y) = R_{\rho_2, A_2}^{\eta_2, M_2}[A_2(g_2(y)) - \rho_2 G(x, y)], \end{cases} \tag{3.2}$$

where $\rho_1, \rho_2 > 0$ *are constants.*

Proof. This directly follows from Definition 2.5. $\square$

REMARK 3.1. The equality (3.2) can be written as

$$\begin{cases} x = x - g_1(x) + R^{\eta_1,M_1}_{\rho_1,A_1}[A_1(g_1(x)) - \rho_1 F(x,y)], \\ \\ y = y - g_2(y) + R^{\eta_2,M_2}_{\rho_2,A_2}[A_2(g_2(y)) - \rho_2 G(x,y)], \end{cases} \tag{3.3}$$

where $\rho_1, \rho_2 > 0$ are constants. This fixed point formulation enables us to suggest the following iterative algorithm.

ALGORITHM 3.1.. For $i = 1,2$, assume that $\eta_i, A_i, g_i, M_i, X_i, F$ and G be as in problem (3.1). Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be two sequences such that $\alpha_n, \beta_n \in [0,1]$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Let $\{(e_n, f_n)\}$ be a sequence in $X_1 \times X_2$ introduced to take into account possible incxact computation. For any given $(x_0, y_0) \in X_1 \times X_2$, define the iterative sequence $\{(x_n, y_n)\}$ by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n[x_n - g_1(x_n) + R^{\eta_1,M_1}_{\rho_1,A_1}(A_1(g_1(x_n)) - \rho_1 F(x_n,y_n))] + \alpha_n e_n, \\ y_{n+1} = (1-\alpha_n)y_n + \alpha_n[y_n - g_2(y_n) + R^{\eta_2,M_2}_{\rho_2,A_2}(A_2(g_2(y_n)) - \rho_2 G(x_n,y_n))] + \alpha_n f_n, \end{cases}$$

for $n = 0,1,2,\cdots$.

Let $\{(u_n, v_n)\}$ be any sequence in $X_1 \times X_2$ and define $\{\varepsilon_n\}$ by

$$\varepsilon_n = \|(u_{n+1}, v_{n+1}) - (A_n, B_n)\|_*, \tag{3.4}$$

where

$$A_n = (1-\alpha_n)u_n + \alpha_n[u_n - g_1(u_n) + R^{\eta_1,M_1}_{\rho_1,A_1}(A_1(g_1(u_n)) - \rho_1 F(u_n,v_n))] + \alpha_n e_n, \tag{3.5}$$

$$B_n = (1-\alpha_n)v_n + \alpha_n[v_n - g_2(v_n) + R^{\eta_2,M_2}_{\rho_2,A_2}(A_2(g_2(v_n)) - \rho_2 G(u_n,v_n))] + \alpha_n f_n, \tag{3.6}$$

for $n = 0,1,2,\cdots$.

REMARK 3.2. If we choose suitable α_n, e_n, f_n, $A_1, A_2, \eta_1, \eta_2, g_1, g_2, M_1, M_2, F, G$ and the spaces X_1, X_2, then Algorithm 3.1 can be degenerated to a number of algorithms involving many known algorithms which due to classes of variational inequalities and variational inclusions (see, for example, [1], [4], [6]–[9], [11], [13]–[15], [22], [27]–[29]).

4. Existence and convergence theorems

In this section, we shall prove existence and uniqueness of solutions for problem (3.1) and the convergence and stability of iterative sequence generated by Algorithm (3.1).

THEOREM 4.1. *For $i = 1,2$, let X_i be q_i-uniformly smooth Banach spaces. Let $\eta_i : X_i \times X_i \to X_i$ be τ_i-Lipschitz continuous, $A_i : X_i \to X_i$ be r_i-strongly η_i-accretive and γ_i-Lipschitz continuous, $g_i : X_i \to X_i$ be (s_i, t_i)-relaxed cocoercive and δ_i-Lipschitz continuous, $M_i : X \to 2^X$ be (A_i, η_i)-accretive mappings. Let $F : X_1 \times X_2 \to X_1$ be (a,b)-relaxed cocoercive with respect to $A_1 \circ g_1$ in the first* *argument and* (μ_1, ν_1)*-Lipschitz continuous,* $G : X_1 \times X_2 \to X_2$ *be* (c,d)*-relaxed cocoercive with respect to* $A_2 \circ g_2$ *in the second argument and* (μ_2, ν_2)*-Lipschitz continuous. If there exist constant* $\rho_1 \in (0, \frac{r_1}{m_1})$ *and* $\rho_2 \in (0, \frac{r_2}{m_2})$ *such that*

$$\begin{cases} k_1 = \theta_1 + \lambda_1 l_1 + \rho_2 \mu_2 l_2 < 1, \\ k_2 = \theta_2 + \lambda_2 l_2 + \rho_1 \nu_1 l_1 < 1. \end{cases} \tag{4.1}$$

where

$$\begin{aligned}
&\theta_1 = (1 - q_1 t_1 + q_1 s_1 \delta_1^{q_1} + C_{q_1} \delta_1^{q_1})^{\frac{1}{q_1}}, \quad \theta_2 = (1 - q_2 t_2 + q_2 s_2 \delta_2^{q_2} + C_{q_2} \delta_2^{q_2})^{\frac{1}{q_2}}, \\
&\lambda_1 = (\gamma_1^{q_1} \delta_1^{q_1} - q_1 \rho_1 b + q_1 \rho_1 a \mu_1^{q_1} + C_{q_1} \rho_1^{q_1} \mu_1^{q_1})^{\frac{1}{q_1}}, \\
&\lambda_2 = (\gamma_2^{q_2} \delta_2^{q_2} - q_2 \rho_2 d + q_2 \rho_2 c \nu_2^{q_2} + C_{q_2} \rho_2^{q_2} \nu_2^{q_2})^{\frac{1}{q_2}}, \\
&l_1 = \frac{\tau_1^{q_1 - 1}}{r_1 - \rho_1 m_1}, \quad l_2 = \frac{\tau_2^{q_2 - 1}}{r_2 - \rho_2 m_2}.
\end{aligned}$$

Then problem (3.1) admits a unique solution.

Proof. For any given $\rho_i > 0 \ \ (i = 1, 2)$, define $T : X_1 \times X_2 \to X_1$ and $S : X_1 \times X_2 \to X_2$ by

$$\begin{aligned}
T(x,y) &= x - g_1(x) + R_{\rho_1, A_1}^{\eta_1, M_1}[A_1(g_1(x)) - \rho_1 F(x,y)], \\
S(x,y) &= y - g_2(y) + R_{\rho_2, A_2}^{\eta_2, M_2}[A_2(g_2(y)) - \rho_2 G(x,y)],
\end{aligned} \tag{4.2}$$

for all $(x,y) \in X_1 \times X_2$.

For any $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$, it follows from (4.2) and Lemma 2.3 that

$$\begin{aligned}
&\|T(x_1, y_1) - T(x_2, y_2)\|_1 \\
&\quad \leqslant \|x_1 - x_2 - (g_1(x_1) - g_1(x_2))\|_1 + \|R_{\rho_1, A_1}^{\eta_1, M_1}[A_1(g_1(x_1)) - \rho_1 F(x_1, y_1)] \\
&\qquad - R_{\rho_1, A_1}^{\eta_1, M_1}[A_1(g_1(x_2)) - \rho_1 F(x_2, y_2)]\|_1 \\
&\quad \leqslant \|x_1 - x_2 - (g_1(x_1) - g_1(x_2))\|_1 + \frac{\tau^{q_1 - 1}}{r_1 - \rho_1 m_1} (\rho_1 \|F(x_2, y_1) - F(x_2, y_2)\|_1 \\
&\qquad + \|A_1(g_1(x_1)) - A_1(g_1(x_2)) - \rho_1 (F(x_1, y_1) - F(x_2, y_1))\|_1).
\end{aligned} \tag{4.3}$$

By assumptions, we have

$$\begin{aligned}
&\|x_1 - x_2 - (g_1(x_1) - g_1(x_2))\|_1^{q_1} \\
&\quad \leqslant \|x_1 - x_2\|_1^{q_1} - q_1 \langle g_1(x_1) - g_2(x_2), J_{q_1}(x_1 - x_2) \rangle + C_{q_1} \|g_1(x_1) - g_1(x_2)\|_1^{q_1} \\
&\quad \leqslant (1 - q_1 t_1 + q_1 s_1 \delta_1^{q_1} + C_{q_1} \delta_1^{q_1}) \|x_1 - x_2\|_1^{q_1}
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
&\|A_1(g_1(x_1)) - A_1(g_1(x_2)) - \rho_1 (F(x_1, y_1) - F(x_2, y_1))\|_1^{q_1} \\
&\quad \leqslant \|A_1(g_1(x_1)) - A_1(g_1(x_2))\|_1^{q_1} + C_{q_1} \rho_1^{q_1} \|F(x_1, y_1) - F(x_2, y_1)\|_1^{q_1} \\
&\qquad - q_1 \rho_1 \langle F(x_1, y_1) - F(x_2, y_1), J_{q_1}(A_1(g_1(x_1)) - A_1(g_1(x_2))) \rangle \\
&\quad \leqslant (\gamma_1^{q_1} \delta_1^{q_1} - q_1 \rho_1 b + q_1 \rho_1 a \mu_1^{q_1} + C_{q_1} \rho_1^{q_1} \mu_1^{q_1}) \|x_1 - x_2\|_1^{q_1}
\end{aligned} \tag{4.5}$$

$$\|F(x_2, y_1) - F(x_2, y_2)\|_1 \leqslant \nu_1 \|y_1 - y_2\|_2. \tag{4.6}$$

From (4.3)–(4.6), we have

$$\|T(x_1,y_1)-T(x_2,y_2)\|_1 \leqslant (\theta_1+\lambda_1 l_1)\|x_1-x_2\|_1 + l_1\rho_1\nu_1\|y_1-y_2\|_2, \tag{4.7}$$

where

$$l_1=\frac{\tau_1^{q_1-1}}{r_1-\rho_1 m_1},\quad \theta_1=(1-q_1t_1+q_1s_1\delta_1^{q_1}+C_{q_1}\delta_1^{q_1})^{\frac{1}{q_1}},$$
$$\lambda_1=(\gamma_1^{q_1}\delta_1^{q_1}-q_1\rho_1 b+q_1\rho_1 a\mu_1^{q_1}+C_{q_1}\rho_1^{q_1}\mu_1^{q_1})^{\frac{1}{q_1}}.$$

Similarly, we can prove that

$$\|S(x_1,y_1)-S(x_2,y_2)\|_2 \leqslant (\theta_2+\lambda_2 l_2)\|y_1-y_2\|_2 + l_2\rho_2\mu_2\|x_1-x_2\|_1. \tag{4.8}$$

where

$$l_2=\frac{\tau_2^{q_2-1}}{r_2-\rho_2 m_2},\quad \theta_2=(1-q_2t_2+q_2s_2\delta_2^{q_2}+C_{q_2}\delta_2^{q_2})^{\frac{1}{q_2}},$$
$$\lambda_2=(\gamma_2^{q_2}\delta_2^{q_2}-q_2\rho_2 d+q_2\rho_2 c\nu_2^{q_2}+C_{q_2}\rho_2^{q_2}\nu_2^{q_2})^{\frac{1}{q_2}}.$$

From (4.7) and (4.8), we have

$$\begin{aligned}&\|T(x_1,y_1)-T(x_2,y_2)\|_1+\|S(x_1,y_1)-S(x_2,y_2)\|_2\\ &\quad\leqslant k_1\|x_1-x_2\|_1+k_2\|y_1-y_2\|_2\\ &\quad\leqslant k(\|x_1-x_2\|_1+\|y_1-y_2\|_2),\end{aligned} \tag{4.9}$$

where $k=\max\{k_1,k_2\}$, $k_1=\theta_1+\lambda_1 l_1+\rho_2\mu_2 l_2$, $k_2=\theta_2+\lambda_2 l_2+\rho_1\nu_1 l_1$.

Define the norm $\|\cdot\|_*$ on $X_1\times X_2$ by

$$\|(x,y)\|_*=\|x\|_1+\|y\|_2,\ (x,y)\in X_1\times X_2. \tag{4.10}$$

It is easy to see that $(X_1\times X_2,\|\cdot\|_*)$ is a Banach space. Define $Q(x,y):X_1\times X_2\to X_1\times X_2$ by

$$Q(x,y)=(T(x,y),S(x,y)),\ \forall (x,y)\in X_1\times X_2.$$

By (4.1), we know that $0<k<1$. It follows from (4.9) and (4.10) that

$$\|Q(x_1,y_1)-Q(x_2,y_2)\|_*\leqslant k\|(x_1,y_1)-(x_2,y_2)\|_*.$$

This proves that $Q(x,y):X_1\times X_2\to X_1\times X_2$ is a contraction mapping. Hence, by Banach contraction principle, there exists a unique $(x^*,y^*)\in X_1\times X_2$ such that $Q(x^*,y^*)=(x^*,y^*)$, which implies that

$$\begin{cases} g_1(x^*)=R_{\rho_1,A_1}^{\eta_1,M_1}[A_1(g_1(x^*))-\rho_1F(x^*,y^*)],\\ g_2(y^*)=R_{\rho_2,A_2}^{\eta_2,M_2}[A_2(g_2(y^*))-\rho_2G(x^*,y^*)].\end{cases}$$

It follows from Lemm 3.1 that (x^*,y^*) is the unique solution of problem (3.1). This completes the proof. □

THEOREM 4.2. *For $i = 1,2$, let $\eta_i, A_i, g_i, M_i, X_i, F$ and G be the same as in Theorem 4.1 and let condition (4.1) of Theorem 4.1 hold. Then:*

(i) If $\lim_{n\to\infty} \|(e_n,f_n)\|_* = 0$, then approximate solution (x_n, y_n) generated by Algorithm 3.1 converges strongly to the unique solution (x^*, y^*) of problem (3.1).

(ii) Moreover, if $0 < \alpha < \alpha_n$, then $\lim_{n\to\infty}(u_n, v_n) = (x^*, y^*)$ if and only if $\lim_{n\to\infty} \varepsilon_n = 0$.

Proof. It follows from Theorem 4.1 that problem (3.1) has the unique solution (x^*, y^*). Hence, by Lemma 3.1, we have

$$\begin{cases} g_1(x^*) = R^{\eta_1,M_1}_{\rho_1,A_1}[A_1(g_1(x^*)) - \rho_1 F(x^*, y^*)], \\ \\ g_2(y^*) = R^{\eta_2,M_2}_{\rho_2,A_2}[A_2(g_2(y^*)) - \rho_2 G(x^*, y^*)], \end{cases} \tag{4.11}$$

From Algorithm 3.1 and Lemma 2.3, we obtain

$$\begin{aligned} &\|x_{n+1} - x^*\|_1 \\ &\quad \leqslant (1-\alpha_n)\|x_n - x^*\|_1 + \alpha_n\|x_n - x^* - (g_1(x_n) - g_1(x^*))\|_1 + \alpha_n\|e_n\|_1 \\ &\qquad + \alpha_n\|R^{\eta_1,M_1}_{\rho_1,A_1}[A_1(g_1(x_n)) - \rho_1 F(x_n,y_n)] - R^{\eta_1,M_1}_{\rho_1,A_1}[A_1(g_1(x^*)) - \rho_1 F(x^*,y^*)]\|_1 \\ &\quad \leqslant (1-\alpha_n)\|x_n - x^*\|_1 + \alpha_n\|x_n - x^* - (g_1(x_1) - g_1(x_2))\|_1 + \alpha_n\|e_n\|_1 \\ &\qquad + \alpha_n l_1(\|A_1(g_1(x_n)) - A_1(g_1(x^*)) - \rho_1(F(x_n,y_n) - F(x^*,y_n))\|_1 \\ &\qquad + \rho_1\|F(x^*,y_n) - F(x^*,y^*)\|_1). \end{aligned} \tag{4.12}$$

where $l_1 = \frac{\tau_1^{q_1-1}}{r_1 - \rho_1 m_1}$.

By Lemma 2.1, δ_1-Lipschitz continuous and (s_1,t_1)-relaxed cocoercive of g_1, we obtain

$$\begin{aligned} &\|x_n - x^* - (g_1(x_n) - g_1(x^*))\|_1^{q_1} \\ &\quad \leqslant \|x_n - x^*\|_1^{q_1} - q_1\langle g_1(x_n) - g_1(x^*), J_{q_1}(x_n - x^*)\rangle + C_{q_1}\|g_1(x_n) - g_1(x^*)\|_1^{q_1} \\ &\quad \leqslant (1 - q_1 t_1 + q_1 s_1 \delta_1^{q_1} + C_{q_1}\delta_1^{q_1})\|x_n - x^*\|_1^{q_1} \end{aligned} \tag{4.13}$$

Since, A_1 is γ_1-Lipschitz continuous, F is (a,b)-relaxed cocoercive with respect to $A_1 \circ g_1$ in the first argument and (μ_1, ν_1)-Lipschitz continuous, then using Lemma 2.1, we have

$$\begin{aligned} &\|A_1(g_1(x_n)) - A_1(g_1(x^*)) - \rho_1(F(x_n,y_n) - F(x^*,y_n))\|_1^{q_1} \\ &\quad \leqslant \|A_1(g_1(x_n)) - A_1(g_1(x^*))\|_1^{q_1} \\ &\qquad - q_1\rho_1\langle F(x_n,y_n) - F(x^*,y_n), J_{q_1}(A_1(g_1(x_n)) - A_1(g_1(x^*)))\rangle \\ &\qquad + C_{q_1}\rho_1^{q_1}\|F(x_n,y_n) - F(x^*,y_n)\|_1^{q_1} \\ &\quad \leqslant (\gamma_1^{q_1}\delta_1^{q_1} - q_1\rho_1 b + q_1\rho_1 a\mu_1^{q_1} + C_{q_1}\rho_1^{q_1}\mu_1^{q_1})\|x_n - x^*\|_1^{q_1} \end{aligned} \tag{4.14}$$

$$\|F(x^*,y_n) - F(x^*,y^*)\|_1 \leqslant \nu_1\|y_n - y^*\|_2. \tag{4.15}$$

From (4.12)–(4.15), we have

$$\begin{aligned} \|x_{n+1} - x^*\|_1 \leqslant\ & (1-\alpha_n)\|x_n - x^*\|_1 + \alpha_n[(\theta_1 + \lambda_1 l_1)\|x_n - x^*\|_1 \\ & + l_1\rho_1\nu_1\|y_n - y^*\|_2] + \alpha_n\|e_n\|_1, \end{aligned} \tag{4.16}$$

where

$$l_1 = \frac{\tau_1^{q_1-1}}{r_1 - \rho_1 m_1}, \quad \theta_1 = (1 - q_1 t_1 + q_1 s_1 \delta_1^{q_1} + C_{q_1}\delta_1^{q_1})^{\frac{1}{q_1}},$$
$$\lambda_1 = (\gamma_1^{q_1}\delta_1^{q_1} - q_1\rho_1 b + q_1\rho_1 a\mu_1^{q_1} + C_{q_1}\rho_1^{q_1}\mu_1^{q_1})^{\frac{1}{q_1}}.$$

Similarly, we have

$$\begin{aligned}\|y_{n+1} - y^*\|_2 \ \leqslant\ & (1-\alpha_n)\|y_n - y^*\|_2 + \alpha_n[(\theta_2 + \lambda_2 l_2)\|y_n - y^*\|_2 \\ & +l_2\rho_2\mu_2\|x_n - x^*\|_1] + \alpha_n\|f_n\|_2, \qquad (4.17)\end{aligned}$$

where

$$l_2 = \frac{\tau_2^{q_2-1}}{r_2 - \rho_2 m_2}, \quad \theta_2 = (1 - q_2 t_2 + q_2 s_2 \delta_2^{q_2} + C_{q_2}\delta_2^{q_2})^{\frac{1}{q_2}},$$
$$\lambda_2 = (\gamma_2^{q_2}\delta_2^{q_2} - q_2\rho_2 d + q_2\rho_2 c\nu_2^{q_2} + C_{q_2}\rho_2^{q_2}\nu_2^{q_2})^{\frac{1}{q_2}}.$$

By (4.16) and (4.17), we obtain

$$\begin{aligned}&\|(x_{n+1}, y_{n+1}) - (x^*, y^*)\|_* \\ &\quad = \|x_{n+1} - x^*\|_1 + \|y_{n+1} - y^*\|_2 \\ &\quad \leqslant (1-\alpha_n)\|(x_n, y_n) - (x^*, y^*)\|_* + \alpha_n \max\{k_1, k_2\}\|(x_n, y_n) - (x^*, y^*)\|_* \\ &\qquad +\alpha_n\|(e_n, f_n)\|_* \\ &\quad = (1 - (1-k)\alpha_n)\|(x_n, y_n) - (x^*, y^*)\|_* + \alpha_n\|(e_n, f_n)\|_*, \qquad (4.18)\end{aligned}$$

where $k = \max\{k_1, k_2\}$, $k_1 = \theta_1 + \lambda_1 l_1 + \rho_2\mu_2 l_2$, $k_2 = \theta_2 + \lambda_2 l_2 + \rho_1\nu_1 l_1$.

Letting

$$a_n = \|(x_{n+1}, y_{n+1}) - (x^*, y^*)\|_*, \quad b_n = (1-k)\alpha_n, \quad c_n = \frac{\|(e_n, f_n)\|_*}{1-k},$$

then (4.18) can written as

$$a_{n+1} \leqslant (1 - b_n)a_n + b_n c_n.$$

It follows from Lemma 2.2 that $a_n \to 0\,(n \to \infty)$, and so (x_n, y_n) converges strongly to the unique solution (x^*, y^*) of problem (3.1).

Now we prove conclusion

(ii) , by (3.4)–(3.6), we obtain

$$\begin{aligned}\|(u_{n+1}, v_{n+1}) - (x^*, y^*)\|_* \ \leqslant\ & \|(u_{n+1}, v_{n+1}) - (A_n, B_n)\|_* + \|(A_n, B_n) - (x^*, y^*)\|_* \\ \leqslant\ & \varepsilon_n + \|A_n - x^*\|_1 + \|B_n - y^*\|_2. \qquad (4.19)\end{aligned}$$

As the proof of in equality (4.16), we have

$$\begin{aligned}\|A_n - x^*\|_1 \ \leqslant\ & (1-\alpha_n)\|u_n - x^*\|_1 + \alpha_n[(\theta_1 + \lambda_1 l_1)\|u_n - x^*\|_1 \\ & +\rho_1\nu_1 l_1\|v_n - y^*\|_2] + \alpha_n\|e_n\|_1, \qquad (4.20)\\ \|B_n - y^*\|_2 \ \leqslant\ & (1-\alpha_n)\|v_n - y^*\|_2 + \alpha_n[(\theta_2 + \lambda_2 l_2)\|v_n - y^*\|_2 \\ & +\rho_2\mu_2 l_2\|u_n - x^*\|_1] + \alpha_n\|f_n\|_2. \qquad (4.21)\end{aligned}$$

Since $0 < \alpha < \alpha_n$, by (4.19)–(4.21) we have

$$\begin{aligned}
&\|(u_{n+1}, v_{n+1}) - (x^*, y^*)\|_* \\
&\quad \leqslant (1 - (1 - \max\{k_1, k_2\})\alpha_n)\|(u_n, v_n) - (x^*, y^*)\|_* + \alpha_n\|(e_n, f_n)\|_* + \varepsilon_n \\
&\quad \leqslant (1 - (1 - k)\alpha_n)\|(u_n, v_n) - (x^*, y^*)\|_* + (1 - k)\alpha_n \left[\frac{\|(e_n, f_n)\|_*}{1 - k} + \frac{\varepsilon_n}{(1 - k)\alpha}\right].
\end{aligned}$$

where $k = \max\{k_1, k_2\}$, $k_1 = \theta_1 + \lambda_1 l_1 + \rho_2\mu_2 l_2$, $k_2 = \theta_2 + \lambda_2 l_2 + \rho_1 \nu_1 l_1$.

Suppose that $\lim_{n\to\infty} \varepsilon_n = 0$. Then from $\sum_{n=0}^{\infty} \alpha_n = \infty$ and Lemma 2.2, we have $\lim_{n\to\infty} (u_n, v_n) = (x^*, y^*)$.

Conversely, if $\lim_{n\to\infty} (u_n, v_n) = (x^*, y^*)$, then we get

$$\begin{aligned}
\varepsilon_n &= \|(u_{n+1}, v_{n+1}) - (A_n, B_n)\|_* \\
&\leqslant \|(u_{n+1}, v_{n+1}) - (x^*, y^*)\|_* + \|A_n - x^*\|_1 + \|B_n - y^*\|_2 \\
&\leqslant \|(u_{n+1}, v_{n+1}) - (x^*, y^*)\|_* + (1 - (1 - k)\alpha_n)\|(u_n, v_n) - (x^*, y^*)\|_* \\
&\quad + \alpha_n\|(e_n, f_n)\|_* \to 0 (n \to \infty),
\end{aligned}$$

i.e., $\lim_{n\to\infty} \varepsilon_n = 0$. This completes the proof. □

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(Received December 27, 2006)

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