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SOME RETARDED GRONWALL TYPE INTEGRAL INEQUALITY AND ITS APPLICATIONS

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Abstract. We consider nonlinear integral inequalities of Gronwall type for functions of one variable or two variables. We also study integral inequalities for proving the boundedness and uniqueness of the solutions to hyperbolic partial differential equations.

1. Introduction

Let $u : [\alpha, \alpha + h] \to R$ be a continuous real-valued function satisfying the inequality

$$0 \leqslant u(t) \leqslant \int_{\alpha}^{t} [a + bu(s)]\, ds \quad \text{for} \quad t \in [\alpha, \alpha + h],$$

where a, b are nonnegative constants. Then $u(t) \leqslant ahe^{bh}$ for $t \in [\alpha, \alpha + h]$. This result was proved by T. H. Gronwall [7] in the year 1919, and is the prototype for the study of several integral inequalities of Volterra type, and also for obtaining explicit bounds of the unknown function. Among the several publications on this subject, the paper of Bellman [3] is very well known: *Let $x(t)$ and $k(t)$ be real valued nonnegative continuous functions for $t \geqslant \alpha$. If a is a constant, $a \geqslant 0$, and*

$$x(t) \leqslant a + \int_{\alpha}^{t} k(s)x(s)\, ds, \quad t \geqslant \alpha,$$

then

$$x(t) \leqslant a \exp\left(\int_{\alpha}^{t} k(s)\, ds\right), \quad \textit{for} \quad t \geqslant \alpha.$$

It is clear that Bellman's result contains that of Gronwall. This is the reason why inequalities of this type were called "Gronwall-Bellman inequalities" or "Inequalities of Gronwall type". The Gronwall type integral inequalities provide a necessary tool for the study of the theory of differential equations, integral equations and inequalities of various types. During the past few years several authors (see references below and

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some of the references cited therein) have established several Gronwall type integral inequalities in two or more independent real variables. In [11], Pachpatte proved the following interesting integral inequality:

THEOREM 1.1. *Let* $a, b \in C(I, R_+)$, $\alpha \in C^1(I, I)$ *be nondecreasing with* $\alpha(t) \leqslant t$ *on* I, *and* $k \geqslant 0$ *be a constants. If* $u \in C(I, R_+)$ *and*

$$u(t) \leqslant k + \int_{t_0}^{t} a(s)u(s)\,ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)u(s)\,ds \tag{1.1}$$

for $t \in I$, *then*

$$u(t) \leqslant k \exp\left(\int_{t_0}^{t} a(s)\,ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)\,ds\right) \tag{1.2}$$

for $t \in I$.

In this paper is to obtain bound in the inequality (1.1) for functions of one or two independent variables when the constant k and the function $u(t)$ in the right-hand side of the inequality (1.1) are replaced by the function $k(t)$ and $u^p(t)$ for $0 < p \leqslant 1$, respectively. We also provide some application of these integral inequalities for finding the boundedness and uniqueness of the solutions to hyperbolic partial differential equations.

2. Integral Inequalities

In this section we consider nonlinear integral inequalities of Gronwall type for functions. We shall introduce her some notation: R denotes the set of real numbers, $R_+ = [0, \infty)$, $I = [t_0, T)$, $J_1 = [x_0, X)$ and $J_2 = [y_0, Y)$ are given subsets of R.

LEMMA 2.1. *Let* $a, b \in C(I, R_+)$, $\alpha \in C^1(I, I)$ *be nondecreasing with* $\alpha(t) \leqslant t$ *on* I, $k \geqslant 1$ *and* $0 < p \leqslant 1$ *be constants. If* $u \in C(I, R_+)$ *and*

$$u(t) \leqslant k + \int_{t_0}^{t} a(s)u^p(s)\,ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)u^p(s)\,ds \tag{2.1}$$

for $t \in I$, *then*

$$u(t) \leqslant k \exp\left(\int_{t_0}^{t} a(s)\,ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)\,ds\right) \tag{2.2}$$

for $t \in I$.

Proof. From the given hypotheses we observe that $\alpha'(t) \geqslant 0$ for $t \in I$. Let $k \geqslant 1$ and define a function $z(t)$ by the right-hand side of (2.1). Then, $z(t) \geqslant 1, z(t_0) = k, u(t) \leqslant z(t)$, and

$$\begin{aligned} z'(t) &= a(t)u^p(t) + b(\alpha(t))u^p(\alpha(t))\alpha'(t) \\ &\leqslant a(t)z^p(t) + b(\alpha(t))z^p(\alpha(t))\alpha'(t) \\ &\leqslant a(t)z(t) + b(\alpha(t))z(\alpha(t))\alpha'(t). \end{aligned}$$

As $\alpha(t) \leqslant t$ on I, we deduce that $z'(t) \leqslant a(t)z(t) + b(\alpha(t))z(t)\alpha'(t)$ and therefore

$$\frac{z'(t)}{z(t)} \leqslant a(t) + b(\alpha(t))\alpha'(t). \tag{2.3}$$

Integrating (2.3) from t_0 to t, where $t \in I$, and applying some change of variables yields

$$z(t) \leqslant k \exp\left(\int_{t_0}^{t} a(s)\,ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)\,ds\right) \tag{2.4}$$

for $t \in I$. Using (2.4) in $u(t) \leqslant z(t)$, we get the inequality (2.2). □

THEOREM 2.2. *Let $a, b \in C(I, R_+)$, $\alpha \in C^1(I, I)$ be nondecreasing with $\alpha(t) \leqslant t$ on I, and $0 < p \leqslant 1$ be a constant. If $k \in C(I, R_+ - \{0\}), u \in C(I, R_+)$ and*

$$u(t) \leqslant k(t) + \int_{t_0}^{t} a(s)u^p(s)\,ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)u^p(s)\,ds \tag{2.5}$$

for $t \in I$, then

$$u(t) \leqslant k(t) + e(t)\exp\left(p\int_{t_0}^{t} a(s)k^{p-1}(s)\,ds + p\int_{\alpha(t_0)}^{\alpha(t)} b(s)k^{p-1}(s)\,ds\right) \tag{2.6}$$

for $t \in I$, where

$$e(t) = \int_{t_0}^{t} a(s)k^p(s)\,ds + p\int_{\alpha(t_0)}^{\alpha(t)} b(s)k^p(s)\,ds \tag{2.7}$$

for $t \in I$.

Proof. From (2.5) we have $u(t) \leqslant k(t) + z(t)$, where the function $z(t)$ is defined by $z(t) = \int_{t_0}^{t} a(s)u^p(s)\,ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)u^p(s)\,ds$. From the above relation we derive

$$z(t) \leqslant \int_{t_0}^{t} a(s)(k(s) + z(s))^p\,ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)(k(s) + z(s))^p\,ds.$$

By applying the following generalizations of Bernoulli's inequality (see, [10, p.65]) $(1+x)^a \leqslant 1 + ax$, where $0 < a \leqslant 1$ and $-1 < x$, it is easy to obtain that

$$\begin{aligned} z(t) &\leqslant \int_{t_0}^{t} a(s)k^p(s)\left(1 + \frac{z(s)}{k(s)}\right)^p ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)k^p(s)\left(1 + \frac{z(s)}{k(s)}\right)^p ds \\ &\leqslant \int_{t_0}^{t} a(s)k^p(s)\left(1 + p\frac{z(s)}{k(s)}\right) ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)k^p(s)\left(1 + p\frac{z(s)}{k(s)}\right) ds \\ &= e(t) + \int_{t_0}^{t} a(s)k^{p-1}(s)z(s)\,ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)k^{p-1}(s)z(s)\,ds, \end{aligned}$$

where $e(t)$ is defined by (2.7). First, we assume that $e(t) > 0$ for $t \in I$. we get

$$\frac{z(t)}{e(t)} \leqslant 1 + p\int_{t_0}^{t} a(s)k^{p-1}(s)\frac{z(s)}{e(s)}\,ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)k^{p-1}(s)\frac{z(s)}{e(s)}\,ds. \tag{2.8}$$

From the Lemma 2.1, the previous inequality (2.8) yields

$$\frac{z(t)}{e(t)} \leqslant \exp\left(p\int_{t_0}^{t} a(s)k^{p-1}(s)\,ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)k^{p-1}(s)\,ds\right). \tag{2.9}$$

Using inequality (2.9) in $u(t) \leqslant k(t)+z(t)$, we get the required inequality in (2.6). If $e(t)$ is nonnegative, then we carry out the above procedure with $e(t)+\varepsilon$ instead of $e(t)$, where $\varepsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\varepsilon \to 0$ to obtain (2.6). □

THEOREM 2.3. *Let a, b, α, p be as in Theorem 2.2 and $c > 1$ be a constant. If $u \in C(I, R_+)$ and*

$$u(t) \leqslant c + \int_{t_0}^{t} a(s)u^p(s)\log u(s)\,ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)u^p(s)\log u(s)\,ds \tag{2.10}$$

for $t \in I$, then

$$u(t) \leqslant c^{[1+(A(t)+B(t))\exp(A(t)+B(t))]} \tag{2.11}$$

for $t \in I$, where

$$A(t) = \int_{t_0}^{t} a(s)\,ds, \quad B(t) = \int_{\alpha(t_0)}^{\alpha(t)} b(s)\,ds \tag{2.12}$$

for $t \in I$.

Proof. From the giben hypotheses we observe that $\alpha'(t) \geqslant 0$ for $t \in I$. Let $c > 1$ and define a function $z(t)$ by the right-hand side of (2.10). Then, $z(t) > 1$, $z(t_0) = c$, $u(t) \leqslant z(t)$, and

$$\begin{aligned} z'(t) &= a(t)u^p(t)\log u(t) + b(\alpha(t))u^p(\alpha(t))\log u(\alpha(t))\alpha'(t) \\ &\leqslant a(t)z^p(t)\log z(t) + b(\alpha(t))z^p(\alpha(t))\log z(\alpha(t))\alpha'(t) \\ &\leqslant a(t)z(t)\log z(t) + b(\alpha(t))z(\alpha(t))\log z(\alpha(t))\alpha'(t). \end{aligned}$$

Because of the fact $\alpha(t) \leqslant t$ on I, we deduce that $z'(t) \leqslant a(t)z(t)\log z(t) + b(\alpha(t))z(t)\log z(\alpha(t))\alpha'(t)$. Therefore

$$\frac{z'(t)}{z(t)} \leqslant a(t)\log z(t) + b(\alpha(t))\log z(\alpha(t))\alpha'(t). \tag{2.13}$$

Integrating (2.13) from t_0 to t, where $t \in I$, and applying some change of variables yields

$$\log z(t) \leqslant \log c + \int_{t_0}^{t} a(s)\log z(s)\,ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)\log z(s)\,ds \tag{2.14}$$

for $t \in I$. Now by a suitable application of the result given in Theorem 2.2 to (2.14), we get

$$\begin{aligned}\log z(t) &\leqslant (\log c)(1 + (A(t) + B(t)) \exp(A(t) + B(t))) \\ &\leqslant \log c^{[1+(A(t)+B(t)) \exp(A(t)+B(t))]},\end{aligned} \tag{2.15}$$

where $A(t), B(t)$ are defined by (2.12). From (2.15) we observe that

$$z(t) \leqslant c^{[1+(A(t)+B(t)) \exp(A(t)+B(t))]}. \tag{2.16}$$

Now by using (2.16) in $u(t) \leqslant z(t)$, the inequality in (2.11) follows. $\square$

In the following theorems we establish two independent-variable versions of Theorems 2.2 and 2.3, which can be used for a qualitative analysis of hyperbolic partial differential equations with retarded arguments. In what follows, $J_1 = [x_0, X)$ and $J_2 = [y_0, Y)$ are given subsets of real numbers R, and denote by $\triangle = J_1 \times J_2$. The first order partial derivatives of $z(x,y)$ defined for $x, y \in R$ with respect to x and y are denoted by $z_x(x,y)$ and $z_y(x,y)$, respectively.

LEMMA 2.4. *Let $a, b \in C(\triangle, R_+)$, $\alpha \in C^1(J_1, J_1), \beta \in C^1(J_2, J_2)$ be nondecreasing function with $\alpha(x) \leqslant x$ on J_1, $\beta(y) \leqslant y$ on J_2, $k \geqslant 1$ and $0 < p \leqslant 1$ be constants. If $u \in C(\triangle, R_+)$ and*

$$u(x,y) \leqslant k + \int_{x_0}^{x} \int_{y_0}^{y} a(s,t) u^p(s,t)\, dt\, ds + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} b(s,t) u^p(s,t)\, dt\, ds \tag{2.17}$$

for $(x,y) \in \triangle$, then

$$u(x,y) \leqslant k \exp\left(\int_{x_0}^{x} \int_{y_0}^{y} a(s,t)\, dt\, ds + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} b(s,t)\, dt\, ds \right) \tag{2.18}$$

for $(x,y) \in \triangle$.

Proof. Let $k \geqslant 1, 0 < p \leqslant 1$ and define a function $z(x,y)$ by the right-hand side of (2.17). Then $z(x,y) \geqslant 1, z(x_0, y) = z(x, y_0) = k$, $u(x,y) \leqslant z(x,y)$, and

$$\begin{aligned} z_x(x,y) &= \int_{y_0}^{y} a(x,t) u^p(x,t)\, dt + \left(\int_{\beta(y_0)}^{\beta(y)} b(\alpha(x),t) u^p(\alpha(x),t)\, dt \right) \alpha'(x) \\ &\leqslant \int_{y_0}^{y} a(x,t) z^p(x,t)\, dt + \left(\int_{\beta(y_0)}^{\beta(y)} b(\alpha(x),t) z^p(\alpha(x),t)\, dt \right) \alpha'(x) \\ &\leqslant z(x,y) \int_{y_0}^{y} a(x,t)\, dt + z(\alpha(x), \beta(y)) \left(\int_{\beta(y_0)}^{\beta(y)} b(\alpha(x),t)\, dt \right) \alpha'(x). \end{aligned}$$

Because of the fact $\alpha(t) \leqslant t$ on I, we deduce that

$$z_x(x,y) \leqslant z(x,y) \left[\int_{y_0}^{y} a(x,t)\, dt + \left(\int_{\beta(y_0)}^{\beta(y)} b(\alpha(x),t)\, dt \right) \alpha'(x) \right].$$

The last estimate reduces to the inequality

$$\frac{z_x(x,y)}{z(x,y)} \leqslant \int_{y_0}^{y} a(x,t)\,dt + \left(\int_{\beta(y_0)}^{\beta(y)} b(\alpha(x),t)\,dt\right)\alpha'(x). \tag{2.19}$$

Keeping y fixed in (2.19), setting $x=\sigma$, and integrating it with respect to σ from x_0 to $x, x \in J_1$, and making the change of variable yields

$$z(x,y) \leqslant k\exp\left(\int_{x_0}^{x}\int_{y_0}^{y} a(s,t)\,dt\,ds + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s,t)\,dt\,ds\right). \tag{2.20}$$

Using (2.20) in $u(x,y) \leqslant z(x,y)$, we get the inequality in (2.18). □

THEOREM 2.5. *Let* $a,b \in C(\triangle, R_+)$, $\alpha \in C^1(J_1,J_1), \beta \in C^1(J_2,J_2)$ *be nondecreasing functions with* $\alpha(x) \leqslant x$ *on* J_1, $\beta(y) \leqslant y$ *on* J_2, *and* $0 < p \leqslant 1$ *be a constant. If* $k \in C(\triangle, R_+ - \{0\}), u \in C(\triangle, R_+)$ *and*

$$u(x,y) \leqslant k(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} a(s,t)u^p(s,t)\,dt\,ds + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s,t)u^p(s,t)\,dt\,ds$$

for $(x,y) \in \triangle$, *then*

$$u(x,y) \leqslant k(x,y) + f(x,y)\exp\big(A_1(x,y)+B_1(x,y)\big)$$

for $(x,y) \in \triangle$, *where*

$$\begin{aligned}
f(x,y) &= \int_{x_0}^{x}\int_{y_0}^{y} a(s,t)k^p(s,t)\,dt\,ds + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s,t)k^p(s,t)\,dt\,ds,\\
A_1(x,y) &= p\int_{x_0}^{x}\int_{y_0}^{y} a(s,t)k^{p-1}(s,t)\,dt\,ds,\\
B_1(x,y) &= p\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s,t)k^{p-1}(s,t)\,dt\,ds
\end{aligned}$$

for $(x,y) \in \triangle$.

Proof. We deduce from the hypothesis on $u(x,y)$ that $u(x,y) \leqslant k(x,y) + z(x,y)$, where the function $z(x,y)$ is defined by

$$z(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} a(s,t)u^p(s,t)\,dt\,ds + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s,t)u^p(s,t)\,dt\,ds.$$

By applying some the generalizations of Bernoulli's inequality, it is easy to observe that

$$\begin{aligned}
u^p(x,y) &\leqslant k^p(x,y)\left(1+\frac{z(x,y)}{k(x,y)}\right)^p\\
&\leqslant k^p(x,y)\left(1+p\frac{z(x,y)}{k(x,y)}\right)
\end{aligned} \tag{2.21}$$

for $0 < p \leqslant 1, k : \triangle \to R_+ - \{0\}$. From (2.21) we get

$$z(x,y) \leqslant f(x,y) + p\int_{x_0}^{x}\int_{y_0}^{y} a(s,t)k^{p-1}(s,t)z(s,t)\,dt\,ds + p\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s,t)k^{p-1}(s,t)z(s,t)\,dt\,ds.$$

The rest of the proof follows by an argument similar to that in the proof of Theorem 2.2 with suitable changes. □

THEOREM 2.6. *Let* a, b, α, β, p *be as in Theorem 2.5 and* $c > 1$ *be a constant. If* $u \in C(\triangle, R_+)$ *and*

$$u(x,y) \leqslant c + \int_{x_0}^{x}\int_{y_0}^{y} a(x,y)u^p(x,y)\log u(x,y)\,dt\,ds + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(x,y)u^p(x,y)\log u(x,y)\,dt\,ds$$

for $(x,y) \in \triangle$, *then*

$$u(x,y) \leqslant c^{[1+(A(x,y)+B(x,y))\exp(A(x,y)+B(x,y))]}$$

for $(x,y) \in \triangle$, *where*

$$A(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} a(x,y)\,dt\,ds, \quad B(x,y) = \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(x,y)\,dt\,ds$$

for $(x,y) \in \triangle$.

Proof. The proof of Theorem 2.6 follows by an argument similar to that given for the proof of Theorem 2.3 with some minor changes. □

3. Some Application

In this section we present applications of the inequality given in Theorem 2.5 for the study of the boundedness and uniquness of the solutions of the initial boundary value problem for hyperbolic partial delay differential equations of the form

$$z_{xy}(x,y) = f\left((x,y,z(x,y),z(x-h_1(x),y-h_2(y))\right), \tag{3.1}$$

$$z(x,y_0) = a_1(x), \quad z(x_0,y) = a_2(y), \quad a_1(x_0) = a_2(y_0) = 0, \tag{3.2}$$

where $f \in C(\triangle \times R^2, R), a_1 \in C^1(J_1,R), a_2 \in C^1(J_2,R), h_1 \in C^1(J_1,R_+), h_2 \in C^1(J_2,R_+)$ such that $x - h_1(x) \geqslant 0, y - h_2(y) \geqslant 0, h_1'(x) < 1, h_2'(y) < 1$, and $h_1(x_0) = h_2(y_0) = 0$. Our first aim is to derive the bound on the solution of the problem (3.1)–(3.2).

THEOREM 3.1. *Assume that* $f : \triangle \times R^2 \to R$ *is a continuous function for which there exist continuous nonnegative functions* $a(x,y), b(x,y)$ *for* $(x,y) \in \triangle$ *such that*

$$| f(x,y,u,v) | \leqslant a(x,y) \mid u \mid + b(x,y) \mid v \mid, \tag{3.3}$$

and

$$| a_1(x) + a_2(y) | \leqslant k(x,y) \tag{3.4}$$

for $k : \triangle \to R_+ - \{0\}$, *and let*

$$M_1 = \max_{x \in J_1}\{1 - h_1'(x)\}, \quad M_2 = \max_{y \in J_2}\{1 - h_2'(y)\}. \tag{3.5}$$

If $z(x,y)$ *is any solution of (3.1)–(3.2), then*

$$| z(x,y) | \leqslant k(x,y) + \overline{f}(x,y) \exp\left(\overline{A}(x,y) + \overline{B}(x,y)\right), \tag{3.6}$$

where

$$\overline{f}(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} a(s,t)k(s,t)\,dt\,ds + \frac{1}{M_1M_2}\int_{\phi(x_0)}^{\phi(x)}\int_{\psi(y_0)}^{\psi(y)} \overline{b}(\sigma,\tau)k(\sigma,\tau)\,d\tau\,d\sigma,$$

$$\overline{A}(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} a(s,t)\,dt\,ds,$$

$$\overline{B}(x,y) = M_1M_2\int_{\phi(x_0)}^{\phi(x)}\int_{\psi(y_0)}^{\psi(y)} \overline{b}(\sigma,\tau)\,d\tau\,d\sigma,$$

in which $\phi(x) = x - h_1(x), x \in J_1, \psi(y) = y - h_2(y), y \in J_2,$ *and* $\overline{b}(\sigma,\tau) = b(\sigma + h_1(s), \tau + h_2(t))$ *for* $\sigma, s \in J_1, \tau, t \in J_2$.

Proof. Under the given conditions the solution $z(x,y)$ of the problem (3.1)–(3.2) satisfies the equivalent integral equation

$$z(x,y) = a_1(x) + a_2(y) + \int_{x_0}^{x}\int_{y_0}^{y} f\left(s,t,z(s,t), z(s-h_1(s), t-h_2(t))\right) dt\,ds. \tag{3.8}$$

Using (3.3), (3.4), and (3.5) in (3.8) and making the change of variables, we obtain

$$\begin{aligned} | z(x,y) | \leqslant & k(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} a(s,t) \mid z(s,t) \mid \,dt\,ds \\ & + \frac{1}{M_1M_2}\int_{\phi(x_0)}^{\phi(x)}\int_{\psi(y_0)}^{\psi(y)} \overline{b}(\sigma,\tau) \mid z(\sigma,\tau) \mid \,d\tau\,d\sigma. \end{aligned} \tag{3.9}$$

Now a suitable application of the inequality given in Theorem 2.5 to (3.9) yields (3.6). The right-hand side (3.6) gives us the bound on the solution $z(x,y)$ of (3.1)–(3.2) in terms of the known functions. Thus, if the right-hand side of (3.6) is bounded, then we assert that the solution of (3.1)–(3.2) is bounded for $(x,y) \in \triangle$. □

In the next we derive from Theorem 2.5 the uniqueness of the solutions of the problem (3.1)–(3.2). Let $z(x,y)$ and $\overline{z}(x,y)$ be two solutions of the problem (3.1)–(3.2) with the function f in (3.1) satisfying the condition

$$|f(x,y,u,v) - f(x,y,\overline{u},\overline{v})| \leqslant a(x,y)|u - \overline{u}| + b(x,y)|v - \overline{v}|, \tag{3.10}$$

where $a, b \in C(\triangle, R_+ - \{0\})$, and under some suitable conditions on the functions $M_1, M_2, \phi, \psi, \overline{b}$ as in Theorem 3.1. Then, one obtains the equivalent integral equation

$$z(x,y) - \overline{z}(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} \left[f\left(s,t,z(s,t),z(s-h_1(s),t-h_2(t))\right) \right. \\ \left. -f\left(s,t,z(s,t),\overline{z}(s-h_1(s),t-h_2(t))\right)\right] \, dt\, ds. \tag{3.11}$$

Using (3.10) in (3.11) and making the change of variables, we have

$$\mid z(x,y) - \overline{z}(x,y) \mid \leqslant \int_{x_0}^{x}\int_{y_0}^{y} a(s,t) \mid z(s,t) - \overline{z}(s,t) \mid \, dt\, ds \\ + \frac{1}{M_1 M_2}\int_{\phi(x_0)}^{\phi(x)}\int_{\psi(y_0)}^{\psi(y)} \overline{b}(\sigma,\tau) \mid z(\sigma,\tau) - \overline{z}(\sigma,\tau) \mid \, d\tau\, d\sigma. \tag{3.12}$$

Now by applying the inequality given in Theorem 2.5 to (3.12) yields

$$|z(x,y) - \overline{z}(x,y)| \leqslant 0.$$

From the last estimate we infer $z(x,y) = \overline{z}(x,y)$; that is, there exist at most one solution of the problem (3.1)–(3.2).

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