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A GENERAL HLAWKA INEQUALITY AND ITS REVERSE INEQUALITY

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Abstract. We investigate a general Hlawka inequality and its reverse inequality on a Banach space in an integral form. We also give an interesting relation between a general Djoković inequality and its reverse inequality on a Banach space in a weighted form.

1. Introduction and results

Let (Ω,μ) be a fixed non-trivial finite measure space. Let X be a Banach space and $L^1(X;\Omega,\mu)$ the space of all Bochner integrable X-valued functions on (Ω,μ). For $f\in L^1(X;\Omega,\mu)$, we consider the following inequality (cf. [6]):

$$(\mu(\Omega)-2)\left\|\int_\Omega f(\omega)d\mu\right\|+\int_\Omega \|f(\omega)\|\,d\mu \geqslant \int_\Omega\left\|f(\omega)-\int_\Omega f(t)d\mu(t)\right\|d\mu. \qquad \text{(GHI)}$$

If Ω consists of the three points and μ is the counting measure, then GHI reduces to the well-known Hlawka inequality (cf. [3]):

$$\|x\|+\|y\|+\|z\|+\|x+y+z\|\geqslant\|x+y\|+\|y+z\|+\|z+x\|.$$

This inequality holds for all x,y,z in any Hilbert space. Therefore we call GHI a general Hlawka inequality. If Ω consists of the $n-$ points ($n\geqslant 3$), then GHI reduces to the following inequality:

$$(\sum_{i=1}^n\mu_i-2)\left\|\sum_{i=1}^n\mu_ix_i\right\|+\sum_{i=1}^n\mu_i\|x_i\|\geqslant\sum_{i=1}^n\mu_i\left\|x_i-\sum_{j=1}^n\mu_jx_j\right\|. \qquad \text{(GDI)}$$

If $\mu_1=\cdots=\mu_n=1$, then GDI reduces to the Djoković inequality (cf. [1],[2] and [5]):

$$(n-2)\left\|\sum_{i=1}^n x_i\right\|+\sum_{i=1}^n\|x_i\|\geqslant\sum_{i=1}^n\|x_1+\cdots+\hat{x}_i+\cdots x_n\|,$$

where the sign ˆ placed over a vector indicates that this vector is to be deleted from the

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sum. This inequality holds for all $x_1, \cdots, x_n$ in any Hilbert space. Therefore we call GDI a general Djoković inequality. If GHI holds for all $f \in L^1(X;\Omega,\mu)$, then we say that GHI holds for X. However, GHI does not necessarily hold (cf. [6]) and so it will be meaningful to consider the reverse of a general Hlawka inequality, say RGHI. Similarly for the reverse of a general Djoković inequality, say RGDI. In this paper, we investigate a condition on (Ω,μ) under which GHI or RGHI hold for a Banach space. We also give an interesting relation between GDI and RGDI.

Throughout this paper, we assume that all linear spaces are nonzero. Recall that μ is said to be purely atomic if $\mu(\Omega_a) = \mu(\Omega)$, where Ω_a is the atomic part of μ. We will show the following results.

THEOREM 1. *Let (Ω,μ) be a fixed non-trivial finite measure space. Then GHI holds for all L^1-spaces X if and only if μ is purely atomic and there exist disjoint atoms $A_1,\cdots,A_n$ in Ω such that $\mu(A_i) \geqslant 1 (i = 1,\cdots,n)$ and $\mu(\Omega) = \mu\left(\cup_{i=1}^n A_i\right)$.*

THEOREM 2. *Let (Ω,μ) be a fixed non-trivial finite measure space. Then RGHI holds for all L^1-spaces X if and only if $\mu(\Omega) \leqslant 1$ or there exist disjoint atoms $A_1,\cdots,A_n$ in Ω such that $\mu(A_i) > 0$ $(i = 1,\cdots,n)$, $\mu(\Omega) = \mu\left(\cup_{i=1}^n A_i\right)$, $\sum_{j\neq k} \mu(A_j) \leqslant 1$ $(k = 1,\cdots,n)$ and $\sum_{j=1}^n \mu(A_j) > 1$.*

THEOREM 3. *Suppose that μ is not purely atomic. Then the following are equivalent:*

(i) *RGHI holds for all Banach spaces.*
(ii) *RGHI holds for $X = l_2^\infty(\mathbb{R})$.*
(iii) $\mu(\Omega) \leqslant 1$.

THEOREM 4. *Let X be a Banach space and $n \geqslant 2$. Then the following are equivalent:*

(I) *GDI holds for all $x_1,\cdots,x_n \in X$ and $\mu_1,\cdots,\mu_n \geqslant 1$.*
(II) *RGDI holds for all $x_1,\cdots,x_n \in X$ and $\mu_1,\mu_2,\cdots,\mu_n > 0$ such that $\sum_{j=1}^n \mu_j > 1$ and $\sum_{j\neq k} \mu_j \leqslant 1 (k = 1,\cdots,n)$.*

In the next section, we prepare some lemmas to show the above theorems.

2. Lemmas

LEMMA 5. *If $\mu_1,\cdots,\mu_n \geqslant 1$, then GDI holds for all L^1-spaces.*

Proof. This follows immediately from [6, Corollary 2]. □

LEMMA 6. *If there exists a measurable set A with $0 < \mu(A) < 1$, then GHI does not hold for any Banach space X.*

Proof. Let X be a Banach space and suppose that $0 < \mu(A) < 1$. Take a norm one element $e \in X$ and set $f(\omega) = \chi_A(\omega)e$ $(\omega \in \Omega)$, where χ_A denotes the characteristic function on A. In this case GHI does not hold for $f \in L^1(X;\Omega,\mu)$. In fact, we have

$$(\mu(\Omega)-2)\left\|\int_\Omega f(\omega)d\mu\right\|+\int_\Omega \|f(\omega)\|\,d\mu = (\mu(\Omega)-2)\mu(A)+\mu(A) = \mu(A)(\mu(\Omega)-1).$$

Also we have

$$\begin{aligned}
&\int_\Omega \left\|f(\omega) - \int_\Omega f(t)d\mu(t)\right\| d\mu(\omega)\\
&\quad = \int_\Omega \|\chi_A(\omega)e - \mu(A)e\| d\mu(\omega)\\
&\quad = \int_A \|\chi_A(\omega)e - \mu(A)e\| d\mu(\omega) + \int_{\Omega-A} \|\chi_A(\omega)e - \mu(A)e\| d\mu(\omega)\\
&\quad = \mu(A)\|e - \mu(A)e\| + \mu(\Omega - A)\| - \mu(A)e\|\\
&\quad = \mu(A)(1 - \mu(A)) + \mu(\Omega - A)\mu(A)\\
&\quad = \mu(A)\left(1 + \mu(\Omega) - 2\mu(A)\right).
\end{aligned}$$

Since $0 < \mu(A) < 1$, it follows that

$$(\mu(\Omega) - 2)\left\|\int_\Omega f(\omega)d\mu\right\| + \int_\Omega \|f(\omega)\|d\mu < \int_\Omega \left\|f(\omega) - \int_\Omega f(t)d\mu(t)\right\| d\mu$$

and so GHI does not hold for $f \in L^1(X;\Omega,\mu)$. □

COROLLARY 7. *If GHI holds for some Banach space, then μ is purely atomic.*

Proof. Suppose that GHI holds for some Banach space. By Lemma 6, $\mu(A) \geqslant 1$ holds for all measurable set A with $\mu(A) \neq 0$. If μ is not purely atomic, then we can find a measurable set B such that $0 < \mu(B) < 1$. This is a contradiction. □

LEMMA 8. *If $\mu(\Omega) \leqslant 1$, then RGHI holds for all Banach spaces.*

Proof. Let X be a Banach space and $f \in L^1(X;\Omega,\mu)$. Assume $\mu(\Omega) \leqslant 1$. Since

$$\|f(\omega)\| \leqslant \left\|f(\omega) - \int_\Omega f(t)d\mu(t)\right\| + \left\|\int_\Omega f(t)d\mu(t)\right\|$$

holds for all $\omega \in \Omega$, it follows from $\mu(\Omega) \leqslant 1$ that

$$\begin{aligned}
\int_\Omega \|f(\omega)\|d\mu(\omega) &\leqslant \int_\Omega \left\|f(\omega) - \int_\Omega f(t)d\mu(t)\right\| d\mu(\omega) + \int_\Omega \left\|\int_\Omega f(t)d\mu(t)\right\| d\mu(\omega)\\
&= \int_\Omega \left\|f(\omega) - \int_\Omega f(t)d\mu(t)\right\| d\mu(\omega) + \mu(\Omega)\left\|\int_\Omega f(t)d\mu(t)\right\|\\
&\leqslant \int_\Omega \left\|f(\omega) - \int_\Omega f(t)d\mu(t)\right\| d\mu(\omega) + (2 - \mu(\Omega))\left\|\int_\Omega f(t)d\mu(t)\right\|
\end{aligned}$$

and hence RGHI holds for X. □

LEMMA 9. *Suppose that RGHI holds for* $X = l_2^\infty(\mathbb{R})$. *Then* $\mu(\Omega - A) = 0$ *holds for all measurable sets* A *with* $\mu(A) > 1$.

Proof. Let A be a measurable set with $\mu(A) > 1$ and let $\alpha > 0$ be sufficiently small. Set

$$f(\omega) = \chi_A(\omega)e_1 + \alpha\chi_{\Omega-A}(\omega)e_2 \ \ (\omega \in \Omega),$$

where $e_1 = (1,0)$ and $e_2 = (0,1)$ in $l_2^\infty(\mathbb{R})$. Note that

$$\left\|\int_\Omega f(\omega)d\mu(\omega)\right\|_\infty = \|(\mu(A), \alpha\mu(\Omega - A))\|_\infty = \max\{\mu(A), \alpha\mu(\Omega - A)\} = \mu(A)$$

and

$$\begin{aligned}\int_\Omega \|f(\omega)\|_\infty d\mu(\omega) &= \int_A \|f(\omega)\|_\infty d\mu(\omega) + \int_{\Omega-A} \|f(\omega)\|_\infty d\mu(\omega) \\ &= \mu(A) + \alpha\mu(\Omega - A).\end{aligned}$$

Moreover, we have

$$\begin{aligned}&\int_\Omega \left\|f(\omega) - \int_\Omega f(t)d\mu(t)\right\|_\infty d\mu(\omega) \\ &= \int_A \left\|f(\omega) - \int_\Omega f(t)d\mu(t)\right\|_\infty d\mu(\omega) + \int_{\Omega-A} \left\|f(\omega) - \int_\Omega f(t)d\mu(t)\right\|_\infty d\mu(\omega) \\ &= \int_A \|e_1 - \mu(A)e_1 - \alpha\mu(\Omega - A)e_2\|_\infty d\mu(\omega) \\ &\qquad\qquad + \int_{\Omega-A} \|\alpha e_2 - \mu(A)e_1 - \alpha\mu(\Omega - A)e_2\|_\infty d\mu(\omega) \\ &= \mu(A)\max\{|1 - \mu(A)|, \alpha\mu(\Omega - A)\} + \mu(\Omega - A)\max\{\mu(A), \alpha|1 - \mu(\Omega - A)|\} \\ &= \mu(A)(\mu(A) - 1) + \mu(A)\mu(\Omega - A) \ \ (\text{since } \mu(A) > 1 \text{ and } \alpha \text{ is sufficiently small}).\end{aligned}$$

Since RGHI holds for $X = l_2^\infty(\mathbb{R})$, it follows from the above equalities that

$$(\mu(\Omega) - 2)\mu(A) + \mu(A) + \alpha\mu(\Omega - A) \leqslant \mu(A)(\mu(A) - 1) + \mu(A)\mu(\Omega - A)$$

and then

$$\mu(\Omega) - 2 + 1 + \alpha\frac{\mu(\Omega - A)}{\mu(A)} \leqslant \mu(A) - 1 + \mu(\Omega - A).$$

Consequently, we obtain that $\alpha\frac{\mu(\Omega-A)}{\mu(A)} \leqslant 0$ and so $\mu(\Omega - A)$ must be zero. □

COROLLARY 10. *Suppose that RGHI holds for* $X = l_2^\infty(\mathbb{R})$. *If* μ *is not purely atomic, then* $\mu(\Omega) \leqslant 1$.

Proof. Suppose that μ is not purely atomic and RGHI holds for $X = l_2^\infty(\mathbb{R})$. In this case, we show that $\mu(\Omega) \leqslant 1$. Suppose contrary and put $\delta = \mu(\Omega) - 1$, hence $\delta > 0$. Since μ is not purely atomic, it follows that there exists a measurable set A such that $0 < \mu(A) < \delta$. Then $\mu(\Omega - A) = \mu(\Omega) - \mu(A) = \delta + 1 - \mu(A) > 1$ and hence $\mu(A) = 0$ by Lemma 9, a contradiction. □

3. Proofs of Theorems

Proof of Theorem 3. This follows immediately from Lemma 8 and Corollary 10. □

Proof of Theorem 2. Necessity. Suppose that RGHI holds for all L^1-spaces and $\mu(\Omega) > 1$. Note that the space $l_2^\infty(\mathbb{R})$ can be isometrically embedded in some L^1-space (cf. [4]). Since RGHI holds for $l_2^\infty(\mathbb{R})$, it follows from Lemma 9 that

$$\mu(\Omega - A) = 0 \text{ holds for all measurable sets } A \text{ with } \mu(A) > 1 \tag{1}$$

and hence μ must be purely atomic as observed in the proof of Corollary 10. Since (Ω, μ) is finite, we can find at most countable disjoint atoms $A_1, A_2, \cdots$ such that $\mu(\Omega) = \sum_{k=1}^{\infty} \mu(A_k)$ and $\mu(A_k) > 0$ for each k. Since $\mu(\Omega) > 1$, there is a number n with $\sum_{k=1}^{n} \mu(A_k) > 1$ and then by (1), $\mu\left(\Omega - \cup_{k=1}^{n} A_k\right) = 0$, hence $\mu(A_k) = 0$ for each $k \geqslant n+1$.Therefore we have from (1) that $\sum_{j \neq k} \mu(A_k) \leqslant 1 \quad (k = 1, \cdots, n)$ because $\mu(A_k) > 0$ for each $1 \leqslant k \leqslant n$.

Sufficiency. If $\mu(\Omega) \leqslant 1$, then RGHI holds for all L^1-spaces by Lemma 8. Next suppose that there exist disjoint atoms $A_1, A_2, \cdots, A_n$ in Ω such that $\mu(A_i) > 0 \quad (i = 1, \cdots, n)$, $\mu(\Omega) = \mu\left(\cup_{k=1}^{n} A_k\right)$, $\sum_{j \neq k} \mu(A_k) \leqslant 1 \quad (k = 1, \cdots, n)$ and $\sum_{j=1}^{n} \mu(A_j) > 1$. In this case, we show that RGHI again holds for all L^1-spaces. To do this, let X be an L^1-space and put $\nu_j = \mu(A_j) \quad (j = 1, \cdots, n)$. Then $\nu_i > 0 \quad (i = 1, \cdots, n)$, $\sum_{j \neq k} \nu_j \leqslant 1 \quad (k = 1, \cdots, n)$ and $\sum_{j=1}^{n} \nu_j > 1$. Also RGHI can be rewritten by the following RGDI:

$$\left(\sum_{i=1}^{n} \nu_i - 2\right) \left\|\sum_{i=1}^{n} \nu_i y_i\right\| + \sum_{i=1}^{n} \nu_i \|y_i\| \leqslant \sum_{i=1}^{n} \nu_i \left\|y_i - \sum_{j=1}^{n} \nu_j y_j\right\|, \tag{2}$$

where $y_1, \cdots, y_n \in X$. However, (2) always holds by Lemma 5 and Theorem 4 which is shown later. □

Proof of Theorem 1. Necessity. This follows immediately from Lemma 6 and Corollary 7.

Sufficiency. Suppose that μ is purely atomic and there exist disjoint atoms $A_1, \cdots A_n$ in Ω such that $\mu(A_j) \geqslant 1 \quad (i = 1, \cdots n)$ and $\mu(\Omega) = \mu\left(\cup_{i=1}^{n} A_i\right)$. Let X be an L^1-space and put $\nu_j = \mu(A_j) \quad (j = 1, \cdots, n)$. Then GHI can be rewritten by the following GDI:

$$\left(\sum_{i=1}^{n} \nu_i - 2\right) \left\|\sum_{i=1}^{n} \nu_i y_i\right\| + \sum_{i=1}^{n} \nu_i \|y_i\| \geqslant \sum_{i=1}^{n} \nu_i \left\|y_i - \sum_{j=1}^{n} \nu_j y_j\right\|, \tag{3}$$

where $y_1, \cdots, y_n \in X$. However since $\nu_j \geqslant 1 \quad (j = 1, \cdots, n)$, it follows from Lemma 5 that (3) always holds. □

Proof of Theorem 4. (I) $\Rightarrow$ (II). Suppose (I). Let $x_1, \cdots, x_n \in X$ and let $\mu_1, \mu_2, \cdots, \mu_n > 0$ be such that $\sum_{j=1}^n \mu_j > 1$ and $\sum_{j \neq k} \mu_j \leqslant 1$ $(k = 1, \cdots, n)$. Set

$$\nu_j = \frac{\mu_j}{\sum_{i=1}^n \mu_i - 1} \text{ and } y_j = \frac{1}{\sum_{i=1}^n \nu_i - 1}\left(x_j - \sum_{i=1}^n \mu_i x_i\right) \ (j = 1, \cdots, n).$$

Then we have the following properties:

(i) $\nu_j \geqslant 1$ $(j = 1, \cdots, n)$.

(ii) $\mu_j = \frac{\nu_j}{\sum_{i=1}^n \nu_i - 1}$ $(j = 1, \cdots, n)$.

(iii) $2 - \sum_{i=1}^n \mu_i = \frac{\sum_{i=1}^n \nu_i - 2}{\sum_{i=1}^n \nu_i - 1}$.

(iv) $x_j = \left(\sum_{i=1}^n \nu_i - 1\right)\left(y_j - \sum_{i=1}^n \nu_i y_i\right)$ $(j = 1, \cdots, n)$.

In fact, (i),(ii) and (iii) follows from an easy observation. Also since

$$\begin{aligned}
\sum_{j=1}^n \nu_j y_j &= \frac{1}{\sum_{i=1}^n \nu_i - 1}\sum_{j=1}^n\left(\nu_j x_j - \nu_j \sum_{i=1}^n \mu_i x_i\right) \quad \text{(by the construction of } y_j\text{)}\\
&= \sum_{j=1}^n \mu_j x_j - \frac{\sum_{j=1}^n \nu_j}{\sum_{i=1}^n \nu_i - 1}\sum_{i=1}^n \mu_i x_i \text{ (by (ii))}\\
&= \left(1 - \frac{\sum_{j=1}^n \nu_j}{\sum_{i=1}^n \nu_i - 1}\right)\sum_{i=1}^n \mu_i x_i\\
&= -\frac{\sum_{i=1}^n \mu_i x_i}{\sum_{i=1}^n \nu_i - 1}\\
&= y_j - \frac{x_j}{\sum_{i=1}^n \nu_i - 1} \text{ (by the construction of } y_j\text{)},
\end{aligned}$$

if follows that (iv) holds. Therefore we have

$$\begin{aligned}
\sum_{j=1}^n \mu_j \|x_j\| &= \frac{\sum_{j=1}^n \nu_j \|x_j\|}{\sum_{i=1}^n \nu_i - 1} \text{ (by (ii))}\\
&= \sum_{j=1}^n \nu_j \left\| y_j - \sum_{i=1}^n \nu_i y_i \right\| \text{ (by (iv))}\\
&\leqslant \left(\sum_{i=1}^n \nu_i - 2\right)\left\|\sum_{i=1}^n \nu_i y_i\right\| + \sum_{i=1}^n \nu_i \|y_i\| \text{ (by (i) and (I))}.
\end{aligned}$$

Note that

$$\begin{aligned}\sum_{j=1}^{n}\mu_j x_j &= \frac{1}{\sum_{i=1}^{n}\nu_i - 1}\sum_{j=1}^{n}\nu_j x_j \text{ (by (ii))}\\ &= \sum_{j=1}^{n}\nu_j\left(y_j - \sum_{i=1}^{n}\nu_i y_i\right) \text{ (by (iv))}\\ &= \sum_{j=1}^{n}\nu_j y_j - \left(\sum_{i=1}^{n}\nu_i\right)\sum_{i=1}^{n}\nu_i y_i\\ &= \left(1 - \sum_{i=1}^{n}\nu_i\right)\sum_{i=1}^{n}\nu_i y_i.\end{aligned}$$

Then we have

$$\begin{aligned}\sum_{j=1}^{n}\mu_j\|x_j\| &\leqslant \left(\sum_{i=1}^{n}\nu_i - 2\right)\frac{\left\|\sum_{j=1}^{n}\mu_j x_j\right\|}{\sum_{i=1}^{n}\nu_i - 1} + \sum_{j=1}^{n}\frac{\nu_j}{\sum_{i=1}^{n}\nu_i - 1}\left\|x_j - \sum_{i=1}^{n}\mu_i x_i\right\|\\ &= \left(2 - \sum_{i=1}^{n}\mu_i\right)\left\|\sum_{j=1}^{n}\mu_j x_j\right\| + \sum_{j=1}^{n}\mu_j\left\|x_j - \sum_{i=1}^{n}\mu_i x_i\right\| \text{ (by (ii) and (iii))}\end{aligned}$$

and hence (II) holds.

(II) $\Rightarrow$ (I). Suppose (II). Let $x_1,\cdots x_n \in X$ and $\mu_1,\cdots\mu_n \geqslant 1$. Set

$$\nu_j = \frac{\mu_j}{\sum_{i=1}^{n}\mu_i - 1} \text{ and } y_j = \left(\sum_{i=1}^{n}\mu_i - 1\right)\left(x_j - \sum_{i=1}^{n}\mu_i x_i\right) \ (j = 1,\cdots,n).$$

Then we can easily see that $\nu_1,\nu_2,\cdots,\nu_n > 0$, $\sum_{j=1}^{n}\nu_j > 1$ and $\sum_{j\neq k}\nu_j \leqslant 1$ $(k = 1,\cdots,n)$. Therefore by hypothesis, we have

$$\left(\sum_{i=1}^{n}\nu_i - 2\right)\left\|\sum_{i=1}^{n}\nu_i y_i\right\| + \sum_{i=1}^{n}\nu_i\|y_i\| \leqslant \sum_{i=1}^{n}\nu_i\left\|y_i - \sum_{j=1}^{n}\nu_j y_j\right\|. \tag{4}$$

Note also that the following equalities hold:

(a) $2 - \sum_{i=1}^{n}\nu_i = \frac{\sum_{i=1}^{n}\mu_i - 2}{\sum_{i=1}^{n}\mu_i - 1}$.

(b) $\sum_{j=1}^{n}\nu_j y_j = \left(1 - \sum_{i=1}^{n}\mu_i\right)\sum_{i=1}^{n}\mu_i x_i$.

(c) $y_j - \sum_{i=1}^{n}\nu_i y_i = \left(\sum_{i=1}^{n}\mu_i - 1\right)x_j \ (j = 1,\cdots,n)$.

Then by (a) and (b) we have

$$\left(\sum_{i=1}^{n} \nu_i - 2\right) \left\|\sum_{i=1}^{n} \nu_i y_i\right\| = \left(2 - \sum_{i=1}^{n} \mu_i\right) \left\|\sum_{j=1}^{n} \mu_i x_i\right\|. \tag{5}$$

By the construction of ν_j and y_j, we have

$$\sum_{i=1}^{n} \nu_i \|y_i\| = \sum_{i=1}^{n} \mu_i \left\|x_i - \sum_{j=1}^{n} \mu_j x_j\right\|. \tag{6}$$

Also by the construction of ν_j and (c), we have

$$\sum_{i=1}^{n} \nu_i \left\|y_i - \sum_{j=1}^{n} \nu_j y_j\right\| = \sum_{i=1}^{n} \mu_i \|x_i\|. \tag{7}$$

Then by (4),(5),(6) and (7), we have

$$\left(\sum_{i=1}^{n} \mu_i - 2\right) \left\|\sum_{i=1}^{n} \mu_i x_i\right\| + \sum_{i=1}^{n} \mu_i \|x_i\| \geqslant \sum_{i=1}^{n} \mu_i \left\|x_i - \sum_{j=1}^{n} \mu_j x_j\right\|$$

and hence (I) holds. □

4. Further remarks

REMARK 1. If GDI holds for some Banach space X, then we have $\mu_1, \cdots \mu_n \geqslant 1$ by Lemma 6. But we can show this fact directly as follows: let $1 \leqslant j \leqslant n$ and take a norm one element $e \in X$. Set $x_i = \delta_{ij} e \ (i = 1, \cdots, n)$, where δ denotes the Kronecker delta. Then GDI reduces to the following inequality:

$$\left(\sum_{i=1}^{n} \mu_i - 2\right) \mu_j + \mu_j \geqslant \mu_j |1 - \mu_j| + \mu_j \sum_{i \neq j} \mu_i.$$

This implies easily that $\mu_j \geqslant 1$. Actually, this fact was a motivation of our paper.

REMARK 2. Lemma 5 is extended as follows:

PROPOSITION 11. *Let X be an L^1-space, $\mu_1, \cdots, \mu_n > 0$ and $\mu = \min\{1, \mu_1, \cdots, \mu_n\}$. Then*

$$\left(\sum_{i=1}^{n} \mu_i - 2\mu\right) \left\|\sum_{i=1}^{n} \mu_i x_i\right\| + \sum_{i=1}^{n} \mu_i \|x_i\| \geqslant \sum_{i=1}^{n} \mu_i \left\|x_i - \sum_{j=1}^{n} \mu_j x_j\right\|$$

holds for all $x_1, \cdots, x_n \in X$.

Proof. Let $x_1, \cdots, x_n \in X$. Since $\frac{\mu_j}{\mu} \geqslant 1 (j = 1, \cdots, n)$, it follows from Lemma 5 that

$$\sum_{i=1}^{n} \mu_i \left\| x_i - \sum_{j=1}^{n} \frac{\mu_j}{\mu} x_j \right\| \leqslant \left(\sum_{i=1}^{n} \mu_i - 2\mu \right) \left\| \sum_{i=1}^{n} \frac{\mu_i}{\mu} x_i \right\| + \sum_{i=1}^{n} \mu_i \|x_i\|. \tag{8}$$

Therefore,

$$\begin{aligned}
&\sum_{i=1}^{n} \mu_i \left\| x_i - \sum_{j=1}^{n} \mu_j x_j \right\| \\
&= \mu \sum_{i=1}^{n} \mu_i \left\| \frac{x_i}{\mu} - \sum_{j=1}^{n} \frac{\mu_j}{\mu} x_j \right\| \\
&\leqslant \mu \sum_{i=1}^{n} \mu_i \left\| x_i - \sum_{j=1}^{n} \frac{\mu_j}{\mu} x_j \right\| + \mu \sum_{i=1}^{n} \mu_i \left\| \frac{x_i}{\mu} - x_i \right\| \\
&\leqslant \mu \left(\left(\sum_{i=1}^{n} \mu_i - 2\mu \right) \left\| \sum_{i=1}^{n} \frac{\mu_i}{\mu} x_i \right\| + \sum_{i=1}^{n} \mu_i \|x_i\| \right) + \mu \sum_{i=1}^{n} \mu_i \left\| \frac{x_i}{\mu} - x_i \right\| \quad \text{(by (8))} \\
&= \left(\sum_{i=1}^{n} \mu_i - 2\mu \right) \left\| \sum_{i=1}^{n} \mu_i x_i \right\| + \sum_{i=1}^{n} \mu_i \left(\mu + |1 - \mu| \right) \|x_i\| \\
&= \left(\sum_{i=1}^{n} \mu_i - 2\mu \right) \left\| \sum_{i=1}^{n} \mu_i x_i \right\| + \sum_{i=1}^{n} \mu_i \|x_i\| \quad (\text{ since } \mu \leqslant 1)
\end{aligned}$$

and so we obtain the desired inequality. □

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