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A CONVERSE OF THE HÖLDER INEQUALITY THEOREM

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Abstract. Let (Ω,Σ,μ) be a measure space such that $0<\mu(A)<1<\mu(B)<\infty$ for some $A,B\in\Sigma$ and let bijections $\varphi_1,\varphi_2,\psi_1,\psi_2:(0,\infty)\to(0,\infty)$ be such that $\frac{\psi_1\circ\varphi_1(t)}{t}\leqslant c\leqslant\frac{t}{\psi_2\circ\varphi_2(t)}$ $(t>0)$. We prove that if

$$\int_\Omega xy\,d\mu\leqslant\psi_1\left(\int_{\Omega(\mathbf{x})}\varphi_1\circ|x|\,d\mu\right)\psi_2\left(\int_{\Omega(\mathbf{y})}\varphi_2\circ|y|\,d\mu\right)$$

for all nonnegative μ-integrable simple functions $\mathbf{x},\mathbf{y}:\Omega\to\mathbb{R}$ (where $\Omega(\mathbf{x})$ stands for the support of $\mathbf{x}$), then then there exists a real $p>1$ such that

$$\frac{\varphi_1(t)}{\varphi_1(1)}=t^p,\quad\frac{\psi_1(t)}{\psi_1(1)}=t^{1/p},\quad\frac{\varphi_2(t)}{\varphi_2(1)}=t^q,\quad\frac{\psi_2(t)}{\psi_2(1)}=t^{1/q},\qquad t>0,$$

where $\frac{1}{p}+\frac{1}{q}=1$. A relevant result for the reversed inequality is also given.

1. Introduction

For a measure space (Ω,Σ,μ) denote by $\mathbf{S}=\mathbf{S}(\Omega,\Sigma,\mu)$ the real linear space of all μ-integrable simple functions $\mathbf{x}:\Omega\to\mathbb{R}$. For two arbitrarily fixed bijections $\varphi,\psi:(0,\infty)\to(0,\infty)$ define the functional $\mathbb{P}_{\varphi,\psi}:\mathbf{S}\to[0,\infty)$ by

$$\mathbb{P}_{\varphi,\psi}(\mathbf{x}):=\begin{cases}\psi\left(\int_{\Omega(\mathbf{x})}\varphi\circ|\mathbf{x}|\,d\mu\right) & \text{if}\quad \mu(\Omega(\mathbf{x}))>0\\ 0 & \text{if}\quad \mu(\Omega(\mathbf{x}))=0\end{cases},$$

where $\Omega(\mathbf{x}):=\{\omega\in\Omega:\mathbf{x}(\omega)\neq0\}$.

Supposing that there are $A,B\in\Sigma$ such that $0<\mu(A)<1<\mu(B)<\infty$ we show that if the bijections $\varphi_1,\varphi_2,\psi_1,\psi_2:(0,\infty)\to(0,\infty)$ are such that

$$\frac{\psi_1\circ\varphi_1(t)}{t}\leqslant c\leqslant\frac{t}{\psi_2\circ\varphi_2(t)},\qquad t>0,\tag{c}$$

for a constant $c>0$, and satisfy the inequality

$$\int_\Omega\mathbf{xy}\,d\mu\leqslant\mathbb{P}_{\varphi_1,\psi_1}(\mathbf{x})\mathbb{P}_{\varphi_2,\psi_2}(\mathbf{y})\tag{H}$$

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for all $\mathbf{x},\mathbf{y} \in \mathbf{S}_+(\Omega,\Sigma,\mu) := \{\ \mathbf{x} \in \mathbf{S} :\ \mathbf{x} \geqslant \mathbf{0}\}$, then there is a real $p > 1$ such that

$$\frac{\varphi_1(t)}{\varphi_1(1)} = t^p, \quad \frac{\psi_1(t)}{\psi_1(1)} = t^{1/p}, \quad \frac{\varphi_2(t)}{\varphi_2(1)} = t^q, \quad \frac{\psi_2(t)}{\psi_2(1)} = t^{1/q}, \qquad t > 0,$$

where $p^{-1} + q^{-1} = 1$ (Theorem 1). Concerning the bijections, no regularity conditions are assumed.

This converse of Hölder's inequality theorem generalizes the main result of [5] where the case $\varphi_1 := \varphi$, $\varphi_2 := \psi$, $\psi_1 := \varphi^{-1}$, $\psi_2 := \psi^{-1}$ (with only two unknown functions) was considered. Note that, in this case, $\frac{\psi_1\circ\varphi_1(t)}{t} = 1 = \frac{t}{\psi_2\circ\varphi_2(t)}$, and condition (c) is obviously satisfied. Since $\varphi_1, \varphi_2, \psi_1, \psi_2$ are defined in $(0,\infty)$, Theorem 1 improves the main result of [5] were φ_1, φ_2 are defined on $[0,\infty)$ and it is assumed that $\varphi_1(0) = \varphi_2(0) = 0$ (cf. Corollary 1).

A relevant result for the reversed inequality is also presented (Theorem 2).

The existence of two sets $A, B \in \Sigma$ such that $0 < \mu(A) < 1 < \mu(B) < \infty$ plays a crucial role. If a measure space fails to satisfy this condition, then there are some broad classes of non-power bijections $\varphi_1, \varphi_2, \psi_1, \psi_2$ for which the functionals $\mathbb{P}_{\varphi_1,\psi_1}$ and $\mathbb{P}_{\varphi_2,\psi_2}$ satisfy the inequality (H).

The suitable results for the Minkowski inequality are given in [3] and [7].

2. Some lemmas

LEMMA 1. ([4], [6]) *Let* real *numbers a, b such that $0 < a < 1 < a + b$ be fixed. Then a function $f : (0,\infty) \to \mathbb{R}$ such that $\limsup_{t\to 0+} f(t) \leqslant 0$ satisfies the inequality*

$$f(as + bt) \leqslant af(s) + bf(t), \qquad s,t > 0,$$

if, and only if, $f(t) = f(1)t$ for all $t > 0$.

Applying this lemma we obtain:

LEMMA 2. *Let real numbers a, b such that $0 < a < 1 < a + b$ be fixed. If a function $F : (0,\infty)^2 \to \mathbb{R}$ satisfies the inequality*

$$F(ax_1 + bx_2, ay_1 + by_2) \leqslant aF(x_1, y_1) + bF(x_2, y_2), \qquad x_1, x_2, y_1, y_2 > 0,$$

and the condition

$$\limsup_{t\to 0+} F(tx, ty) \leqslant 0, \qquad x, y > 0,$$

then F is positively homogeneous, i.e.

$$F(tx, ty) = tF(x, y), \qquad t, x, y > 0.$$

REMARK 1. A finite dimensional counterpart of this lemma is also true (cf. [4], Theorem 2, and [6]). Let a positive integer $n \geqslant 2$ real numbers a, b such that $0 < a < 1 < a + b$ be fixed. If $F : (0,\infty)^n \to \mathbb{R}$ satisfies the condition

$$\limsup_{t\to 0+} F(tx_1, \ldots, tx_n) \leqslant 0, \qquad (x_1, \ldots, x_n) \in (0,\infty)^n,$$

and the inequality

$$F(ax_1+by_1,...,ax_n+by_n) \leqslant aF(x_1,...,x_n)+bF(y_1,...,y_n)$$

for all $x_1,...,x_n,y_1,...,y_n > 0$, then

$$F(tx_1,...,tx_n) = tF(x_1,...,x_n), \qquad t,x_1,...,x_n > 0.$$

3. The converse of Hölder's inequality

The main result of this paper reads as follows.

THEOREM 1. *Let (Ω,Σ,μ) be a measure space such that there are two sets $A,B \in \Sigma$ satisfying the condition*

$$0 < \mu(A) < 1 < \mu(B) < \infty.$$

Suppose that $\varphi_1,\varphi_2,\psi_1,\psi_2 : (0,\infty) \to (0,\infty)$ are bijective functions such that

$$\frac{\psi_1\circ\varphi_1(t)}{t} \leqslant c \leqslant \frac{t}{\psi_2\circ\varphi_2(t)}, \qquad t>0, \tag{1}$$

for a constant $c > 0$. Then the following conditions are equivalent:

(i) the functions $\varphi_1,\varphi_2,\psi_1,\psi_2$ satisfy the inequality

$$\int_\Omega \mathbf{x}\mathbf{y}\,d\mu \leqslant \mathbb{P}_{\varphi_1,\psi_1}(\mathbf{x})\mathbb{P}_{\varphi_2,\psi_2}(\mathbf{y}) \ \ \textit{for all} \ \ \mathbf{x},\mathbf{y}\in\mathbf{S}_+; \tag{2}$$

(ii) there is a real $p > 1$ such that

$$\frac{\varphi_1(t)}{\varphi_1(1)} = t^p, \quad \frac{\psi_1(t)}{\psi_1(1)} = t^{1/p}, \quad \frac{\varphi_2(t)}{\varphi_2(1)} = t^q, \quad \frac{\psi_2(t)}{\psi_2(1)} = t^{1/q}, \qquad t>0,$$

and

$$\psi_1(1)\psi_2(1)\left(\varphi_1(1)\right)^{1/p}\left(\varphi_2(1)\right)^{1/q} = 1,$$

where

$$\frac{1}{p}+\frac{1}{q} = 1.$$

Proof. To show the implication $(i) \Rightarrow (ii)$ suppose that (i) holds true and put $a := \mu(A)$, $b := \mu(B\backslash A)$. Then, obviously,

$$0 < a < 1 < a+b.$$

For all $x_1,x_2 > 0$, the function $\mathbf{x} := x_1\chi_A + x_2\chi_{B\backslash A}$ belongs to $\mathbf{S}_+$. Setting

$$\mathbf{x} := x_1\chi_A + x_2\chi_{B\backslash A}, \qquad \mathbf{y} := y_1\chi_A + y_2\chi_{B\backslash A}, \qquad x_1,x_2,y_1,y_2 > 0,$$

in inequality (2) we get, for all $x_1, x_2, y_1, y_2 > 0$,

$$ax_1y_1 + bx_2y_2 \leqslant \psi_1\left(a\varphi_1(x_1) + b\varphi_1(x_2)\right)\psi_2\left(a\varphi_2(y_1) + b\varphi_2(y_2)\right). \tag{3}$$

Replacing here x_i by $\varphi_1^{-1}(x_i)$, y_i by $\varphi_2^{-1}(y_i)$ for $i = 1,2$, we obtain

$$a\varphi_1^{-1}(x_1)\varphi_2^{-1}(y_1) + b\varphi_1^{-1}(x_2)\varphi_2^{-1}(y_2) \leqslant \psi_1\left(ax_1 + bx_2\right)\psi_2(ay_1 + by_2) \tag{4}$$

for all $x_1, x_2, y_1, y_2 > 0$.

From assumption (1) we have

$$\frac{\psi_1(t)}{\varphi_1^{-1}(t)} \leqslant c \leqslant \frac{\varphi_2^{-1}(t)}{\psi_2(t)}, \qquad t > 0,$$

whence

$$\frac{\psi_1(s)}{\varphi_1^{-1}(s)} \leqslant \frac{\varphi_2^{-1}(t)}{\psi_2(t)}, \qquad s,t > 0,$$

and, consequently,

$$\psi_1(s)\psi_2(t) \leqslant \varphi_1^{-1}(s)\varphi_2^{-1}(t), \qquad s,t > 0. \tag{5}$$

This inequality and (4) imply that, for all $x_1, y_1, x_2, y_2 > 0$,

$$a\psi_1(x_1)\psi_2(y_1) + b\psi_1(x_2)\psi_2(y_2) \leqslant \psi_1\left(ax_1 + bx_2\right)\psi_2\left(ay_1 + by_2\right). \tag{6}$$

Thus $F : (0,\infty)^2 \to (-\infty, 0)$ defined by

$$F(x,y) = -\psi_1(x)\psi_2(y), \qquad x,y > 0,$$

has negative values and satisfies the inequality

$$F(ax_1 + bx_2, ay_1 + by_2) \leqslant aF(x_1, y_1) + bF(x_2, y_2), \qquad x_1, x_2, y_1, y_2 > 0.$$

By Lemma 2, the function F is positively homogeneous, i.e.

$$\psi_1(tx)\psi_2(ty) = t\psi_1(x)\psi_2(y), \qquad t,x,y > 0. \tag{7}$$

Setting here $y = 1$ we obtain

$$\psi_2(t) = \psi_2(1)t\frac{\psi_1(x)}{\psi_1(tx)}, \qquad t,x > 0.$$

Since the right-hand side does not depend on x,

$$\psi_2(1)t\frac{\psi_1(x)}{\psi_1(tx)} = \psi_2(1)t\frac{\psi_1(y)}{\psi_1(ty)}, \qquad t,x,y > 0,$$

whence

$$\frac{\psi_1(x)}{\psi_1(tx)} = \frac{\psi_1(y)}{\psi_1(ty)}, \qquad t,x,y > 0.$$

Setting here $y = 1$, we get

$$\frac{\psi_1(x)}{\psi_1(tx)} = \frac{\psi_1(1)}{\psi_1(t)}, \qquad t, x > 0,$$

which can be written in the form

$$\frac{\psi_1(tx)}{\psi_1(1)} = \frac{\psi_1(t)}{\psi_1(1)} \cdot \frac{\psi_1(x)}{\psi_1(1)}, \qquad t, x > 0.$$

This shows that the function $k_1 : (0,\infty) \to (0,\infty)$ defined by

$$k_1(t) := \frac{\psi_1(t)}{\psi_1(1)}, \qquad t > 0,$$

is multiplicative, i.e.

$$k_1(st) = k_1(s)k_1(t), \qquad s, t > 0.$$

In the same way we can show that the function

$$k_2(t) := \frac{\psi_2(t)}{\psi_2(1)}, \qquad t > 0,$$

is multiplicative. Since

$$\psi_1(t) = \psi_1(1)k_1(t), \qquad \psi_2(t) = \psi_2(1)k_2(t) \qquad t > 0, \tag{8}$$

setting these functions into (7) gives

$$k_1(t)k_2(t) = t, \qquad t > 0. \tag{9}$$

From (4) and (5) we obtain, for all $x_1, x_2, y_1, y_2 > 0$,

$$a\varphi_1^{-1}(x_1)\varphi_2^{-1}(y_1) + b\varphi_1^{-1}(x_2)\varphi_2^{-1}(y_2) \leqslant \varphi_1^{-1}(ax_1 + bx_2)\,\varphi_2^{-1}(ay_1 + by_2).$$

Thus the function $F : (0,\infty)^2 \to (-\infty, 0)$ defined by

$$F(x,y) = -\varphi_1^{-1}(x)\varphi_2^{-1}(y), \qquad x, y > 0,$$

satisfies the assumptions of Lemma 2. Now, repeating the above reasoning, we can show that there are some multiplicative functions $l_1, l_2 : (0,\infty) \to (0,\infty)$ such that

$$\varphi_1^{-1}(t) = \varphi_1^{-1}(1)l_1(t), \qquad \varphi_2^{-1}(t) = \varphi_2^{-1}(1)l_2(t), \qquad t > 0, \tag{10}$$

and

$$l_1(t)l_2(t) = t, \qquad t > 0. \tag{11}$$

Put

$$k := k_1, \qquad l := l_1. \tag{12}$$

From (9) and (11) we get

$$k_2(t) := \frac{t}{k(t)}, \qquad l_2(t) := \frac{t}{l(t)}, \qquad t > 0. \tag{13}$$

Making use of (5), (8) and (10), we get

$$\psi_1(1)\psi_2(1)k(s)\frac{t}{k(t)} \leqslant \varphi_1^{-1}(1)\varphi_2^{-1}(1)l(s)\frac{t}{l(t)}, \qquad s,t > 0,$$

whence

$$d\frac{k(s)}{l(s)} \leqslant \frac{k(t)}{l(t)}, \qquad s,t > 0, \tag{14}$$

where

$$d := \frac{\psi_1(1)\psi_2(1)}{\varphi_1^{-1}(1)\varphi_2^{-1}(1)}. \tag{15}$$

Inequality (14) implies that the function $\frac{k}{l}$ is globally bounded. Since $\frac{k}{l}$ is a multiplicative function, it must be constant and $\frac{k}{l} \equiv 1$. Consequently,

$$l(t) = k(t), \qquad t > 0. \tag{16}$$

From (4), taking into account (8), (10), (12), (13) and (16), we obtain

$$\varphi_1^{-1}(1)\varphi_2^{-1}(1)\left(ak(x_1)\frac{y_1}{k(y_1)} + bk(x_2)\frac{y_2}{k(y_2)}\right) \leqslant \psi_1(1)\psi_2(1)k(ax_1+bx_2)\frac{ay_1+by_2}{k(ax_1+bx_2)}$$

whence, taking into account (15), for all $x_1,x_2,y_1,y_2 > 0$,

$$ak(x_1)\frac{y_1}{k(y_1)} + bk(x_2)\frac{y_2}{k(y_2)} \leqslant dk(ax_1+bx_2)\frac{ay_1+by_2}{k(ay_1+by_2)}.$$

By the multiplicativity of k we can write the last inequality in the form

$$ay_1k\left(\frac{ax_1}{ay_1}\right) + by_2k\left(\frac{bx_2}{by_2}\right) \leqslant d\frac{ay_1+by_2}{k(ay_1+by_2)}k(ax_1+bx_2),$$

which means that

$$y_1k\left(\frac{x_1}{y_1}\right) + y_2k\left(\frac{x_2}{y_2}\right) \leqslant d(y_1+y_2)k\left(\frac{x_1+x_2}{y_1+y_2}\right), \qquad x_1,x_2,y_1,y_2 > 0.$$

Setting here $x_1 = x_2 = y_1 = y_2 = 1$ we get $1 \leqslant d$. On the other hand, setting $s = t$ in (14) we get $d \leqslant 1$. Thus

$$d = 1,$$

and, consequently,

$$y_1k\left(\frac{x_1}{y_1}\right) + y_2k\left(\frac{x_2}{y_2}\right) \leqslant (y_1+y_2)k\left(\frac{x_1+x_2}{y_1+y_2}\right), \qquad x_1,x_2,y_1,y_2 > 0.$$

Putting here $x_1 = tx,\ x_2 = (1-t)y,\ y_1 := t,\ y_2 := 1-t$, and taking into account that $k(1) = 1$, we obtain

$$tk(x) + (1-t)k(y) \leqslant k(tx + (1-t)y), \qquad x, y > 0;\ t \in (0,1).$$

Thus k is concave and, consequently, continuous. It follows that there is $p \geqslant 1$ such that (cf. [2], p. 311)

$$k_1(t) = k(t) = t^{1/p}, \qquad t > 0,$$

whence, by the first formula of (8),

$$\psi_1(t) = \psi_1(1)t^{1/p}, \qquad t > 0.$$

The concavity of k implies that $p \geqslant 1$. Since the roles of the functions ψ_1 and ψ_2 are symmetric, we conclude that $k_2(t) = t^{1/q}$ for some $q \geqslant 1$ whence, by the second formula of (8),

$$\psi_2(t) = \psi_2(1)t^{1/q}, \qquad t > 0.$$

From (9) we have

$$\frac{1}{p} + \frac{1}{q} = 1,$$

which implies that $p > 1$. Now from (10), (11), (12) and (16) we get

$$\varphi_1(t) = \varphi_1(1)t^p, \qquad \varphi_2(t) = \varphi_2(1)t^q, \qquad t > 0.$$

The equality $d = 1$ implies that

$$\psi_1(1)\psi_2(1)\left(\varphi_1(1)\right)^{1/p})\left(\varphi_2(1)\right)^{1/q} = 1.$$

This completes the proof of the implication $(i) \Rightarrow (ii)$. Since the converse implication is a consequence of Hölder's inequality, the proof is complete.

REMARK 2. Theorem 1 remains true if inequality (2) is assumed to hold for all $\mathbf{x}, \mathbf{y}$ from the two dimensional cone

$$\mathbf{S}_+(A,B) := \left\{x_1\chi_A + x_2\chi_{B\setminus A} \in \mathbf{S} : x_1, x_2 > 0\right\}.$$

COROLLARY 1. *Let* (Ω, Σ, μ) *be a measure space with* $A, B \in \Sigma$ *such that*

$$0 < \mu(A) < 1 < \mu(B) < \infty,$$

and let $\varphi, \psi : (0,\infty) \to (0,\infty)$ *be bijective functions.*

Then the following conditions are equivalent:

(i) *the functions* φ *and* ψ *satisfy the inequality*

$$\int_\Omega \mathbf{x}\mathbf{y}\,d\mu \leqslant \varphi^{-1}\left(\int_{\Omega(\mathbf{x})} \varphi \circ \mathbf{x}\,d\mu\right) \psi^{-1}\left(\int_{\Omega(\mathbf{y})} \psi \circ \mathbf{y}\,d\mu\right), \qquad \mathbf{x}, \mathbf{y} \in \mathbf{S}_+ \tag{17}$$

(ii) *there is a real* $p > 1$ *such that*

$$\frac{\varphi(t)}{\varphi(1)} = t^p, \quad \frac{\psi(t)}{\psi(1)} = t^{1/q}, \qquad t > 0,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. With $\varphi_1 := \varphi,\ \varphi_2 := \psi,\ \psi_1 := \varphi^{-1},\ \psi_2 := \psi^{-1}$ inequality (17) becomes (2). Since

$$\frac{\psi_1(t)}{\varphi_1^{-1}(t)} = \frac{\varphi^{-1}(t)}{\varphi^{-1}(t)} = 1, \qquad \frac{\varphi_2^{-1}(t)}{\psi_2(t)} = \frac{\psi^{-1}(t)}{\psi^{-1}(t)} = 1, \qquad t > 0,$$

condition (1) is satisfied with $c = 1$.

REMARK 3. This improves the main result of [5] where it is assumed that $\varphi(0) = \psi(0) = 0$.

4. The converse of the accompanying Hölder's inequality

THEOREM 2. *Let (Ω,Σ,μ) be a measure space such that there are two sets $A,B \in \Sigma$ satisfying the condition*

$$0 < \mu(A) < 1 < \mu(B) < \infty.$$

Suppose that $\varphi_1,\varphi_2,\psi_1,\psi_2 : (0,\infty) \to (0,\infty)$ are bijective functions such that

$$\frac{t}{\psi_2 \circ \varphi_2(t)} \leqslant c \leqslant \frac{\psi_1 \circ \varphi_1(t)}{t}, \qquad t > 0, \tag{18}$$

for some constant $c > 0$, and, for all $\alpha,\beta > 0$,

$$\lim_{t\to 0+} \psi_1(\alpha t)\psi_2(\beta t) = 0. \tag{19}$$

Then the following conditions are equivalent:

(i) the functions $\varphi_1,\varphi_2,\psi_1,\psi_2$ satisfy the inequality

$$\int_\Omega \mathbf{x}\mathbf{y}d\mu \geqslant \mathbb{P}_{\varphi_1,\psi_1}(\mathbf{x})\mathbb{P}_{\varphi_2,\psi_2}(\mathbf{y}) \text{ for all } \mathbf{x},\mathbf{y} \in \mathbf{S}_+ ; \tag{20}$$

(ii) there is a real $p < 1$ such that

$$\frac{\varphi_1(t)}{\varphi_1(1)} = t^p, \quad \frac{\psi_1(t)}{\psi_1(1)} = t^{1/p}, \quad \frac{\varphi_2(t)}{\varphi_2(1)} = t^q, \quad \frac{\psi_2(t)}{\psi_2(1)} = t^{1/q}, \qquad t > 0,$$

$$\psi_1(1)\psi_2(1)\,(\varphi_1(1))^{1/p}\,(\varphi_2(1))^{1/q} = 1,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. To show the implication $(i) \Rightarrow (ii)$ we put $a := \mu(A),\ b := \mu(B\backslash A)$. Setting

$$\mathbf{x} := x_1\chi_A + x_2\chi_{B\backslash A}, \qquad \mathbf{y} := y_1\chi_A + y_2\chi_{B\backslash A}, \qquad x_1,x_2,y_1,y_2 > 0,$$

in (20) we get, for all $x_1,x_2,y_1,y_2 > 0$,

$$ax_1y_1 + bx_2y_2 \geqslant \psi_1\left(a\varphi_1(x_1) + b\varphi_1(x_2)\right)\psi_2\left(a\varphi_2(y_1) + b\varphi_2(y_2)\right).$$

Replacing here x_i by $\varphi_1^{-1}(x_i)$, y_i by $\varphi_2^{-1}(y_i)$ for $i=1,2$, we obtain

$$a\varphi_1^{-1}(x_1)\varphi_2^{-1}(y_1)+b\varphi_1^{-1}(x_2)\varphi_2^{-1}(y_2) \geqslant \psi_1\left(ax_1+bx_2\right)\psi_2\left(ay_1+by_2\right)), \tag{21}$$

for all $x_1,x_2,y_1,y_2>0$. From (18) we have

$$\frac{\psi_1(s)}{\varphi_1^{-1}(s)} \geqslant \frac{\varphi_2^{-1}(t)}{\psi_2(t)}, \qquad s,t>0,$$

whence

$$\psi_1(s)\psi_2(t) \geqslant \varphi_1^{-1}(s)\varphi_2^{-1}(t), \qquad s,t>0. \tag{22}$$

Inequalities (21) and (22) imply that, for all $x_1,y_1,x_2,y_2>0$,

$$a\psi_1(x_1)\psi_2(y_1)+b\psi_1(x_2)\psi_2(y_2) \geqslant \psi_1\left(ax_1+bx_2\right)\psi_2\left(ay_1+by_2\right)), \tag{23}$$

which proves that the function $F:(0,\infty)^2\to(0,\infty)$ defined by

$$F(x,y)=\psi_1(x)\psi_2(y), \qquad x,y>0,$$

satisfies the inequality

$$F(ax_1+bx_2,ay_1+by_2) \leqslant aF(x_1,y_1)+bF(x_2,y_2), \quad x_1,x_2,y_1,y_2>0. \tag{24}$$

Taking into account (19) and Lemma 2, we infer that F is homogeneous, i.e.

$$\psi_1(tx)\psi_2(ty)=t\psi_1(x)\psi_2(y), \qquad t,x,y>0.$$

Now, repeating the relevant part of the proof of Theorem 1, we can show that there are multiplicative functions $k_1,k_2:(0,\infty)\to(0,\infty)$ such that

$$\psi_1(t)=\psi_1(1)k_1(t), \qquad \psi_2(t)=\psi_2(1)k_2(t) \qquad t>0, \tag{25}$$

and

$$k_1(t)k_2(t)=t, \qquad t>0. \tag{26}$$

Inequalities (21) and (22) imply that the function $F:(0,\infty)^2\to(0,\infty)$ defined by

$$F(x,y)=\varphi_1^{-1}(x)\varphi_2^{-1}(y), \qquad x,y>0,$$

satisfies inequality (24). Inequality (22) and condition (19) imply that

$$\lim_{t\to 0}F(tx,ty)=0, \qquad x,y>0.$$

By Remark 1, the function F is positively homogeneous. This allows us to conclude that there are the multiplicative functions $l_1,l_2:(0,\infty)\to(0,\infty)$ such that

$$\varphi_1^{-1}(t)=\varphi_1^{-1}(1)l_1(t), \qquad \varphi_2^{-1}(t)=\varphi_2^{-1}(1)l_2(t), \qquad t>0, \tag{27}$$

and

$$l_1(t)l_2(t) = t, \qquad t > 0. \tag{28}$$

Setting

$$k := k_1, \qquad l := l_1 \tag{29}$$

and arguing in the same way as in the proof of Theorem 1, we can show that

$$l(t) = k(t), \qquad t > 0, \tag{30}$$

and

$$tk(x) + (1-t)k(y) \geqslant dk(tx + (1-t)y), \qquad x,y > 0;\ t \in (0,1),$$

where

$$d := \frac{\psi_1(1)\psi_2(1)}{\varphi_1^{-1}(1)\varphi_2^{-1}(1)} = 1.$$

This implies that k is convex and, consequently, for some $p \leqslant 1$,

$$k_1(t) = k(t) = t^{1/p}, \qquad t > 0.$$

Similarly we have

$$k_2(t) = t^{1/q}, \qquad t > 0.$$

for some $q \leqslant 1$. In view of (26),

$$\frac{1}{p} + \frac{1}{q} = 1,$$

which implies that $p < 1$. Now, from (25),

$$\psi_1(t) = \psi_1(1)t^{1/p}, \qquad \psi_2(t) = \psi_2(1)t^{1/q} \qquad t > 0.$$

From (27), (28), (29) and (30) we get

$$\varphi_1(t) = \varphi_1(1)t^p, \qquad \varphi_2(t) = \varphi_2(1)t^q, \qquad t > 0.$$

The equality $d = 1$ implies that

$$\psi_1(1)\psi_2(1)\,(\varphi_1(1))^{1/p})\,(\varphi_2(1))^{1/q} = 1.$$

This completes the proof of the implication $(i) \Rightarrow (ii)$. The converse implication is a consequence of the accompanying Hölder's inequality.

5. Remarks on the basic assumptions

REMARK 4. To see that, in Theorem 1, condition (1) is essential, note that the functions $\varphi_1(t) = t^p$, $\psi_1(t) = t^{1/p}$, $\varphi_2(t) = t^q$ and a non-power function $\psi_2(t) = t^{1/q} + t$, where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, satisfy inequality (2) and, as

$$\frac{\psi_1 \circ \varphi_1(t)}{t} = 1 > \frac{t}{\psi_2 \circ \varphi_2(t)} = \frac{1}{1 + t^{q-1}}, \qquad t > 0,$$

condition (1) is not satisfied.

Moreover, taking $p=q=1$ and (Ω,Σ,μ) such that $\Omega=\{1,2\}$, $\Sigma=2^{\Omega}$, and μ such that $\mu(\{0\})=\frac{1}{2}$, $\mu(\{1\})=2$, it easy to verify that inequality (2). This shows that conditions (1) and the basic assumption on the measure space are independent.

The next two remarks show that the assumption of the underlying measure space is indispensable. They can be treated as some generalizations of the Hölder inequality.

REMARK 5. Suppose that (Ω,Σ,μ) is a measure space such that $\mu(\Omega)\leqslant 1$ with at least one set $A\in\Sigma$ such that $0<\mu(A)<1$. Let $\alpha,\beta:(0,\infty)\to(0,\infty)$ be bijective increasing and such that the function of two variables

$$(s,t)\to\alpha^{-1}(s)\beta^{-1}(t)$$

is concave in $(0,\infty)^2$ (cf. inequality (6) for $b=1-a$). Then (cf. [1], [5]),

$$\int_{\Omega}\mathbf{x}\mathbf{y}d\mu\leqslant\mathbb{P}_{\alpha,\alpha^{-1}}(\mathbf{x})\mathbb{P}_{\beta,\beta^{-1}}(\mathbf{y}),\qquad \mathbf{x},\mathbf{y}\in\mathbf{S}(\Omega,\Sigma,\mu). \tag{31}$$

For arbitrary bijective increasing functions $\varphi_1,\ \varphi_2,\ \psi_1,\ \psi_2:(0,\infty)\to(0,\infty)$ such that

$$\psi_1^{-1}\leqslant\alpha\leqslant\varphi_1,\qquad \psi_2^{-1}\leqslant\beta\leqslant\varphi_2, \tag{32}$$

we, obviously, have

$$\mathbb{P}_{\alpha,\alpha^{-1}}(\mathbf{x})\leqslant\mathbb{P}_{\varphi_1,\psi_1}(\mathbf{x}),\qquad \mathbb{P}_{\beta,\beta^{-1}}(\mathbf{y})\leqslant\mathbb{P}_{\varphi_2,\psi_2}(\mathbf{y}),\qquad \mathbf{x},\mathbf{y}\in\mathbf{S}(\Omega,\Sigma,\mu),$$

and, consequently, a generalized Hölder inequality (2) is satisfied.

REMARK 6. Suppose that (Ω,Σ,μ) is a measure space such that for every $A\in\Sigma$, either $\mu(A)=0$ or $\mu(A)\geqslant 1$. Let $\alpha,\beta:(0,\infty)\to(0,\infty)$ be increasing bijective functions and $p,q>0$, $p^{-1}+q^{-1}=1$, such that the functions

$$t\to\frac{\alpha(t)}{t^p},\qquad t\to\frac{\beta(t)}{t^q}$$

are nonincreasing. Then (cf. [5], Theorem 6) inequality (31) is satisfied. Let $\varphi_1,\ \varphi_2,\ \psi_1,\ \psi_2:(0,\infty)\to(0,\infty)$ be arbitrary increasing bijections satisfying conditions (32). Now, similarly as in the previous remark, we infer that (2) holds true.

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