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# THE SHARPENING OF SOME INEQUALITIES VIA ABSTRACT CONVEXITY

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*Abstract.* One of the application areas of abstract convexity is inequality theory. In this work, the authors seek to derive new inequalities by sharpening well-known inequalities by the use of abstract convexity. Cauchy-Schwarz inequality, Minkowski inequality and well-known mean inequalities are studied in this sense, concrete results are obtained for some of them.

## 1. Introduction

The applications of abstract convexity are seen in many different areas. (See [3], [4], [5], [6], [7], [8], [9], [10]). One of them is the application to the inequality theory. For instance, for different function classes, Hermite-Hadamard type inequalities have been derived by the several authors [3, 4, 5, 9]. Another use of abstract convexity in inequality theory is to sharpen known inequalities [8].

In this paper, some famous inequalities such as weighted harmonic-geometric-aritmethic mean, Cauchy-Schwarz and Minkowski inequalities are studied and investigated in the frame of abstract convexity.

The structure of the paper is as follows: In the first part of the second section, the basic concepts of abstract convexity and an important theorem related to optimization theory are given. In the second part of the second section $M_t(x,\alpha)$ mean is introduced and its properties are given. In the third section, the inequalities are considered separately and investigated, the results are presented as theorems. In the fourth section, the results are summarized.

We shall use the following notations:

$R$ is the real line; $R_{+\infty} := R \cup \{+\infty\}$; $R_{-\infty} := R \cup \{-\infty\}$; $\bar{R} := R \cup \{-\infty, +\infty\}$;

$R^n$ is a $n$-dimensional Euclidean Space;

$R^n_+$ is the set of points with nonnegative coordinates;

$R^n_{++}$ is the set of points with strictly positive coordinates;

$X$ is a Hilbert space with the inner product $[.,.]$ and the norm $\|x\| = \sqrt{[x,x]}$;

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$B(y,r) = \{x \in X : \|x-y\| \leqslant r\}$;
If $f : \Omega \to \underline{R}$, then $\mathrm{dom} f := \{x \in \Omega : -\infty < f(x) < +\infty\}$;
If $f : \Omega \to R$ and $g : \Omega \to R$, then $f \leqslant g$ means that $f(x) \leqslant g(x)$ for all $x \in \Omega$.

## 2. Preliminaries

### 2.1. Abstract convexity, abstract concavity and an application to the optimization theory

Let $\Omega$ be a set and $H$ be a set of functions $h : \Omega \to R_{-\infty}$. A function $f : \Omega \to R_{+\infty}$ is called abstract convex with respect to $H$ (or $H$-*convex*) if there exists a set $U \subset H$ such that

$$f(x) = \sup_{h \in U} h(x)$$

for all $x \in \Omega$.

Let $H$ be a set of functions $h : \Omega \to R_{+\infty}$. A function $f : \Omega \to R_{-\infty}$ is called abstract concave with respect to $H$ (or $H$-*concave*) if there exists a set $U \subset H$ such that

$$f(x) = \inf_{h \in U} h(x)$$

for all $x \in \Omega$.

The set $H$ is called the set of elementary functions.

Let $X$ be Hilbert space, let $\Omega \subset \Omega' \subset X$, $f : \Omega \to R_{+\infty}$ and $x_0 \in \mathrm{dom} f$ and $L$ be a set of functions $l : \Omega' \to R_{-\infty}$. An element $l \in L$ is called an $L$-subgradient of $f$ at the point $x_0$ if $x_0 \in \mathrm{dom}\, l$ and

$$f(x) \geqslant f(x_0) + l(x) - l(x_0)$$

The set $\partial_L f(x_0)$ of all $L$-subgradient of $f$ at $x_0$ is referred to as $L$-subdifferential of $f$ at $x_0$.

$f : \Omega \to R_{+\infty}$ is a lower semicontinuous convex function and $x \in \mathrm{dom} f$, then $\partial_L f(x) = \partial f(x)$, where $\partial f(x)$ is the subdifferential in the sense of convex analysis.

Let $H$ be the set of all quadratic functions $h$ of the form

$$h(x) = a\|x\|^2 + [l,x] + c, \quad x \in X$$

where $a > 0$, $l \in X$ and $c \in R$. We say that a function $f : \Omega \to R_{-\infty}$ is majorized by $H$ if there exists $h \in H$ such that $h \geqslant f$.

Let $\Omega \subset X$ and let $H$ be the set of quadratic functions. Then a function $f : \Omega \to R_{-\infty}$ is $H-$*concave* if and only if $f$ is majorized by $H$ and $f$ is upper semicontinuous (see [7]).

The following result holds (see [8]).

PROPOSITION 1. *Let $\Omega \subset X$ be a convex set and let $f$ be a differentiable function defined on an open set containing $\Omega$. Assume that the mapping $x \mapsto \nabla f(x)$ is Lipschitz continuous on $\Omega$:*

$$K := \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{\|\nabla f(x) - \nabla f(y)\|}{\|x-y\|} < +\infty$$

*Let* $a \geqslant K$*. For each* $t \in \Omega$ *consider the function*

$$f_t(x) = f(t) + [\nabla f, x - t] + a\|x - t\|^2, \quad x \in X.$$

*Then* $f(x) = \min_{t\in\Omega} f_t(x), \; x \in \Omega.$

In [8], the global minimization of a function $f$ over a convex set that can be represented as the infimum of a family $(f_t)_{t\in T}$ of convex functions is considered and some necessary and sufficient ( or only sufficient ) conditions for the global minimum has been obtained.

In the simplest case of the unconstrained minimization of a function $f : X \to R$ such that $\|\nabla f(x) - \nabla f(y)\| \leqslant a\|x - y\|$ for all $x, y \in X$, the following result is obtained: If a point $x^*$ is a global minimizer of $f$ over $X$, then

$$f(x) - f(x^*) \geqslant \frac{1}{4a}\|\nabla f(x)\|^2 \tag{1}$$

for all $x \in X$.

The following theorem which gives the more general case of the inequality (1) is proved in [8].

THEOREM 1. *Consider an* $n$ *dimensional space* $\mathbb{R}^n$ *with norms* $\|.\|$ *and* $\|.\|_\circ$*. Let* $\Omega \subset \mathbb{R}^n$ *be a set with int* $\Omega \neq \emptyset$ *and let* $f \in C^1(\Omega)$*. Assume that the mapping* $x \longmapsto \nabla f(x)$ *is Lipschitz on* $\Omega$ *:*

$$K := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{\|\nabla f(x) - \nabla f(y)\|}{\|x - y\|} < \infty$$

*Let* $x^* \in int\ \Omega$ *be a global minimizer of* $f$ *over* $\Omega$*. Consider the ball*

$$B_\circ(x^*, r) = \{x : \|x - x^*\|_\circ \leqslant r\} \subset int\Omega$$

*and let*

$$M := \max\left\{\|\nabla f(x)\|_\circ : x \in B_\circ(x^*, r)\right\}.$$

*Let* $q > 0$ *be a number such that* $B_\circ(x^*, r + q) \subset \Omega$ *and let* $a \geqslant \max\left(K, \frac{M}{2q}\right)$*. Then*

$$\frac{1}{4a}\|\nabla f(x)\|^2 \leqslant f(x) - f(x^*), \; x \in B_\circ(x^*, r).$$

## 2.2. $\mathbf{M}_t(\mathbf{x}, \alpha)$ mean

Well-known means (arithmetic mean, geometric mean,harmonic mean etc.) are the rings of a certain mean chain and the relation among them is determined by their places in this chain.

Let $(x) \equiv (x_1, x_2, ..., x_n)$ and $(\alpha) \equiv (\alpha_1, \alpha_2, ..., \alpha_n)$ be positive numbers, such that $\sum_{i=1}^{n} \alpha_i = 1$, and $t \neq 0$ be real number. The following expression is called the $t^{th}$ order mean of $(x_1, x_2, ..., x_n)$ with the weights $(\alpha_1, \alpha_2, ..., \alpha_n)$ :

$$M_t(x, \alpha) = \left(\sum_{i=1}^{n} \alpha_i x_i^t\right)^{\frac{1}{t}}.$$

Especially, if $t$ is chosen to be equal to $-1,\ 1$ and 2 respectively, $M_t(x,\alpha)$ gives the weighted harmonic, the weighted arithmetic and the weighted quadratic means, respectively:

$$\frac{1}{\sum\limits_{i=1}^{n}\frac{\alpha_i}{x_i}}, \qquad \sum_{i=1}^{n}\alpha_i x_i, \qquad \left(\sum_{i=1}^{n}\alpha_i x_i^2\right)^{\frac{1}{2}}$$

By applying L' Hospital law, it is derived that

$$M_0(x,\alpha) \equiv \lim_{t\to 0} M_t(x,\alpha) = \prod_{i=1}^{n} x_i^{\alpha_i},$$

that is, it can be easily shown that $0^{th}$ order mean corresponds to the weighted geometric mean.

Also let's note the following important situations:

$$\begin{aligned} M_{+\infty}(x,\alpha) &\equiv \lim_{t\to+\infty} M_t(x,\alpha) = \max\{x_1,x_2,...,x_n\} \\ M_{-\infty}(x,\alpha) &\equiv \lim_{t\to-\infty} M_t(x,\alpha) = \min\{x_1,x_2,...,x_n\} \end{aligned}$$

REMARK 1. In the case that for some $i\in\{1,2,..,n\}$ $x_i$ is zero, $M_t(x,\alpha)$ is accepted to be equal to zero for $t\leqslant 0$.

For given positive numbers $(x)$ and given weights $(\alpha)$, the mean $M_t(x,\alpha)$ is an increasing function of $t$ in $\bar{R}$. If all $x_i$ $(i\in\overline{1,n})$ does not take the same value, then it is a strictly increasing function of $t$ (see [2]). Thus, for arbitrary positive $(x)$ and $(\alpha)$, when $t_1\leqslant t_2$, the inequality $M_{t_1}(x,\alpha)\leqslant M_{t_2}(x,\alpha)$ holds. So the relation between the different means is determined by making use of this property.

## 3. Main results

Many inequalities can be represented in the form $f(x)\geqslant 0$, where $f$ is a certain function. We say that the inequality $f(x)\geqslant u(x)$ with $u(x)\geqslant 0$ is sharper than the inequality $f(x)\geqslant 0$ if there exists $x$ with $u(x)>0$.

Some certain conditions for global minimum can be used for sharpening some special inequalities. Using the optimality conditions which are obtained via abstract convexity in the previous section, we will study some well-known inequalities in terms of sharpening.

### 3.1. The sharpening of the weighted arithmetic-geometric mean inequality

It is known from the previous section that $M_t(x,\alpha)$ mean is an increasing function with respect to $t$, moreover, if all $x_i$ values do not take the same value, then $M_t(x,\alpha)$ is a strictly increasing function, i.e., when $t_1<t_2$, $M_{t_1}(x,\alpha)<M_{t_2}(x,\alpha)$ holds. In particular, if $t_1=0$ and $t_2=1$, then $M_0(x,\alpha)<M_1(x,\alpha)$ is obtained, that is, the

famous inequality between the weighted arithmetic mean and the weighted geometric mean is derived:

$$\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n > x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}$$

where $x \in R_+^n$, $x \neq \lambda \mathbf{1}$ with $\lambda \geqslant 0$, $\alpha_i \geqslant 0$, $\forall i \in \overline{1,n}$, $\sum_{i=1}^{n} \alpha_i = 1$ and $\mathbf{1} = (1,1,...,1) \in R_+^n$.

By making use of Theorem1, this inequality will be sharpened.

THEOREM 2. *Let $\lambda > r$ be positive numbers. Let*

$$a_{\lambda,r} = \min_{r<d<\lambda} \max \left\{ (m^2 + m - 2p)^{\frac{1}{2}} \frac{\lambda + d}{(\lambda - d)^2}, \frac{M_0}{2(d-r)} \right\}$$

*where*

$$M_0 = \max_{1 \leqslant i \leqslant n} \left\{ \alpha_i \left[ \left( \frac{\lambda + r}{\lambda - r} \right)^{1-\alpha_i} - 1 \right] \right\}, \quad m = \sum_{i=1}^{n} \alpha_i^2, \quad p = \sum_{i=1}^{n} \alpha_i^3.$$

*Then for all $x \in \mathbb{R}_+^n$ such that $\|x - \lambda \mathbf{1}\|_\infty \leqslant r$ the following inequality holds:*

$$\sum_{i=1}^{n} \alpha_i x_i \geqslant \prod_{i=1}^{n} x_i^{\alpha_i} + \frac{1}{4a_{\lambda,r}} \sum_{i=1}^{n} \alpha_i^2 \left( 1 - \frac{\prod_{j=1}^{n} x_j^{\alpha_j}}{x_i} \right)^2$$

*where* $\sum_{i=1}^{n} \alpha_i = 1$, $\alpha_i \geqslant 0$.

*Proof.* Let

$$f(x) = \sum_{i=1}^{n} \alpha_i x_i - \prod_{i=1}^{n} x_i^{\alpha_i}$$

where $\sum_{i=1}^{n} \alpha_i = 1$, $\alpha_i \geqslant 0$, $x = (x_1, x_2, ..., x_n) \in \mathbb{R}_+^n$. Then $f(x) \geqslant 0$ and $f(x) = 0$ if and only if $x = \lambda \mathbf{1}$, where $\lambda > 0$ and $\mathbf{1} = (1,1,...,1)$. So the vectors $\lambda \mathbf{1}$ are the global minimizers of $f$ over $\mathbb{R}_+^n$. We will sharpen the weighted geometric-arithmetic mean inequality by applying Theorem1 to the inequality $f(x) \geqslant 0$. It is easily calculated that

$$\nabla f(x) = \left[ \alpha_1 \left( 1 - \frac{\prod_{j=1}^{n} x_j^{\alpha_j}}{x_1} \right), \alpha_2 \left( 1 - \frac{\prod_{j=1}^{n} x_j^{\alpha_j}}{x_2} \right), ..., \alpha_n \left( 1 - \frac{\prod_{i=1}^{n} x_j^{\alpha_j}}{x_n} \right) \right]$$

Hence

$$\|\nabla f(x)\|^2 = \sum_{i=1}^{n} \alpha_i^2 \left( 1 - \frac{\prod_{j=1}^{n} x_j^{\alpha_j}}{x_i} \right)^2$$

Later we will use not only the norm $\|.\| = \|.\|_2$ but also the norm $\|.\|_\infty$. For $\lambda > d > 0$ consider the ball

$$\begin{aligned} V_{\lambda,d} &= B_\infty(\lambda\mathbf{1},d) = \{x \in \mathbb{R}^n : \|\lambda\mathbf{1}-x\|_\infty \leqslant d\} \\ &= \{x \in \mathbb{R}^n : \lambda - d \leqslant x_i \leqslant \lambda + d, i = 1, ..., n\}. \end{aligned}$$

Since $d < \lambda$ it follows that $V_{\lambda,d} \subset \mathbb{R}^n_{++}$. Let $\rho_i(x) = \frac{\prod_{j=1}^n x_j^{\alpha_j}}{x_i}$. We need to estimate $\|\nabla\rho_i(x)\|$ for $x \in V_{\lambda,d}$. For this reason we observe the following inequalities:

$$\left|\frac{\partial\rho_i}{\partial x_i}(x)\right| = \left|(\alpha_i - 1)\frac{\prod_{j=1}^n x_j^{\alpha_j}}{x_i^2}\right| \leqslant (1-\alpha_i)\frac{\lambda+d}{(\lambda-d)^2}$$

$$\left|\frac{\partial\rho_i}{\partial x_j}(x)\right| = \left|\alpha_j\frac{\prod_{k=1}^n x_k^{\alpha_k}}{x_i x_j}\right| \leqslant \alpha_j\frac{\lambda+d}{(\lambda-d)^2}$$

and consequently

$$\begin{aligned} \|\nabla\rho_i(x)\| &\leqslant \left((1-\alpha_i)^2\left(\frac{\lambda+d}{(\lambda-d)^2}\right)^2 + \sum_{\substack{j=1 \\ j\neq i}}^n \alpha_j^2\left(\frac{\lambda+d}{(\lambda-d)^2}\right)^2\right)^{\frac{1}{2}} \\ &\leqslant \left(1 - 2\alpha_i + \sum_{i=1}^n \alpha_i^2\right)^{\frac{1}{2}} \frac{\lambda+d}{(\lambda-d)^2} \quad (x \in V_{\lambda,d}). \qquad (2) \end{aligned}$$

Now let $x, y \in V_{\lambda,d}$. Applying the mean value theorem and Cauchy-Schwarz inequality, we conclude that there exist numbers $\theta_i \in (0,1)$, $i = 1, ..., n$ such that

$$\begin{aligned} \|\nabla f(x) - \nabla f(y)\| &= \|\alpha_1[\rho_1(y) - \rho_1(x)], \alpha_2[\rho_2(y) - \rho_2(x)], ..., \alpha_n[\rho_n(y) - \rho_n(x)]\| \\ &= \left(\sum_{i=1}^n \alpha_i^2[\rho_i(y) - \rho_i(x)]^2\right)^{\frac{1}{2}} \\ &= \left(\sum_{i=1}^n \alpha_i^2[\nabla\rho_i(x+\theta_i(y-x))(x-y)]^2\right)^{\frac{1}{2}} \\ &\leqslant \left(\sum_{i=1}^n \alpha_i^2\|\nabla\rho_i(x+\theta_i(y-x))\|^2\|x-y\|^2\right)^{\frac{1}{2}} \\ &= \left[\sum_{i=1}^n \alpha_i^2\|\nabla\rho_i(x+\theta_i(y-x))\|^2\right]^{\frac{1}{2}}\|x-y\| \end{aligned}$$

Since $x, y \in V_{\lambda,d}$ it follows that $x+\theta_i(y-x) \in V_{\lambda,d}$ for all $i$. Applying the inequality (2), we conclude that

$$\|\nabla f(x) - \nabla f(y)\| \leqslant a_1(\lambda, d)\, \|x-y\|, \ \ x, y \in V_{\lambda,d}$$

where

$$a_1(\lambda, d) = \left[\sum_{i=1}^{n} \alpha_i^2 \left(1 - 2\alpha_i + \sum_{i=1}^{n} \alpha_i^2\right)\right]^{\frac{1}{2}} \frac{\lambda+d}{(\lambda-d)^2} = \left(m^2 - 2p + m\right)^{\frac{1}{2}} \frac{\lambda+d}{(\lambda-d)^2}$$

with

$$m = \sum_{i=1}^{n} \alpha_i^2, \qquad p = \sum_{i=1}^{n} \alpha_i^3$$

Hence mapping $x \to \nabla f(x)$ is Lipschitz continuous on $V_{\lambda,d}$ with the Lipschitz constant $K \leqslant a_1(\lambda, d)$. We will apply Theorem1 to a set $\Omega = V_{\lambda,d}$ where $d < \lambda$ and the global minimizer $x^* = \lambda\mathbf{1}$ of the function $f$. Assume that the norm $\|.\|_\circ$ that was used in Theorem1 coincides with $\|.\|_\infty$. Let $r \in (0, d)$ and $q = d - r$. Let's estimate $M = \max\{\|\nabla f(x)\|_\infty : x \in V_{\lambda,r}\}$ as follows:

$$\begin{aligned}
M &= \max_{x\in V_{\lambda,r}} \{\|\nabla f(x)\|_\infty\} = \max_{x\in V_{\lambda,r}} \left\{\max_{1\leqslant i\leqslant n} \left|\alpha_i\left(1 - \frac{x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}}{x_i}\right)\right|\right\} \\
&= \max_{1\leqslant i\leqslant n} \left\{\max_{x\in V_{\lambda,r}} \alpha_i \left|1 - \frac{x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}}{x_i}\right|\right\} \\
&= \max_{1\leqslant i\leqslant n} \left\{\max_{x\in V_{\lambda,r}} \alpha_i \left|1 - \left(\frac{x_1}{x_i}\right)^{\alpha_1} \ldots \left(\frac{x_{i-1}}{x_i}\right)^{\alpha_{i-1}} \left(\frac{x_{i+1}}{x_i}\right)^{\alpha_{i+1}} \ldots \left(\frac{x_n}{x_i}\right)^{\alpha_n}\right|\right\} \\
&\leqslant \max_{1\leqslant i\leqslant n} \alpha_i \left\{\max\left\{1 - \left(\frac{\lambda-r}{\lambda+r}\right)^{1-\alpha_i}, \left(\frac{\lambda+r}{\lambda-r}\right)^{1-\alpha_i} - 1\right\}\right\}
\end{aligned}$$

It can be easily shown that the following inequality holds for given $\alpha_i$ $(i \in \{1, 2, ..., n\})$ :

$$\left(\frac{\lambda+r}{\lambda-r}\right)^{1-\alpha_i} - 1 \geqslant 1 - \left(\frac{\lambda-r}{\lambda+r}\right)^{1-\alpha_i}.$$

So

$$\max\left\{1 - \left(\frac{\lambda-r}{\lambda+r}\right)^{1-\alpha_i}, \left(\frac{\lambda+r}{\lambda-r}\right)^{1-\alpha_i} - 1\right\} = \left(\frac{\lambda+r}{\lambda-r}\right)^{1-\alpha_i} - 1.$$

Hence

$$M \leqslant \max_{1\leqslant i\leqslant n} \left\{\alpha_i \left[\left(\frac{\lambda+r}{\lambda-r}\right)^{1-\alpha_i} - 1\right]\right\} \equiv M_0.$$

Let

$$a_2(\lambda, d, r) = \frac{M_0}{2(d-r)}$$

and

$$a(\lambda,d,r) = \max\{a_1(\lambda,d), a_2(\lambda,d,r)\} = \max\left\{\left(m^2+m-2p\right)^{\frac{1}{2}}\frac{\lambda+d}{(\lambda-d)^2}, \frac{M_0}{2(d-r)}\right\}$$

where

$$M_0 = \max_{1\leqslant i\leqslant n}\left\{\alpha_i\left[\left(\frac{\lambda+r}{\lambda-r}\right)^{1-\alpha_i}-1\right]\right\},\ m=\sum_{i=1}^n \alpha_i^2,\ p=\sum_{i=1}^n \alpha_i^3.$$

Note that $\lim\limits_{d\to\lambda^-} a(\lambda,d,r) = \lim\limits_{d\to r^+} a(\lambda,d,r) = +\infty$ so the function $d \longmapsto a(\lambda,d,r)$ attends its minimum on the segment $(r,\lambda)$. Let $a_{\lambda,r} = \min\limits_{r<d<\lambda} a(\lambda,d,r)$. Applying Theorem1 we conclude that

$$\sum_{i=1}^n \alpha_i x_i \geqslant \prod_{i=1}^n x_n^{\alpha_j} + \frac{1}{4a_{\lambda,r}}\sum_{i=1}^n \alpha_i^2\left(1-\frac{\prod\limits_{j=1}^n x_j^{\alpha_j}}{x_i}\right)^2 \quad \text{for } x\in V_{\lambda,r.}$$

□

REMARK 2. Although $a_{\lambda,r}$ in the Theorem2 is a complicated expression, $a_{\lambda,r}$ can be calculated easily for all $\lambda,r$. Indeed, $a_1(\lambda,d)$ has finite value on the point $d=r$ and goes to infinity when $d$ goes to $\lambda$. Also, it can be shown that $a_1(\lambda,d)$ is monotone increasing function of $d$ on the interval $(r,\lambda)$. On the other hand, $a_2(\lambda,d,r)$ has finite value on the point $d=\lambda$ and goes to infinity when $d$ goes to $r$. Also, it can be shown that $a_2(\lambda,d,r)$ is monotone decreasing function of $d$ on $(r,\lambda)$. So $a(\lambda,d,r)$ attains the minimum value on the point which satisfies the inequality $a_1(\lambda,d) = a_2(\lambda,d,r)$.

For equal weights, i.e., $\alpha_i = \frac{1}{n},\ i=1,2,...,n,$ the result is as follows:

COROLLARY 1. *Let $\lambda > r$ be positive numbers. Let*

$$a_{\lambda,r} = \min_{r<d<\lambda}\max\left\{\frac{(n-1)^{\frac{1}{2}}}{n}\frac{\lambda+d}{(\lambda-d)^2}, \frac{\left(\frac{\lambda+r}{\lambda-r}\right)^{\frac{n-1}{n}}-1}{2(d-r)n}\right\}$$

*Then for all $x\in\mathbb{R}_+^n$ such that $\|x-\lambda\mathbf{1}\|_\infty \leqslant r$ the following inequality holds:*

$$\frac{1}{n}\sum_{i=1}^n x_i \geqslant \left(\prod_{i=1}^n x_i\right)^{\frac{1}{n}} + \frac{1}{4a_{\lambda,r}n^2}\sum_{i=1}^n\left(1-\frac{\left(\prod\limits_{j=1}^n x_j\right)^{\frac{1}{n}}}{x_i}\right)^2.$$

This special case is derived in [8].

REMARK 3. In [1], the inequality between the arithmetic mean and the geometric mean has been analyzed from a different viewpoint. A relation between both means, whose validity has been evaluated statistically, has been derived .

### 3.2. The sharpening of the weighted harmonic-geometric mean inequality

In section 2.2, if we take $M_t(x,\alpha)$ into account with $t=-1$ and $t=0$, the weighted harmonic-geometric mean inequality is derived:

$$x_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n} > \frac{1}{\frac{\alpha_1}{x_1}+\frac{\alpha_2}{x_2}+...+\frac{\alpha_n}{x_n}}$$

where $x \in R^n_{++}$, $x \neq \lambda \mathbf{1}$ with $\lambda \geqslant 0$, $\alpha_i \geqslant 0$, $\forall i \in \overline{1,n}$, $\sum\limits_{i=1}^{n} \alpha_i = 1$ and $\mathbf{1} = (1,1,...,1) \in R^n_{++}$.

By using the similar way in the Theorem2, the following theorem which sharpens the inequality above is proved.

THEOREM 3. *Let $\lambda > r$ be positive numbers. Let*

$$a_{\lambda,r} = \min_{r<d<\lambda} \max \left\{ \frac{(\lambda+d)}{(\lambda-d)^2} \left[ \sum_{i=1}^{n} \left[ \left(1-3\alpha_i+2\frac{\lambda+d}{\lambda-d}\right)^2 + \left(m-\alpha_i^2\right)\left(1+\frac{2(\lambda+d)^2}{(\lambda-d)^2}\right)^2 \right] \alpha_i^2 \right]^{\frac{1}{2}}, \frac{M_0}{2(d-r)} \right\}$$

*where*

$$m = \sum_{i=1}^{n} \alpha_i^2, \quad M_0 = \max_{1\leqslant i\leqslant n} \left\{ \alpha_i \left[ \left(\frac{\lambda+r}{\lambda-r}\right)^2 - \left(\frac{\lambda-r}{\lambda+r}\right)^{1-\alpha_i} \right] \right\}$$

*Then for all $x \in \mathbb{R}^n_{++}$ such that $\|x-\lambda\mathbf{1}\|_\infty \leqslant r$ the following inequality holds:*

$$\prod_{i=1}^{n} x_i^{\alpha_i} \geqslant \frac{1}{\sum\limits_{i=1}^{n}\frac{\alpha_i}{x_i}} + \frac{1}{4\alpha_{\lambda,r}} \sum_{i=1}^{n} \left[ \alpha_i \frac{\prod\limits_{j=1}^{n} x_j^{\alpha_j}}{x_i} - \frac{\alpha_i}{x_i^2} \frac{1}{\left(\sum\limits_{j=1}^{n}\frac{\alpha_j}{x_j}\right)^2} \right]^2 \quad \textit{for } x \in V_{\lambda,r}.$$

*Proof.* Let

$$f(x) = \prod_{j=1}^{n} x_j^{\alpha_j} - \frac{1}{\sum\limits_{j=1}^{n}\frac{\alpha_j}{x_j}}$$

where $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n_{++}$.

It is obvious that $x = \lambda\mathbf{1}$ with $\lambda > 0$ and $\mathbf{1} = (1,1,...,1)$ are the global minimum points of $f(x)$. To sharpen the weighted geometric-arithmetic mean inequality via

Theorem1, the following calculations are pursued:

$$\nabla f(x) = \left[ \alpha_1 \frac{\prod_{j=1}^{n} x_j^{\alpha_j}}{x_1} - \frac{\alpha_1}{x_1^2} \frac{1}{\left( \sum_{j=1}^{n} \frac{\alpha_j}{x_j} \right)^2}, \ldots, \alpha_n \frac{\prod_{j=1}^{n} x_j^{\alpha_j}}{x_n} - \frac{\alpha_n}{x_n^2} \frac{1}{\left( \sum_{j=1}^{n} \frac{\alpha_j}{x_j} \right)^2} \right]$$

Hence

$$\|\nabla f(x)\|^2 = \sum_{i=1}^{n} \left[ \alpha_i \frac{\prod_{j=1}^{n} x_j^{\alpha_j}}{x_i} - \frac{\alpha_i}{x_i^2} \frac{1}{\left( \sum_{j=1}^{n} \frac{\alpha_j}{x_j} \right)^2} \right]^2 .$$

We take $\|.\| = \|.\|_2$, $\|.\|_\circ = \|.\|_\infty$. Let's define $V_{\lambda,d}$ with $\lambda > d > 0$ as follows:

$$\begin{aligned} V_{\lambda,d} &= B_\infty(\lambda \mathbf{1}, d) = \{x \in \mathbb{R}^n : \|\lambda \mathbf{1} - x\|_\infty \leqslant d\} \\ &= \{x \in \mathbb{R}^n : \lambda - d \leqslant x_i \leqslant \lambda + d, i = 1, ..., n\}. \end{aligned}$$

It is clear that $V_{\lambda,d} \subset \mathbb{R}^n_{++}$. Let $\rho_i(x) = \frac{\prod_{j=1}^{n} x_j^{\alpha_j}}{x_i} - \frac{1}{x_i^2} \frac{1}{\left( \sum_{j=1}^{n} \frac{\alpha_j}{x_j} \right)^2}$. The estimation of $\|\nabla \rho_i(x)\|$ on $V_{\lambda,d}$ is required, so

$$\begin{aligned} \frac{\partial \rho_i}{\partial x_i}(x) &= (\alpha_i - 1) \frac{\prod_{j=1}^{n} x_j^{\alpha_j}}{x_i^2} + \frac{2}{x_i^3} \frac{1}{\left( \sum_{j=1}^{n} \frac{\alpha_j}{x_j} \right)^2} \left[ 1 - \frac{\alpha_i}{x_i \sum_{j=1}^{n} \frac{\alpha_j}{x_j}} \right] \\ \frac{\partial \rho_i}{\partial x_j}(x) &= \alpha_j \frac{\prod_{j=1}^{n} x_j^{\alpha_j}}{x_i x_j} - \frac{2\alpha_j}{x_i^2 x_j^2} \frac{1}{\left( \sum_{j=1}^{n} \frac{\alpha_j}{x_j} \right)^3}. \end{aligned}$$

It is easily seen that

$$\begin{aligned}\left|\frac{\partial\rho_i}{\partial x_i}(x)\right| &\leqslant \left|(\alpha_i-1)\frac{\prod_{j=1}^{n}x_j^{\alpha_j}}{x_i^2}\right| + \left|\frac{2}{x_i^3}\frac{1}{\left(\sum_{j=1}^{n}\frac{\alpha_j}{x_j}\right)^2}\left[1-\frac{\alpha_i}{x_i\sum_{j=1}^{n}\frac{\alpha_j}{x_j}}\right]\right| \\ &\leqslant (1-\alpha_i)\frac{\lambda+d}{(\lambda-d)^2}+\frac{2(\lambda+d)^2}{(\lambda-d)^3}\left(1-\alpha_i\frac{\lambda-d}{\lambda+d}\right) \\ &= \frac{\lambda+d}{(\lambda-d)^2}\left(1-3\alpha_i+2\frac{\lambda+d}{\lambda-d}\right)\end{aligned}$$

and

$$\left|\frac{\partial\rho_i}{\partial x_j}(x)\right| \leqslant \left|\alpha_j\frac{\prod_{j=1}^{n}x_j^{\alpha_j}}{x_ix_j}\right| + \left|\frac{2\alpha_j}{x_i^2x_j^2}\frac{1}{\left(\sum_{j=1}^{n}\frac{\alpha_j}{xj}\right)^3}\right| \leqslant \alpha_j\frac{\lambda+d}{(\lambda-d)^2}\left[1+\frac{2(\lambda+d)^2}{(\lambda-d)^2}\right]$$

whence

$$\begin{aligned}\|\nabla\rho_i(x)\| &\leqslant \left(\left[\frac{\lambda+d}{(\lambda-d)^2}\left(1-3\alpha_i+2\frac{\lambda+d}{\lambda-d}\right)\right]^2+\sum_{\substack{j=1\\ j\neq i}}^{n}\left[\alpha_j\frac{\lambda+d}{(\lambda-d)^2}\left(1+\frac{2(\lambda+d)^2}{(\lambda-d)^2}\right)\right]^2\right)^{\frac{1}{2}} \\ &= \frac{(\lambda+d)}{(\lambda-d)^2}\left[\left(1-3\alpha_i+2\frac{\lambda+d}{\lambda-d}\right)^2+\sum_{\substack{j=1\\ j\neq i}}^{n}\alpha_j^2\left(1+\frac{2(\lambda+d)^2}{(\lambda-d)^2}\right)^2\right]^{\frac{1}{2}} \qquad (3) \\ &\quad (x\in V_{\lambda,d}).\end{aligned}$$

Now let $x,y\in V_{\lambda,d}$. There exist $\theta_i\in(0,1)$, $i=1...,n$ such that

$$\rho_i(x)-\rho_i(y)=\nabla\rho_i(x+\theta_i(y-x))(x-y),$$

then

$$\begin{aligned}\|\nabla f(x)-\nabla f(y)\| &= \left(\sum_{i=1}^{n}\alpha_i^2\left[\rho_i(x)-\rho_i(y)\right]^2\right)^{\frac{1}{2}} \\ &= \left(\sum_{i=1}^{n}\alpha_i^2\left[\nabla\rho_i(x+\theta_i(y-x))(x-y)\right]^2\right)^{\frac{1}{2}} \\ &\leqslant \left[\sum_{i=1}^{n}\alpha_i^2\left\|\nabla\rho_i(x+\theta_i(y-x)\right\|^2\right]^{\frac{1}{2}}\|x-y\|\end{aligned}$$

Since $x,y \in V_{\lambda,d}$ it follows that $x+\theta_i(y-x) \in V_{\lambda,d}$ for all $i$. Using the inequality (3), it is concluded that

$$\|\nabla f(x) - \nabla f(y)\| \leqslant a_1(\lambda,d)\,\|x-y\|\,, \quad \text{for } x,y \in V_{\lambda,d}$$

where

$$a_1(\lambda,d) = \frac{(\lambda+d)}{(\lambda-d)^2}\left\{\sum_{i=1}^{n}\alpha_i^2\left(\left[1-3\alpha_i+2\frac{\lambda+d}{\lambda-d}\right]^2+\left(m-\alpha_i^2\right)\left[1+\frac{2(\lambda+d)^2}{(\lambda-d)^2}\right]^2\right)\right\}^{\frac{1}{2}}$$

with

$$m=\sum_{j=1}^{n}\alpha_j^2$$

whence it is seen that mapping $x \to \nabla f(x)$ is Lipschitz continuous on $V_{\lambda,d}$ with the Lipschitz constant $K \leqslant a_1(\lambda,d)$. Let's apply Theorem1 to a set $\Omega = V_{\lambda,r}$, let $r \in (0,d)$ and $q=d-r$.

Let's estimate $M = \max\left\{\|\nabla f(x)\|_\infty : x \in V_{\lambda,r}\right\}$ as follows:

$$M = \max_{x\in V_{\lambda,r}}\left\{\|\nabla f(x)\|_\infty\right\} = \max_{x\in V_{\lambda,r}}\left\{\max_{1\leqslant i\leqslant n}\left|\alpha_i\frac{\prod_{j=1}^{n}x_j^{\alpha_j}}{x_i}-\frac{\alpha_i}{x_i^2}\frac{1}{\left(\sum_{j=1}^{n}\frac{\alpha_j}{x_j}\right)^2}\right|\right\}$$

$$= \max_{1\leqslant i\leqslant n}\left\{\max_{x\in V_{\lambda,r}}\left|\alpha_i\frac{\prod_{j=1}^{n}x_j^{\alpha_j}}{x_i}-\frac{\alpha_i}{x_i^2}\frac{1}{\left(\sum_{j=1}^{n}\frac{\alpha_j}{x_j}\right)^2}\right|\right\}$$

$$= \max_{1\leqslant i\leqslant n}\left\{\max_{x\in V_{\lambda,r}}\left|\alpha_i\left(\frac{x_1}{x_i}\right)^{\alpha_1}\cdots\left(\frac{x_{i-1}}{x_i}\right)^{\alpha_{i-1}}\left(\frac{x_{i+1}}{x_i}\right)^{\alpha_{i+1}}\cdots\left(\frac{x_n}{x_i}\right)^{\alpha_n}-\frac{\alpha_i}{x_i^2}\frac{1}{\left(\sum_{j=1}^{n}\frac{\alpha_j}{x_j}\right)^2}\right|\right\}$$

$$\leqslant \max_{1\leqslant i\leqslant n}\left\{\max\left\{\alpha_i\left[\left(\frac{\lambda+r}{\lambda-r}\right)^2-\left(\frac{\lambda-r}{\lambda+r}\right)^{1-\alpha_i}\right],\alpha_i\left[\left(\frac{\lambda+r}{\lambda-r}\right)^{1-\alpha_i}-\left(\frac{\lambda-r}{\lambda+r}\right)^2\right]\right\}\right\}$$

By taking into account that $g(t)=c^t+c^{-t}$ is monotone increasing function on $[0,\infty)$ for $c>1$, the following inequality holds:

$$\left(\frac{\lambda+r}{\lambda-r}\right)^2-\left(\frac{\lambda-r}{\lambda+r}\right)^{1-\alpha_i} \geqslant \left(\frac{\lambda+r}{\lambda-r}\right)^{1-\alpha_i}-\left(\frac{\lambda-r}{\lambda+r}\right)^2.$$

So

$$\max\left\{\alpha_i\left[\left(\frac{\lambda+r}{\lambda-r}\right)^2-\left(\frac{\lambda-r}{\lambda+r}\right)^{1-\alpha_i}\right],\alpha_i\left[\left(\frac{\lambda+r}{\lambda-r}\right)^{1-\alpha_i}-\left(\frac{\lambda-r}{\lambda+r}\right)^2\right]\right\}$$
$$=\alpha_i\left[\left(\frac{\lambda+r}{\lambda-r}\right)^2-\left(\frac{\lambda-r}{\lambda+r}\right)^{1-\alpha_i}\right]$$

Hence

$$M\leqslant\max_{1\leqslant i\leqslant n}\left\{\alpha_i\left[\left(\frac{\lambda+r}{\lambda-r}\right)^2-\left(\frac{\lambda-r}{\lambda+r}\right)^{1-\alpha_i}\right]\right\}\equiv M_{0.}$$

Let

$$a_2(\lambda,d,r)=\frac{M_0}{2(d-r)}$$

and

$$a(\lambda,d,r)=\max\left(a_1(\lambda,d),a_2(\lambda,d,r)\right).$$

Since $\lim_{d\to\lambda-0}a(\lambda,d,r)=\lim_{d\to r+0}a(\lambda,d,r)=+\infty$, the function $d\longmapsto a(\lambda,d,r)$ takes its minimum value on the interval $(r,\lambda)$. Let $a_{\lambda,r}=\min_{r<d<\lambda}a(\lambda,d,r)$. Thus, it is concluded that

$$\prod_{i=1}^n x_i^{\alpha_i}\geqslant\frac{1}{\sum_{i=1}^n\frac{\alpha_i}{x_i}}+\frac{1}{4a_{\lambda,r}}\sum_{i=1}^n\left[\alpha_i\frac{\prod_{j=1}^n x_j^{\alpha_j}}{x_i}-\frac{\alpha_i}{x_i^2}\frac{1}{\left(\sum_{j=1}^n\frac{\alpha_j}{x_j}\right)^2}\right]^2\quad\text{for } x\in V_{\lambda,r.}$$

□

As a special case, if $\alpha_i=\frac{1}{n},\ i=1,2,...,n,$ the following result can be obtained as follows:

COROLLARY 2. *Let $\lambda>r$ be positive numbers. Let*

$$a_{\lambda,r}=\min_{r<d<\lambda}\max\left\{\frac{1}{\sqrt{n}}\frac{(\lambda+d)}{(\lambda-d)^2}\left[\left(1-\frac{3}{n}+2\frac{\lambda+d}{(\lambda-d)}\right)^2+\frac{(n-1)}{n^2}\left(1+\frac{2(\lambda+d)^2}{(\lambda-d)^2}\right)^2\right]^{\frac{1}{2}},\right.$$
$$\left.\frac{r}{2(d-r)n}\left[\left(\frac{\lambda+r}{\lambda-r}\right)^2-\left(\frac{\lambda-r}{\lambda+r}\right)^{\frac{n-1}{n}}\right]\right\}$$

*Then for all $x\in\mathbb{R}^n_{++}$ such that $\|x-\lambda\mathbf{1}\|_\infty\leqslant r$ the following inequality holds:*

$$\prod_{i=1}^n x_i^{\frac{1}{n}}\geqslant\frac{n}{\sum_{i=1}^n\frac{1}{x_i}}+\frac{1}{4\alpha_{\lambda,r}}\sum_{i=1}^n\left[\frac{1}{n}\frac{\prod_{j=1}^n x_j^{\frac{1}{n}}}{x_i}-\frac{n}{x_i^2}\frac{1}{\left(\sum_{j=1}^n\frac{1}{x_j}\right)^2}\right]^2\quad\textit{for } x\in V_{\lambda,r.}$$

### 3.3. Analyzing the inequality between the arithmetic and the quadratic means

The scheme which is used to sharpen the inequalities in the proofs of Theorem 2 and Theorem3 may not give worthwhile results always. For example, if $M_t(x,\alpha)$ is considered with $t_1 = 1$, $t_2 = 2$ (for the simplicity, $\alpha_i = \frac{1}{n}$, $i = 1,2,...,n$) then the following well-known inequality exists:

$$\sqrt{\frac{x_1^2 + x_2^2 + ... + x_n^2}{n}} \geqslant \frac{x_1 + x_2 + ... + x_n}{n}$$

Let's follow the similar scheme to sharpen this inequality.Let

$$f(x) = \sqrt{\frac{x_1^2 + x_2^2 + ... + x_n^2}{n}} - \frac{x_1 + x_2 + ... + x_n}{n}$$

where $\sum_{i=1}^{n} \alpha_i = 1$, $\alpha_i \geqslant 0$, $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n_+$.

Then,

$$\nabla f(x) = \left[\frac{1}{n}\left(\frac{\sqrt{n}x_1}{\left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}} - 1\right), \frac{1}{n}\left(\frac{\sqrt{n}x_2}{\left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}} - 1\right), ..., \frac{1}{n}\left(\frac{\sqrt{n}x_n}{\left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}} - 1\right)\right]$$

Hence

$$\|\nabla f(x)\|^2 = \frac{2}{n\sqrt{n}}\left(\sqrt{n} - \frac{\sum_{i=1}^{n} x_i}{\left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}}\right).$$

For $\lambda > d > 0$, define $V_{\lambda,d}$ as given before:

$$V_{\lambda,d} = \{x \in \mathbb{R}^n : \|\lambda \mathbf{1} - x\|_\infty \leqslant d\}$$

Let $\rho_i(x) = \frac{x_i}{\left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}}$. Let's make the following calculations to estimate $\|\nabla \rho_i(x)\|$ for $x \in V_{\lambda,d}$:

$$\left|\frac{\partial \rho_i}{\partial x_i}(x)\right| = \left|1 - \frac{x_i^2}{(x_1^2 + x_2^2 + ... + x_n^2)^{\frac{3}{2}}}\right| \leqslant \max\left\{\left|1 - \frac{(\lambda - d)^2}{n^{\frac{3}{2}}(\lambda + d)^3}\right|, \left|1 - \frac{(\lambda + d)^2}{n^{\frac{3}{2}}(\lambda - d)^3}\right|\right\}$$

$$\left|\frac{\partial \rho_i}{\partial x_j}(x)\right| = \left|-\frac{x_i x_j}{(x_1^2 + x_2^2 + ... + x_n^2)^{\frac{3}{2}}}\right| \leqslant \frac{(\lambda + d)^2}{n^{\frac{3}{2}}(\lambda - d)^3}.$$

Then

$$\|\nabla \rho_i(x)\| \leqslant \left[(n-1)s_2^2 + s_1^2\right]^{1/2} \qquad (x \in V_{\lambda,d}). \tag{4}$$

where

$$s_1 = \max\left\{\left|1 - \frac{(\lambda-d)^2}{n^{\frac{3}{2}}(\lambda+d)^3}\right|, \left|1 - \frac{(\lambda+d)^2}{n^{\frac{3}{2}}(\lambda-d)^3}\right|\right\} \text{ and } s_2 = \frac{(\lambda+d)^2}{n^{\frac{3}{2}}(\lambda-d)^3}.$$

Now let $x, y \in V_{\lambda,d}$. From the mean value theorem, we know the existence of $\theta_i \in (0,1)$, $i = 1, ..., n$ such that

$$\begin{aligned}
\|\nabla f(x) - \nabla f(y)\| &= \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n}[\rho_i(y) - \rho_i(x)]^2\right)^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n}[\nabla\rho_i(x+\theta_i(y-x))(x-y)]^2\right)^{\frac{1}{2}} \\
&\leqslant \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n}\|\nabla\rho_i(x+\theta_i(y-x))\|^2\|x-y\|^2\right)^{\frac{1}{2}} \\
&\leqslant [(n-1)s_2^2 + s_1^2]^{1/2}\|x-y\|
\end{aligned}$$

It is clear that $x + \theta_i(y-x) \in V_{\lambda,d}$ for all $i$ and $x, y \in V_{\lambda,d}$. Making use of the inequality (4), we come to a conclusion that

$$\|\nabla f(x) - \nabla f(y)\| \leqslant a_1(\lambda,d)\|x-y\|, \;\; x,y \in V_{\lambda,d}$$

where

$$a_1(\lambda,d) = [(n-1)s_2^2 + s_1^2]^{1/2}.$$

Hence we see that $\nabla f(x)$ is Lipschitz continuous on $V_{\lambda,d}$ with the Lipschitz constant $K \leqslant a_1(\lambda,d)$.

Let $\|.\|_\circ = \|.\|_\infty$, $r \in (0,d)$ and $q = d - r$. Let's estimate $M = \max\{\|\nabla f(x)\|_\infty : x \in V_{\lambda,r}\}$ as follows:

$$\begin{aligned}
M &= \max_{x\in V_{\lambda,r}}\{\|\nabla f(x)\|_\infty\} = \max_{x\in V_{\lambda,r}}\left\{\max_{1\leqslant i\leqslant n}\left|\frac{1}{n}\left(\frac{\sqrt{n}x_i}{\sqrt{x_1^2+x_2^2+...+x_n^2}} - 1\right)\right|\right\} \\
&= \frac{1}{n}\max_{1\leqslant i\leqslant n}\left\{\max_{x\in V_{\lambda,r}}\left|\frac{\sqrt{n}x_i}{\sqrt{x_1^2+x_2^2+...+x_n^2}} - 1\right|\right\} \\
&\leqslant \frac{1}{n}\max_{1\leqslant i\leqslant n}\left\{\max\left\{\frac{\lambda+r}{\lambda-r} - 1, 1 - \frac{\lambda-r}{\lambda+r}\right\}\right\} \\
&= \frac{1}{n}\max_{1\leqslant i\leqslant n}\left\{\frac{\lambda+r}{\lambda-r} - 1\right\} = \frac{2r}{n(\lambda-r)}
\end{aligned}$$

Let

$$a_2(\lambda,d,r) = \frac{r}{n(\lambda-r)(d-r)}$$

and

$$a(\lambda,d,r) = \max\{a_1(\lambda,d), a_2(\lambda,d,r)\} = \max\left\{[(n-1)s_2^2 + s_1^2]^{1/2}, \frac{r}{n(\lambda-r)(d-r)}\right\}$$

The function $d \longmapsto a(\lambda,d,r)$ takes its minimum value on the interval $(r,\lambda)$, because $\lim_{d\to\lambda^-} a(\lambda,d,r) = \lim_{d\to r^+} a(\lambda,d,r) = +\infty$. Let $a_{\lambda,r} = \min_{r<d<\lambda} a(\lambda,d,r)$. Using Theorem1 we conclude that

$$f(x) \geqslant \frac{1}{4a_{\lambda,r}} \frac{2}{\sqrt{n}} \frac{1}{\left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}} f(x) \geqslant \frac{1}{2na_{\lambda,r}(\lambda+r)} f(x) \quad \text{for } x \in V_{\lambda,r,}$$

that is, $f(x) \geqslant 0$. Thus, this is not a new inequality.

REMARK 4. By using the scheme which is presented, the inequality between the means with any two order $t_1$ and $t_2$ can be studied.

### 3.4. Analyzing the Cauchy-Schwarz and Minkowski inequalities

The problem of sharpening the inequalities which obey the conditions of Theorem1 can be surveyed in the given way above. The famous two of these inequalities are Cauchy-Schwarz and Minkowski inequalities. It is shown below that given scheme does not give worthwhile results for these inequalities.

Let

$$f_y(x) = f(x,y) = \|x\|\,\|y\| - [x,y]$$

where $y \in \mathbb{R}^n_+$ is a fixed vector, $x \in \mathbb{R}^n_+$ and $[x,y]$ is the inner product of $x$ and $y$.

Cauchy-Schwarz inequality requires that $f_y(x) \geqslant 0$. Only $f_y(x) = 0$ when $x = \lambda y,\ \lambda \in R$. Let's apply the scheme to $f_y$. First, let's show that $\nabla f_y$ is Lipschitz function for fixed $y$.

$$\nabla f_y(x) = \left[\frac{\|y\|}{\|x\|}x_1 - y_1, \frac{\|y\|}{\|x\|}x_2 - y_2, \ldots, \frac{\|y\|}{\|x\|}x_n - y_n\right]$$

Let $\rho_i(x) = \frac{x_i}{\|x\|}$. We have

$$\left|\frac{\partial \rho_i}{\partial x_i}\right| = \frac{\|x\|^2 - x_i^2}{\|x\|^3}$$

$$\left|\frac{\partial \rho_i}{\partial x_j}\right| = \left|\frac{x_i x_j}{\|x\|^3}\right|,\ i \neq j$$

then it is known that there exist numbers $\theta_i \in (0,1),\ i = \overline{1,n}$ such that

$$\begin{aligned}
\|\nabla f_y(x) - \nabla f_y(z)\| &= \|y\| \sqrt{\sum_{i=1}^{n} \left(\frac{x_i}{\|x\|} - \frac{z_i}{\|z\|}\right)^2} = \|y\| \sqrt{\sum_{i=1}^{n} [\rho_i(x) - \rho_i(z)]^2} \\
&= \|y\| \sqrt{\sum_{i=1}^{n} [\nabla \rho_i(x + \theta_i(z-x))^2 (x-z)]^2} \\
&\leqslant \|y\| \sqrt{\sum_{i=1}^{n} \|\nabla \rho_i(x + \theta_i(z-x))\|^2\, \|x-z\|}
\end{aligned}$$

It is clear that $f_y(x)$ has the minimum on the ray $\lambda y,\ \lambda > 0$. It is easily seen from above that for given $\lambda > 0$ there exists a closed neighborhood $V_{\lambda,d}$ of $\lambda y$ not including origin such that $\nabla f_y$ is Lipschitz function, that is, there exists $M_1 > 0$ such that

$$\|\nabla f_y(x) - \nabla f_y(z)\| \leqslant M_1 \|x - z\| .$$

So Theorem1 can be applied to this situation. Consequently,

$$\|\nabla f_y(x)\|^2 = 2\frac{\|y\|}{\|x\|}\left(\|x\|\,\|y\| - [x,y]\right) = 2\frac{\|y\|}{\|x\|}f_y(x)$$

and since a closed neighborhood $V_{\lambda,d}$ of $\lambda y$ doesn't include the origin, there exists $M_2$ such that $2\frac{\|y\|}{\|x\|} \leqslant M_2$ for all $x, y \in V_{\lambda,d}$.

Thus, it is easily seen that there exists $M_3 < 1$ depending on $M_1, M_2$ such that

$$f_y(x) \geqslant M_3 f_y(x).$$

that is, $f_y(x) \geqslant 0$. This is not a new inequality.

Minkowski inequality can be studied in a similar way.

### 3.5. Numerical Computations

Some computations have been done in order to evaluate the results obtained numerically. The weighted geometric-arithmetic mean has been considered and compared with the new derived inequality for some weights and values.

As it is expressed in Remark 2, $a_{\lambda,r}$ can be computed easily. So the inequality

$$a_1(\lambda,d) = a_2(\lambda,d,r)$$

is solved with respect to $d$. The solution $d^*$ is put in the expression $a_1(\lambda,d)$ or $a_2(\lambda,d)$. Thus, $a(\lambda,d)$ is derived, that is,

$$a_{\lambda,r} = a_1(\lambda,d^*) = a_2(\lambda,d^*,r).$$

For the values $\lambda = 1$, $r = 0.5$, $n = 5$ and $(x)_1 = (0.6, 0.8, 1, 1.2, 1.4)$, $(x)_2 = (0.6, 0.7, 0.8, 0.9, 1)$, $(x)_3 = (1, 1.1, 1.2, 1.3, 1.4)$, $(x)_4 = (0.9, 0.95, 1, 1.05, 1.1)$ with the different weights, the weighted geometric-arithmetic mean inequality is compared with the new inequality.

The functions $f(x)$ and $u(x)$ in tables are as follows:

$$f(x) = \sum_{i=1}^{n} \alpha_i x_i - \prod_{i=1}^{n} x_i^{\alpha_i}, \quad u(x) = \frac{1}{4a_{\lambda,r}} \sum_{i=1}^{n} \alpha_i^2 \left(1 - \frac{\prod_{j=1}^{n} x_j^{\alpha_j}}{x_i}\right)^2$$

*Table 1.* Results for the equal weights $(\alpha) = (0.2, 0.2, 0.2, 0.2, 0.2)$

| | $\sum_{i=1}^{n} \alpha_i x_i$ | $\prod_{i=1}^{n} x_i^{\alpha_i}$ | $f(x)$ | $u(x)$ | $\frac{u(x)}{f(x)}$ |
|---|---|---|---|---|---|
| $(x)_1$ | 1.000000 | 0.957878 | 0.042122 | 0.009820 | 0.233137 |
| $(x)_2$ | 0.800000 | 0.787257 | 0.012743 | 0.003185 | 0.249942 |
| $(x)_3$ | 1.200000 | 1.191596 | 0.008404 | 0.001332 | 0.158446 |
| $(x)_4$ | 1.000000 | 0.997492 | 0.002508 | 0.000035 | 0.013864 |

*Table 2.* Results for random weights $(\alpha) = (0.05, 0.05, 0.2, 0.3, 0.4)$

| | $\sum_{i=1}^{n} \alpha_i x_i$ | $\prod_{i=1}^{n} x_i^{\alpha_i}$ | $f(x)$ | $u(x)$ | $\frac{u(x)}{f(x)}$ |
|---|---|---|---|---|---|
| $(x)_1$ | 1.190000 | 1.164844 | 0.025156 | 0.003788 | 0.150559 |
| $(x)_2$ | 0.895000 | 0.887265 | 0.007735 | 0.001476 | 0.190843 |
| $(x)_3$ | 1.295000 | 1.289853 | 0.005147 | 0.000677 | 0.131586 |
| $(x)_4$ | 1.047500 | 1.045953 | 0.001547 | 0.000019 | 0.012391 |

*Table 3.* Results for the case in which one of the weights is dominant on others $(\alpha) = (0.8, 0.1, 0.06, 0.03, 0.01)$

| | $\sum_{i=1}^{n} \alpha_i x_i$ | $\prod_{i=1}^{n} x_i^{\alpha_i}$ | $f(x)$ | $u(x)$ | $\frac{u(x)}{f(x)}$ |
|---|---|---|---|---|---|
| $(x)_1$ | 0.670000 | 0.655642 | 0.014358 | 0.002924 | 0.203672 |
| $(x)_2$ | 0.635000 | 0.630730 | 0.004270 | 0.000918 | 0.215099 |
| $(x)_3$ | 1.035000 | 1.032214 | 0.002786 | 0.000370 | 0.132640 |
| $(x)_4$ | 0.917500 | 0.916676 | 0.000824 | 0.000009 | 0.011363 |

Last two columns of tables show the sharpening ratio of inequality for different values with the different weights.

## 4. Conclusion

One of the large application areas of abstract convexity is the inequality theory. In this area, besides different applications, the sharpening of well-known inequalities takes place.

In this paper, the situations of $M_t(x, \alpha)$ which gives the important and famous mean inequalities are considered separately and studied in terms of sharpening. By sharpening the weighted harmonic-geometric mean inequality and the weighted geometric-arithmetic mean inequality, new inequalities are derived. Numerical results which show the amount of sharpening are given.

By using the same method, Arithmetic-quadratic mean inequality, Cauchy-Schwarz inequality and Minkowski inequality are studied. It is shown that these inequalities obeys the conditions of Theorem1, but the performed scheme does not give new inequality.

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