Mathematical
Inequalities
& Applications

Volume 12, Number 1 (2009), 67–80

ON SOME REVERSE WEIGHTED $L_p(\mathbb{R}^n)$–NORM INEQUALITIES IN CONVOLUTIONS AND THEIR APPLICATIONS

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(*communicated by S. Saitoh*)

Abstract. In this paper, we give the reverse weighted $L_p(\mathbb{R}^n)$ norm inequalities and their important applications to studying stability of some inverse problems. Especially, we will see their applications to inverse problems in non-homogeneous linear differential equations.

1. Introduction

In a series of papers, S. Saitoh, V.K. Tuan, M. Yamamoto ([9], [10], [11]) derived new type norm reverse inequalities in convolutions in some several weighted L_p spaces using the following famous reverse Hölder's inequality

PROPOSITION 1. ([5]) *For two positive functions f and g satisfying*

$$0 < m \leqslant \frac{f}{g} \leqslant M < \infty \tag{1.1}$$

on the set X, and for $p,q > 1$, $p^{-1} + q^{-1} = 1$,

$$\left(\int_X f\,d\mu\right)^{\frac{1}{p}} \left(\int_X g d\mu\right)^{\frac{1}{q}} \leqslant A_{p,q}\left(\frac{m}{M}\right) \int_X f^{\frac{1}{p}} g^{\frac{1}{q}} d\mu, \tag{1.2}$$

if the right hand side integral converges. Here

$$A_{p,q}(t) = p^{-\frac{1}{p}} q^{-\frac{1}{q}} \frac{t^{-\frac{1}{pq}}(1-t)}{\left(1-t^{\frac{1}{p}}\right)^{\frac{1}{p}} \left(1-t^{\frac{1}{q}}\right)^{\frac{1}{q}}}.$$

In the same way, we ([7]) gave the new type of reverse convolution inequality in weighted $L_p(\mathbb{R}^2, \rho)(p > 1)$ spaces

Mathematics subject classification (2000): 44A35, 35A22, 26D20.

Keywords and phrases: Convolution, weighted L_p inequality, reverse Hölder inequality, inverse problems, Green's function, integral transform, partial differential equations, stability.

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Paper MIA-12-06

PROPOSITION 2. *Let F_1 and F_2 be positive functions satisfying*

$$0 < m_1^{\frac{1}{p}} \leqslant F_1(\xi,\tau) \leqslant M_1^{\frac{1}{p}} < \infty, \quad 0 < m_2^{\frac{1}{p}} \leqslant F_2(\xi,\tau) \leqslant M_2^{\frac{1}{p}} < \infty, \quad (\xi,\tau) \in \mathbb{R}^2. \tag{1.3}$$

Then for any positive continuous functions ρ_1 and ρ_2 we have the reverse $L_p(p>1)-$ weighted convolution inequality

$$\begin{aligned} &\left\| ((F_1\rho_1)*(F_2\rho_2))(\rho_1*\rho_2)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}^2)} \\ &\qquad \geqslant A_{p,q}^{-2}\left(\frac{m_1m_2}{M_1M_2}\right) \|F_1\|_{L_p(\mathbb{R}^2,\rho_1)} \|F_2\|_{L_p(\mathbb{R}^2,\rho_2)}. \end{aligned} \tag{1.4}$$

Recently, we ([8]) introduced the inequalities in convolutions in weighted $L_p(\mathbb{R}^n,\rho)$ $(p>1)$ spaces

$$\left\| ((F_1\rho_1)*(F_2\rho_2))(\rho_1*\rho_2)^{\frac{1}{p}-1} \right\|_p \leqslant \|F_1\|_{L_p(\mathbb{R}^n,|\rho_1|)} \|F_2\|_{L_p(\mathbb{R}^n,|\rho_2|)} \tag{1.5}$$

for functions $F_j \in L_p(\mathbb{R}^n,\rho_j)(j=1,2)$.

In this paper, by using the the reverse Hölder inequality, we give the reverse weighted L_p norm inequalities and their important applications to studying stability of some inverse problems.

2. The Main Results

For brevity of presentation we shall use the following notation.

2.1. Notation

By $\mathbb{R}^n$ we denote the $n-$ *dimensional Euclidean space*, $n \in \mathbb{N}$. This is the set of all $n-$ tuples of real numbers, $\mathbf{x} = (x_1,...,x_n)$, $x_j \in \mathbb{R}$, $j = 1,2,...,n$ with the linear operations

$$\begin{aligned} \mathbf{x}+\mathbf{y} &= (x_1+y_1,...,x_n+y_n), \ \ \mathbf{x},\mathbf{y} \in \mathbb{R}^n, \\ \lambda\mathbf{x} &= (\lambda x_1,...,\lambda x_n), \ \ \lambda \in \mathbb{R}, \ \ \mathbf{x} \in \mathbb{R}^n, \end{aligned} \tag{2.6}$$

the scalar product

$$\mathbf{x}\mathbf{y} = x_1y_1+\cdots+x_ny_n, \ \ \mathbf{x},\mathbf{y} \in \mathbb{R}^n, \tag{2.7}$$

and the norm

$$\|\mathbf{x}\| = (\mathbf{x}\mathbf{x})^{\frac{1}{2}} = (x_1^2+\cdots+x_n^2)^{\frac{1}{2}}, \ \ \mathbf{x} \in \mathbb{R}^n. \tag{2.8}$$

We shall write $\mathbf{x} > \mathbf{y}$ instead of $x_j > y_j$, $j = 1,2,...,n$. Anologously one has to understand $\mathbf{x} \geqslant \mathbf{y}$, $\mathbf{x} < \mathbf{y}$, $\mathbf{x} \leqslant \mathbf{y}$. In particular let

$$\mathbf{1} = (1,1,...,1), \quad \mathbf{2} = (2,2,...,2),... \tag{2.9}$$

We shall denote some subsets of $\mathbb{R}^n$

$$\begin{aligned} \mathbb{R}^n_+ &= \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n, \mathbf{x} > \mathbf{0}\}, \\ \mathbb{R}^n_+(\mathbf{t}) &= \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n, \mathbf{0} < \mathbf{x} < \mathbf{t}\}. \end{aligned} \tag{2.10}$$

Now let $\mathbf{z},\mathbf{a}\in\mathbb{R}^n$. Then we set $\mathbf{z}^{\mathbf{i}}=(z_1,...,z_{i-1},z_{i+1},...,z_n)\in\mathbb{R}^{n-1}$, $i=1,2,...,n$, and

$$\mathbf{z}^{\mathbf{a}}=\prod_{j=1}^{n} z_j^{a_j}. \tag{2.11}$$

Finally, we shall denote some integrals

$$\begin{aligned}\int_{\mathbb{R}^n} f(\mathbf{x})d\mathbf{x} &= \int_{\mathbb{R}}\cdots\int_{\mathbb{R}} f(x_1,...,x_n)dx_1\cdots dx_n,\\ \int_{\mathbb{R}^n_+(\mathbf{t})} f(\mathbf{x})d\mathbf{x} &= \int_0^{t_1}\cdots\int_0^{t_n} f(x_1,...,x_n)dx_1\cdots dx_n,\\ \int_{\mathbb{R}^{n-1}} f(\mathbf{x})d\mathbf{x}^{\mathbf{i}} &= \int_{\mathbb{R}}\cdots\int_{\mathbb{R}} f(x_1,...,x_n)dx_1\cdots dx_{i-1}dx_{i+1}\cdots dx_n.\end{aligned} \tag{2.12}$$

2.2. The Inequalities

For the sake of convenience, we first introduce the following corollary

COROLLARY 1. *Let F_1 and F_2 be positive functions satisfying*

$$0<m_1^{\frac{1}{p}}\leqslant F_1(\mathbf{z})\leqslant M_1^{\frac{1}{p}}<\infty,\quad 0<m_2^{\frac{1}{p}}\leqslant F_2(\mathbf{z})\leqslant M_2^{\frac{1}{p}}<\infty,\quad \mathbf{z}\in\mathbb{R}^n. \tag{2.13}$$

Then for any positive continuous functions ρ_1 and ρ_2 on $\mathbb{R}^n$, we have the inequality

$$\begin{aligned}&A_{p,q}^n\left(\frac{m_1m_2}{M_1M_2}\right)\int_{\mathbb{R}^n} F_1(\mathbf{z})\rho_1(\mathbf{z})F_2(\mathbf{x}-\mathbf{z})\rho_2(\mathbf{x}-\mathbf{z})d\mathbf{z}\\ &\geqslant\left\{\int_{\mathbb{R}^n} F_1^p(\mathbf{z})\rho_1(\mathbf{z})F_2^p(\mathbf{x}-\mathbf{z})\rho_2(\mathbf{x}-\mathbf{z})d\mathbf{z}\right\}^{\frac{1}{p}}\left\{\int_{\mathbb{R}^n}\rho_1(\mathbf{z})\rho_2(\mathbf{x}-\mathbf{z})d\mathbf{z}\right\}^{1-\frac{1}{p}}.\end{aligned} \tag{2.14}$$

Proof. We use induction on n. When $n=1$, the inequality (2.14) is reduced to the reverse Hölder's inequality (1.2). Now suppose (2.14) holds for some integer $n-1\geqslant 2$, we claim that it also holds for n. Put

$$f(z_i)=\int_{\mathbb{R}^{n-1}} F_1^p(\mathbf{z})\rho_1(\mathbf{z})F_2^p(\mathbf{x}-\mathbf{z})\rho_2(\mathbf{x}-\mathbf{z})d\mathbf{z}^{\mathbf{i}},\ \ g(z_i)=\int_{\mathbb{R}^{n-1}}\rho_1(\mathbf{z})\rho_2(\mathbf{x}-\mathbf{z})d\mathbf{z}^{\mathbf{i}}.$$

The condition (2.13) implies

$$m_1m_2\leqslant\frac{f(z_i)}{g(z_i)}\leqslant M_1M_2,\quad z_i\in\mathbb{R}.$$

Thus by the Fubini's theorem, the induction hypothesis and the reverse Hölder's inequality, we have

$$\begin{aligned}&A_{p,q}^n\left(\frac{m_1m_2}{M_1M_2}\right)\int_{\mathbb{R}^n} F_1(\mathbf{z})\rho_1(\mathbf{z})F_2(\mathbf{x}-\mathbf{z})\rho_2(\mathbf{x}-\mathbf{z})d\mathbf{z}\\ &=A_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)\int_{\mathbb{R}}\left\{A_{p,q}^{n-1}\left(\frac{m_1m_2}{M_1M_2}\right)\int_{\mathbb{R}^{n-1}} F_1(\mathbf{z})\rho_1(\mathbf{z})F_2(\mathbf{x}-\mathbf{z})\rho_2(\mathbf{x}-\mathbf{z})d\mathbf{z}^{\mathbf{i}}\right\}dz_i\end{aligned}$$

$$\geqslant A_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)\int_{\mathbb{R}}\left\{\int_{\mathbb{R}^{n-1}}F_1^p(\mathbf{z})\rho_1(\mathbf{z})F_2^p(\mathbf{x}-\mathbf{z})\rho_2(\mathbf{x}-\mathbf{z})d\mathbf{z}^{\mathbf{i}}\right\}^{\frac{1}{p}}$$

$$\times\left\{\int_{\mathbb{R}^{n-1}}\rho_1(\mathbf{z})\rho_2(\mathbf{x}-\mathbf{z})d\mathbf{z}^{\mathbf{i}}\right\}^{1-\frac{1}{p}}dz_i$$

$$\geqslant\left\{\int_{\mathbb{R}^n}F_1^p(\mathbf{z})\rho_1(\mathbf{z})F_2^p(\mathbf{x}-\mathbf{z})\rho_2(\mathbf{x}-\mathbf{z})d\mathbf{z}\right\}^{\frac{1}{p}}\left\{\int_{\mathbb{R}^n}\rho_1(\mathbf{z})\rho_2(\mathbf{x}-\mathbf{z})d\mathbf{z}\right\}^{1-\frac{1}{p}}.$$

The proof is complete. □

Next, by using the Corollary 1 and Fubini's theorem, we obtain the reverse weighted L_p norm inequalities.

THEOREM 1. *Let F_1 and F_2 be positive functions satisfying*

$$0<m_1^{\frac{1}{p}}\leqslant F_1(\mathbf{z})\leqslant M_1^{\frac{1}{p}}<\infty,\quad 0<m_2^{\frac{1}{p}}\leqslant F_2(\mathbf{z})\leqslant M_2^{\frac{1}{p}}<\infty,\quad \mathbf{z}\in\mathbb{R}^n. \tag{2.15}$$

Then for any positive continuous functions ρ_1 and ρ_2 on $\mathbb{R}^n$, we have the reverse weighted $L_p(p>1)-$norm convolution inequality

$$\begin{aligned}&\left\|((F_1\rho_1)*(F_2\rho_2))(\rho_1*\rho_2)^{\frac{1}{p}-1}\right\|_{L_p(\mathbb{R}^n)}\\&\qquad\geqslant A_{p,q}^{-n}\left(\frac{m_1m_2}{M_1M_2}\right)\|F_1\|_{L_p(\mathbb{R}^n,\rho_1)}\|F_2\|_{L_p(\mathbb{R}^n,\rho_2)}.\end{aligned}\tag{2.16}$$

Inequality (2.16) and others should be understood in the sense that if the left hand side is finite, then so is the right hand side, and in this case the inequality holds.

Proof. By the Corollary 1, we have directly

$$\frac{\left(\int_{\mathbb{R}^n}F_1^p(\mathbf{z})\rho_1(\mathbf{z})F_2^p(\mathbf{x}-\mathbf{z})\rho_2(\mathbf{x}-\mathbf{z})d\mathbf{z}\right)^p}{\left(\int_{\mathbb{R}^n}\rho_1(\mathbf{z})\rho_2(\mathbf{x}-\mathbf{z})d\mathbf{z}\right)^{p-1}}$$

$$\geqslant A_{p,q}^{-pn}\left(\frac{m_1m_2}{M_1M_2}\right)\int_{\mathbb{R}^n}F_1^p(\mathbf{z})\rho_1(\mathbf{z})F_2^p(\mathbf{x}-\mathbf{z})\rho_2(\mathbf{x}-\mathbf{z})d\mathbf{z}.$$

Therefore,

$$\begin{aligned}&\int_{\mathbb{R}^n}\frac{\left(\int_{\mathbb{R}^n}F_1^p(\mathbf{z})\rho_1(\mathbf{z})F_2^p(\mathbf{x}-\mathbf{z})\rho_2(\mathbf{x}-\mathbf{z})d\mathbf{z}\right)^p}{\left(\int_{\mathbb{R}^n}\rho_1(\mathbf{z})\rho_2(\mathbf{x}-\mathbf{z})d\mathbf{z}\right)^{p-1}}d\mathbf{x}\\&\qquad\geqslant A_{p,q}^{-pn}\left(\frac{m_1m_2}{M_1M_2}\right)\int_{\mathbb{R}^n}F_1^p(\mathbf{z})\rho_1(\mathbf{z})d\mathbf{z}\int_{\mathbb{R}^n}F_2^p(\mathbf{z})\rho_2(\mathbf{z})d\mathbf{z}.\end{aligned}\tag{2.17}$$

Raising both sides of the inequality (2.17) to power $\frac{1}{p}$ yields the inequality (2.16). □

REMARK 1. In formula (2.16) replacing ρ_2 by 1, and $F_2(\mathbf{x}-\mathbf{z})$ by $G(\mathbf{x}-\mathbf{z})$, and integrating with respect to $\mathbf{x}$ from $\mathbf{a}$ to $\mathbf{b}$, we arrive at the following inequality

$$\begin{aligned}&\int_{\mathbf{a}}^{\mathbf{b}}\left(\int_{\mathbb{R}^n} F(\mathbf{z})\rho(\mathbf{z})G(\mathbf{x}-\mathbf{z})d\mathbf{z}\right)^p d\mathbf{x}\\ &\quad\geqslant A_{p,q}^{-np}\left(\frac{m}{M}\right)\left(\int_{\mathbb{R}^n}\rho(\mathbf{z})d\mathbf{z}\right)^{p-1}\int_{\mathbb{R}^n}F^p(\mathbf{z})\rho(\mathbf{z})d\mathbf{z}\int_{\mathbf{a}-\mathbf{z}}^{\mathbf{b}-\mathbf{z}}G^p(\mathbf{y})d\mathbf{y},\end{aligned}\tag{2.18}$$

if positive continuous functions ρ, F and G satisfy

$$0<m^{\frac{1}{p}}\leqslant F(\mathbf{z})G(\mathbf{x}-\mathbf{z})<M^{\frac{1}{p}}<\infty,\quad \mathbf{a}\leqslant\mathbf{x}\leqslant\mathbf{b},\quad \mathbf{z}\in\mathbb{R}^n.\tag{2.19}$$

Moreover, by the reverse Hölder's inequality and Fubini's theorem and by changing the variables in integral, we obtain the following inequalities

THEOREM 2. *Let F_1 and F_2 be positive functions satisfying*

$$0<m_1^{\frac{1}{p}}\leqslant F_1(\mathbf{z},t)\leqslant M_1^{\frac{1}{p}}<\infty,\ \ 0<m_2^{\frac{1}{p}}\leqslant F_2(\mathbf{z},t)\leqslant M_2^{\frac{1}{p}}<\infty,\ \ (\mathbf{z},t)\in\mathbb{R}^{n-1}\times\mathbb{R}\tag{2.20}$$

and

$$0<m_3\leqslant\frac{\left\{\int_{\mathbb{R}^{n-1}}F_1^p(\mathbf{z},t)\rho_1(\mathbf{z},t)d\mathbf{z}\right\}^p}{\left\{\int_{\mathbb{R}^{n-1}}F_2^p(\mathbf{z},t)\rho_2(\mathbf{z},t)d\mathbf{z}\right\}^q}\leqslant M_3,\ p>1,\ p^{-1}+q^{-1}=1,\ (\mathbf{z},t)\in\mathbb{R}^{n-1}\times\mathbb{R}.\tag{2.21}$$

Then for any positive continuous functions ρ_1 and ρ_2 on $\mathbb{R}^n$, we have the inequality

$$\begin{aligned}&\int_{\mathbb{R}^{n-1}}\frac{\left(\int_{\mathbb{R}^n}F_1(\mathbf{z},t)\rho_1(\mathbf{z},t)F_2(\mathbf{x}-\mathbf{z},t)\rho_2(\mathbf{x}-\mathbf{z},t)d\mathbf{z}dt\right)^p}{\left(\int_{\mathbb{R}^n}\rho_1(\mathbf{z},t)\rho_2(\mathbf{x}-\mathbf{z},t)d\mathbf{z}dt\right)^{p-1}}d\mathbf{x}\\ &\geqslant A_{p,q}^{-np}\left(\frac{m_1m_2}{M_1M_2}\right)A_{p,q}^{-1}\left(\frac{m_3}{M_3}\right)\left\|\|F_1\|^p_{L_p(\mathbb{R}^{n-1},\rho_1d\mathbf{z})}\right\|_{L_p(\mathbb{R},dt)}\left\|\|F_2\|^p_{L_p(\mathbb{R}^{n-1},\rho_2d\mathbf{z})}\right\|_{L_q(\mathbb{R},dt)}.\end{aligned}\tag{2.22}$$

Proof. Similar to proof of Corollary 1, we have

$$\begin{aligned}&A_{p,q}^{np}\left(\frac{m_1m_2}{M_1M_2}\right)\frac{\left(\int_{\mathbb{R}^n}F_1(\mathbf{z},t)\rho_1(\mathbf{z},t)F_2(\mathbf{x}-\mathbf{z},t)\rho_2(\mathbf{x}-\mathbf{z},t)d\mathbf{z}dt\right)^p}{\left(\int_{\mathbb{R}^n}\rho_1(\mathbf{z},t)\rho_2(\mathbf{x}-\mathbf{z},t)d\mathbf{z}dt\right)^{p-1}}\\ &\qquad\geqslant\int_{\mathbb{R}^n}F_1^p(\mathbf{z},t)\rho_1(\mathbf{z},t)F_2^p(\mathbf{x}-\mathbf{z},t)\rho_2(\mathbf{x}-\mathbf{z},t)d\mathbf{x}dt.\end{aligned}\tag{2.23}$$

Taking integration of both sides of (2.23) with respect to $\mathbf{x}$ on $\mathbb{R}^{n-1}$, we obtain

$$\begin{aligned}&A_{p,q}^{np}\left(\frac{m_1m_2}{M_1M_2}\right)\int_{\mathbb{R}^{n-1}}\frac{\left(\int_{\mathbb{R}^n}F_1(\mathbf{z},t)\rho_1(\mathbf{z},t)F_2(\mathbf{x}-\mathbf{z},t)\rho_2(\mathbf{x}-\mathbf{z},t)d\mathbf{z}dt\right)^p}{\left(\int_{\mathbb{R}^n}\rho_1(\mathbf{z},t)\rho_2(\mathbf{x}-\mathbf{z},t)d\mathbf{z}dt\right)^{p-1}}d\mathbf{x}\\ &\qquad\geqslant\int_{\mathbb{R}}\left\{\int_{\mathbb{R}^{n-1}}F_1^p(\mathbf{z},t)\rho_1(\mathbf{z},t)d\mathbf{z}\right\}\left\{\int_{\mathbb{R}^{n-1}}F_2^p(\mathbf{z},t)\rho_2(\mathbf{z},t)d\mathbf{z}\right\}dt.\end{aligned}\tag{2.24}$$

From the condition (2.21), we apply the reverse Hölder inequality (1.2) to get

$$\begin{aligned} &\int_{\mathbb{R}} \left\{ \int_{\mathbb{R}^{n-1}} F_1^p(\mathbf{z},t)\rho_1(\mathbf{z},t)d\mathbf{z} \right\} \left\{ \int_{\mathbb{R}^{n-1}} F_2^p(\mathbf{z},t)\rho_2(\mathbf{z},t)d\mathbf{z} \right\} dt \\ &\quad \geqslant A_{p,q}^{-1} \left(\frac{m_3}{M_3} \right) \left\| \|F_1\|^p_{L_p(\mathbb{R}^{n-1},\rho_1 d\mathbf{z})} \right\|_{L_p(\mathbb{R},dt)} \left\| \|F_2\|^p_{L_p(\mathbb{R}^{n-1},\rho_2 d\mathbf{z})} \right\|_{L_q(\mathbb{R},dt)}. \end{aligned} \tag{2.25}$$

Combining (2.24) and (2.25) gives the inequality (2.22). The proof is complete. □

REMARK 2. In formula (2.22) replacing ρ_2 by 1, and $F_2(\mathbf{x}-\mathbf{z},t)$ by $G(\mathbf{x}-\mathbf{z},t)$, we arrive at the following inequality

$$\begin{aligned} &\int_{\mathbb{R}^{n-1}} \left(\int_a^b dt \int_{\mathbb{R}^{n-1}} F(\mathbf{z},t)\rho(\mathbf{z},t)G(\mathbf{x}-\mathbf{z},t)d\mathbf{z} \right)^p d\mathbf{x} \\ &\quad \geqslant A_{p,q}^{-np} \left(\frac{m_1}{M_1} \right) A_{p,q}^{-1} \left(\frac{m_2}{M_2} \right) \left(\int_a^b dt \int_{\mathbb{R}^{n-1}} \rho(\mathbf{z},t)d\mathbf{z} \right)^{p-1} \\ &\quad \times \left[\int_a^b \left\{ \int_{\mathbb{R}^{n-1}} F^p(\mathbf{z},t)\rho(\mathbf{z},t)d\mathbf{z} \right\}^p dt \right]^{\frac{1}{p}} \left[\int_a^b \left\{ \int_{\mathbb{R}^{n-1}} G^p(\mathbf{z},t)d\mathbf{z} \right\}^q dt \right]^{\frac{1}{q}}, \end{aligned} \tag{2.26}$$

if positive continuous functions ρ, F and G satisfy

$$0 < m_1^{\frac{1}{p}} \leqslant F(\mathbf{z},t)G(\mathbf{x}-\mathbf{z},t) < M_1^{\frac{1}{p}} < \infty, \quad \mathbf{x} \in \mathbb{R}^{n-1}, \quad \mathbf{z} \in \mathbb{R}^{n-1}, \quad t \in [a,b] \tag{2.27}$$

and

$$0 < m_2 \leqslant \frac{\left\{ \int_{\mathbb{R}^{n-1}} F^p(\mathbf{z},t)\rho(\mathbf{z},t)d\mathbf{z} \right\}^p}{\left\{ \int_{\mathbb{R}^{n-1}} G^p(\mathbf{z},t)d\mathbf{z} \right\}^q} \leqslant M_2, \quad p > 1, \;\; p^{-1}+q^{-1} = 1, \quad t \in [a,b]. \tag{2.28}$$

REMARK 3. In formula (2.24) replacing ρ_2 by 1, and $F_2(\mathbf{x}-\mathbf{z},t)$ by $G(\mathbf{x}-\mathbf{z},t)$, and integrating with respect to $\mathbf{x}$ from $\mathbf{c}$ to $\mathbf{d}$ we have

$$\begin{aligned} &A_{p,q}^{np} \left(\frac{m}{M} \right) \int_{\mathbf{c}}^{\mathbf{d}} \left(\int_a^b dt \int_{\mathbb{R}^{n-1}} F(\mathbf{z},t)\rho(\mathbf{z},t)G(\mathbf{x}-\mathbf{z},t)d\mathbf{z} \right)^p d\mathbf{x} \\ &\quad \geqslant \left(\int_a^b dt \int_{\mathbb{R}^{n-1}} \rho(\mathbf{z},t)d\mathbf{z} \right)^{p-1} \int_a^b dt \int_{\mathbb{R}^{n-1}} F^p(\mathbf{z},t)\rho(\mathbf{z},t)d\mathbf{z} \int_{\mathbf{c}-\mathbf{z}}^{\mathbf{d}-\mathbf{z}} G^p(\mathbf{x},t)d\mathbf{x}, \end{aligned} \tag{2.29}$$

if positive continuous functions ρ, F and G satisfy

$$0 < m^{\frac{1}{p}} \leqslant F(\mathbf{z},t)G(\mathbf{x}-\mathbf{z},t) < M^{\frac{1}{p}} < \infty, \quad \mathbf{c} \leqslant \mathbf{x} \leqslant \mathbf{d}, \quad (\mathbf{z},t) \in \mathbb{R}^{n-1} \times \mathbb{R}. \tag{2.30}$$

Inequality (1.2) reverses the side if $0 < p < 1$. Hence, inequality (2.16) and inequality (2.22) reverse the side if $0 < p < 1$.

Inequality (2.18) is especially important when $G(\mathbf{x}-\mathbf{z})$ is a Green's function.

In the next section, we will consider some applications to integral transforms and partial differential equation ([1], [2],[3],[4],[12]).

3. Applications

3.1. Laplace Transformation

We consider the Laplace transform

$$u(\mathbf{x}) = \mathrm{L}[\mathrm{F}\rho](\mathbf{x}) = \int_{\mathbb{R}^n_+} \mathrm{e}^{-\mathbf{xz}}\mathrm{F}(\mathbf{z})\rho(\mathbf{z})\mathrm{d}\mathbf{z}. \tag{3.31}$$

Take $G(\mathbf{z}) = e^{-\mathbf{xz}}$. Let

$$\mathbf{0} \leqslant \mathbf{x} \leqslant \mathbf{c}, \quad \mathbf{0} < \mathbf{a} \leqslant \mathbf{z} \leqslant \mathbf{b}, \quad \mathbf{bc} \leqslant \frac{1}{p}\log\left(\frac{M}{m}\right).$$

From

$$e^{-\mathbf{bc}} \leqslant e^{-\mathbf{xz}} \leqslant 1,$$

we have

$$0 < m^{\frac{1}{p}} \leqslant F(\mathbf{z})e^{-\mathbf{xz}} \leqslant M^{\frac{1}{p}}, \quad \mathbf{z} \in \mathbb{R}^n_+ \tag{3.32}$$

if

$$0 < m^{\frac{1}{p}} e^{\mathbf{bc}} \leqslant F(\mathbf{z}) \leqslant M^{\frac{1}{p}}. \tag{3.33}$$

Thus, the inequality (2.22) yields

$$\int_{\mathbb{R}^n_+(\mathbf{c})} u^p(\mathbf{x})d\mathbf{x} \geqslant A_{p,q}^{-pn}\left(\frac{m}{M}\right)\frac{1}{p^n}\frac{\mathbf{1}-e^{p\mathbf{ac}}}{\mathbf{a}}\left(\int_{\mathbf{a}}^{\mathbf{b}}\rho(\mathbf{z})d\mathbf{z}\right)^{p-1}\int_{\mathbf{a}}^{\mathbf{b}}F^p(\mathbf{z})\rho(\mathbf{z})d\mathbf{z}, \tag{3.34}$$

where a positive continuous function ρ on $[\mathbf{a},\mathbf{b}]$ and F satisfies (3.33).

3.2. Abel's Integral Transform

We consider the Abel's integral transform

$$f(\mathbf{x}) = \int_{\mathbb{R}^n_+(\mathbf{x})} \frac{F(\mathbf{z})\rho(\mathbf{z})}{(\mathbf{x}-\mathbf{z})^{\mathbf{a}}}d\mathbf{z}, \quad \mathbf{0} < \mathbf{a} < \mathbf{1}. \tag{3.35}$$

Take $G(\mathbf{z}) = \frac{\mathbf{1}}{\mathbf{z}^{\mathbf{a}}}$. Since

$$\mathbf{0} < \frac{\mathbf{1}}{(\mathbf{c}-\mathbf{z})^{\mathbf{a}}} \leqslant \frac{\mathbf{1}}{(\mathbf{x}-\mathbf{z})^{\mathbf{a}}} \leqslant \frac{\mathbf{1}}{(\mathbf{b}-\mathbf{z})^{\mathbf{a}}}, \quad \mathbf{0} < \mathbf{z} < \mathbf{b} \leqslant \mathbf{x} \leqslant \mathbf{c},$$

we see that the condition (2.19)

$$0 < m^{\frac{1}{p}} \leqslant \frac{F(\mathbf{z})}{(\mathbf{x}-\mathbf{z})^{\mathbf{a}}} \leqslant M^{\frac{1}{p}},$$

holds if

$$0 < m^{\frac{1}{p}}(\mathbf{c}-\mathbf{z})^{\mathbf{a}} \leqslant F(\mathbf{z}) \leqslant M^{\frac{1}{p}}(\mathbf{b}-\mathbf{z})^{\mathbf{a}}. \tag{3.36}$$

Thus, we have the inequality

$$\int_{\mathbf{b}}^{\mathbf{c}} f(\mathbf{x})^p d\mathbf{x} \geqslant \left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-np} \frac{\mathbf{1}}{\mathbf{1}-p\mathbf{a}} \left(\int_{\mathbb{R}^n_+(\mathbf{b})} \rho(\mathbf{z})d\mathbf{z}\right)^{p-1} \int_{\mathbb{R}^n_+(\mathbf{b})} F(\mathbf{z})^p \rho(\mathbf{z}) \left\{(\mathbf{c}-\mathbf{z})^{\mathbf{1}-p\mathbf{a}} - (\mathbf{b}-\mathbf{z})^{\mathbf{1}-p\mathbf{a}}\right\} d\mathbf{z} \quad (p\mathbf{a} < \mathbf{1}), \tag{3.37}$$

where ρ is a positive continuous function on $\mathbb{R}^n_+(\mathbf{b})$ and F satisfies (3.36).

3.3. Heat Equation

In the integral transform

$$u(\mathbf{x},t) = \frac{1}{(2c\sqrt{\pi t})^n} \int_{\mathbb{R}^n} F(\mathbf{z})\rho(\mathbf{z}) \exp\left\{-\frac{|\mathbf{x}-\mathbf{z}|^2}{4c^2 t}\right\} d\mathbf{z} \tag{3.38}$$

which gives the solution $u(\mathbf{x},t)$ of the heat equation

$$u_t = c^2 \Delta_n u(\mathbf{x},t) \quad (\mathbf{x},t) \in \mathbb{R}^n \times \mathbb{R}_+ \tag{3.39}$$

satisfying the condition

$$u(\mathbf{x},0) = F(\mathbf{x})\rho(\mathbf{x}). \tag{3.40}$$

Take

$$G(\mathbf{z}) = \exp\left\{-\frac{|\mathbf{z}|^2}{4c^2 t}\right\}.$$

Let

$$-\mathbf{a} \leqslant \mathbf{x} \leqslant \mathbf{a}, \quad -\mathbf{b} \leqslant \mathbf{z} \leqslant \mathbf{b}, \quad |\mathbf{a}+\mathbf{b}|^2 \geqslant \frac{4c^2 t}{p} \log\frac{M}{m}.$$

From

$$1 \leqslant \exp\left\{\frac{|\mathbf{x}-\mathbf{z}|^2}{4c^2 t}\right\} \leqslant \exp\left\{\frac{|\mathbf{a}+\mathbf{b}|^2}{4c^2 t}\right\},$$

we have

$$0 < m^{\frac{1}{p}} \leqslant F(\mathbf{z}) \exp\left\{-\frac{|\mathbf{x}-\mathbf{z}|^2}{4c^2 t}\right\} \leqslant M^{\frac{1}{p}}, \tag{3.41}$$

if

$$m^{\frac{1}{p}} \exp\left\{\frac{|\mathbf{a}+\mathbf{b}|^2}{4c^2 t}\right\} \leqslant F(\mathbf{z}) \leqslant M^{\frac{1}{p}}, \quad -\mathbf{b} \leqslant \mathbf{z} \leqslant \mathbf{b}. \tag{3.42}$$

We have

$$\int_{\mathbf{c}-\mathbf{z}}^{\mathbf{d}-\mathbf{z}} G^p(\mathbf{y})d\mathbf{y} = \left(\frac{c\sqrt{\pi t}}{\sqrt{p}}\right)^n \left[\operatorname{erf}\left(\frac{\sqrt{p}(\mathbf{d}-\mathbf{z})}{2c\sqrt{t}}\right) - \operatorname{erf}\left(\frac{\sqrt{p}(\mathbf{c}-\mathbf{z})}{2c\sqrt{t}}\right)\right],$$

where

$$\operatorname{erf}(\mathbf{x}) = \left(\frac{2}{\sqrt{\pi}}\right)^n \int_{\mathbb{R}^n_+(\mathbf{x})} e^{-\mathbf{z}^2} d\mathbf{z}$$

is the error function. Therefore, for $-\mathbf{a} \leqslant \mathbf{c} < \mathbf{d} \leqslant \mathbf{a}$, the inequality (2.18) yields

$$\int_{\mathbf{c}}^{\mathbf{d}} u(\mathbf{x},t)^p d\mathbf{x} \geqslant \frac{1}{2^{np}(\sqrt{p})^n(c\sqrt{\pi t})^{n(p-1)}} \left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-np} \left(\int_{-\mathbf{b}}^{\mathbf{b}} \rho(\mathbf{z})d\mathbf{z}\right)^{p-1} \int_{-\mathbf{b}}^{\mathbf{b}} F^p(\mathbf{z})\rho(\mathbf{z}) \left[\operatorname{erf}\left(\frac{\sqrt{p}(\mathbf{d}-\mathbf{z})}{2c\sqrt{t}}\right) - \operatorname{erf}\left(\frac{\sqrt{p}(\mathbf{c}-\mathbf{z})}{2c\sqrt{t}}\right)\right] d\mathbf{z}, \tag{3.43}$$

where ρ is a positive continuous function on $[-\mathbf{b},\mathbf{b}]$, and F satisfies (3.42).

Next, we consider the integral transform

$$u(\mathbf{x},t) = \frac{1}{(2c\sqrt{\pi})^n} \int_0^t \frac{1}{(t-\tau)^{\frac{n}{2}}} \int_{\mathbb{R}^n} F(\mathbf{z},t)\rho(\mathbf{z},t) \exp\left\{-\frac{|\mathbf{x}-\mathbf{z}|^2}{4c^2(t-\tau)}\right\} d\mathbf{z}d\tau \tag{3.44}$$

which gives the solution $u(\mathbf{x},t)$ of the non-homogeneous heat equation

$$u_t - c^2\Delta_n u(\mathbf{x},t) = F(\mathbf{x},t)\rho(\mathbf{x},t) \quad (\mathbf{x},t) \in \mathbb{R}^n \times \mathbb{R}_+ \tag{3.45}$$

satisfying the condition

$$u(\mathbf{x},0) = 0. \tag{3.46}$$

Take

$$G(\mathbf{z},\tau) = \frac{1}{(t-\tau)^{\frac{n}{2}}} \exp\left\{-\frac{|\mathbf{z}|^2}{\mathbf{4c^2(t-\tau)}}\right\}.$$

Let

$$-\mathbf{a} \leqslant \mathbf{x} \leqslant \mathbf{a}, \quad -\mathbf{b} \leqslant \mathbf{z} \leqslant \mathbf{b}, \quad |\mathbf{a}+\mathbf{b}|^2 \geqslant \frac{4c^2(t-\tau)}{p} \log\left(\frac{M}{m}\right), \quad 0 \leqslant \tau < t.$$

From

$$1 \leqslant \exp\left\{\frac{|\mathbf{x}-\mathbf{z}|^2}{4c^2(t-\tau)}\right\} \leqslant \exp\left\{\frac{|\mathbf{a}+\mathbf{b}|^2}{4c^2(t-\tau)}\right\},$$

we have

$$0 < m^{\frac{1}{p}} \leqslant F(\mathbf{z})\frac{1}{(t-\tau)^{\frac{n}{2}}} \exp\left\{-\frac{|\mathbf{x}-\mathbf{z}|^2}{4c^2(t-\tau)}\right\} \leqslant M^{\frac{1}{p}}, \tag{3.47}$$

if

$$m^{\frac{1}{p}}(t-\tau)^{\frac{n}{2}} \exp\left\{\frac{|\mathbf{a}+\mathbf{b}|^2}{4c^2(t-\tau)}\right\} \leqslant F(\mathbf{z}) \leqslant M^{\frac{1}{p}}(t-\tau)^{\frac{n}{2}}, \quad -\mathbf{b} \leqslant \mathbf{z} \leqslant \mathbf{b}, \quad 0 \leqslant \tau < t. \tag{3.48}$$

Therefore, for $-\mathbf{a} \leqslant \mathbf{c} < \mathbf{d} \leqslant \mathbf{a}$, the inequality (2.18) yields

$$\int_{\mathbf{c}}^{\mathbf{d}} u(\mathbf{x},t)^p d\mathbf{x} \geqslant \frac{1}{2^{np}(\sqrt{p})^n(c\sqrt{\pi})^{n(p-1)}} \left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-np} \left(\int_0^t \int_{-\mathbf{b}}^{\mathbf{b}} \rho(\mathbf{z},\tau)d\mathbf{z}d\tau\right)^{p-1}$$

$$\int_0^t \frac{1}{\sqrt{t-\tau}^{n(p-1)}} \int_{-\mathbf{b}}^{\mathbf{b}} F^p(\mathbf{z},\tau)\rho(\mathbf{z},\tau) \left[\operatorname{erf}\left(\frac{\sqrt{p}(\mathbf{d}-\mathbf{z})}{2c\sqrt{t-\tau}}\right) - \operatorname{erf}\left(\frac{\sqrt{p}(\mathbf{c}-\mathbf{z})}{2c\sqrt{t-\tau}}\right)\right] d\mathbf{z}d\tau, \tag{3.49}$$

where ρ is a positive continuous function and F satisfies (3.48).

3.4. Laplace Equation in a half-space of $\mathbb{R}^{n+1}$

We consider the Dirichlet problem for the Laplace equation in a half-space of $\mathbb{R}^{n+1}$, i.e. the determination of the bounded solution of

$$\Delta_{n+1}u(\mathbf{x},t) = 0, \quad (\mathbf{x},t) \in \mathbb{R}^n \times \mathbb{R}_+ \tag{3.50}$$

with the boundary condition

$$u(\mathbf{x},0) = F(\mathbf{x})\rho(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n, \tag{3.51}$$

we have the solution of the Dirichlet problem (3.50), (3.51) in the form

$$u(\mathbf{x},t) = \frac{2t}{\omega_{n+1}} \int_{\mathbb{R}^n} \frac{F(\mathbf{z})\rho(\mathbf{z})}{(t^2+|\mathbf{x}-\mathbf{z}|^2)^{\frac{n+1}{2}}} d\mathbf{z}, \tag{3.52}$$

where

$$\omega_n = \frac{1}{2\pi^{\frac{n+1}{2}}}\Gamma\left(\frac{n+1}{2}\right).$$

Take

$$G(\mathbf{z}) = \frac{1}{[t^2+|\mathbf{z}|^2]^{\frac{n+1}{2}}}.$$

Let

$$\mathbf{a} \leqslant \mathbf{x} \leqslant \mathbf{b}, \quad \mathbf{c} \leqslant \mathbf{z} \leqslant \mathbf{d}.$$

Denote

$$\alpha = \max\{|\mathbf{a}^\mathbf{i}-\mathbf{c}^\mathbf{i}|, |\mathbf{a}^\mathbf{i}-\mathbf{d}^\mathbf{i}|, |\mathbf{b}^\mathbf{i}-\mathbf{c}^\mathbf{i}|, |\mathbf{b}^\mathbf{i}-\mathbf{d}^\mathbf{i}|\},$$
$$\beta = \max\{|\mathbf{a}-\mathbf{c}|, |\mathbf{a}-\mathbf{d}|, |\mathbf{b}-\mathbf{c}|, |\mathbf{b}-\mathbf{d}|\}.$$

Then, we have

$$\frac{1}{[t^2+\alpha^2+(x_i-z_i)^2]^{\frac{n+1}{2}}} \leqslant \frac{1}{[t^2+|\mathbf{x}-\mathbf{z}|^2]^{\frac{n+1}{2}}} \leqslant \frac{1}{[t^2+(x_i-z_i)^2]^{\frac{n+1}{2}}}$$

and

$$\frac{1}{[t^2+\beta^2]^{\frac{n+1}{2}}} \leqslant \frac{1}{[t^2+|\mathbf{x}-\mathbf{z}|^2]^{\frac{n+1}{2}}} \leqslant \frac{1}{t^{n+1}}.$$

Hence, for a function F satisfying

$$\left[t^2+\alpha^2+(x_i-z_i)^2\right]^{\frac{n+1}{2}} m^{\frac{1}{p}} \leqslant F(\mathbf{z}) \leqslant \left[t^2+(x_i-z_i)^2\right]^{\frac{n+1}{2}} M^{\frac{1}{p}}$$

and for a positive continuous function ρ on $[\mathbf{c},\mathbf{d}]$, we obtain

$$\int_{\mathbf{a}^\mathbf{i}}^{\mathbf{b}^\mathbf{i}} u^p(\mathbf{x},t)d\mathbf{x}^\mathbf{i} \geqslant (\mathbf{b}^\mathbf{i}-\mathbf{a}^\mathbf{i}) \left(\frac{2t}{\omega_{n+1}}\right)^p \left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-np} \left(\int_{\mathbf{c}}^{\mathbf{d}} \rho(\mathbf{z})d\mathbf{z}\right)^{p-1} \int_{\mathbf{c}}^{\mathbf{d}} \frac{F^p(\mathbf{z})\rho(\mathbf{z})}{[t^2+\alpha^2+(x_i-z_i)^2]^{\frac{p(n+1)}{2}}} d\mathbf{z}. \tag{3.53}$$

Moreover, if the function F satisfies

$$\left[t^2+\beta^2\right]^{\frac{n+1}{2}} m^{\frac{1}{p}} \leqslant F(\mathbf{z}) \leqslant t^{n+1} M^{\frac{1}{p}}$$

and for a positive continuous function ρ on $[\mathbf{c},\mathbf{d}]$, we obtain

$$\begin{aligned}\int_{\mathbf{a}}^{\mathbf{b}} u(\mathbf{x},t)^p d\mathbf{x} \geqslant \left(\frac{2t}{\omega_{n+1}}\right)^p \frac{(\mathbf{b}-\mathbf{a})}{[t^2+\beta^2]^{\frac{p(n+1)}{2}}} \left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-np} \\ \left(\int_{\mathbf{c}}^{\mathbf{d}} \rho(\mathbf{z}) d\mathbf{z}\right)^{p-1} \int_{\mathbf{c}}^{\mathbf{d}} F^p(\mathbf{z})\rho(\mathbf{z}) d\mathbf{z}.\end{aligned} \tag{3.54}$$

In the conjugate Poisson integral transform

$$v_i(\mathbf{x},t) = \frac{2}{\omega_{n+1}} \int_{\mathbb{R}^n} F(\mathbf{z})\rho(\mathbf{z}) \frac{x_i - z_i}{[t^2+|\mathbf{x}-\mathbf{z}|^2]^{\frac{n+1}{2}}} d\mathbf{z}, \tag{3.55}$$

take

$$G(\mathbf{x}) = \frac{x_i - z_i}{[t^2+|\mathbf{x}|^2]^{\frac{n+1}{2}}}.$$

Let

$$\mathbf{a} \leqslant \mathbf{x} \leqslant \mathbf{b}, \quad \mathbf{c} \leqslant \mathbf{z} \leqslant \mathbf{d}, \quad (d_i \leqslant a_i).$$

Denote

$$\alpha = \max\{|\mathbf{a^i}-\mathbf{c^i}|, |\mathbf{a^i}-\mathbf{d^i}|, |\mathbf{b^i}-\mathbf{c^i}|, |\mathbf{b^i}-\mathbf{d^i}|\}.$$

Then, we have

$$\frac{x_i - z_i}{[t^2+\alpha^2+(x_i-z_i)^2]^{\frac{n+1}{2}}} \leqslant \frac{x_i - z_i}{[t^2+|\mathbf{x}-\mathbf{z}|^2]^{\frac{n+1}{2}}} \leqslant \frac{x_i - z_i}{[t^2+(x_i-z_i)^2]^{\frac{n+1}{2}}}$$

and

$$\frac{a_i - d_i}{[t^2+\alpha^2+(b_i-c_i)^2]^{\frac{n+1}{2}}} \leqslant \frac{x_i - z_i}{[t^2+|\mathbf{x}-\mathbf{z}|^2]^{\frac{n+1}{2}}} \leqslant \frac{b_i - c_i}{[t^2+(a_i-d_i)^2]^{\frac{n+1}{2}}}.$$

Hence, for a function F satisfying

$$\frac{\left[t^2+\alpha^2+(x_i-z_i)^2\right]^{\frac{n+1}{2}}}{x_i-z_i} m^{\frac{1}{p}} \leqslant F(\mathbf{z}) \leqslant \frac{\left[t^2+(x_i-z_i)^2\right]^{\frac{n+1}{2}}}{x_i-z_i} M^{\frac{1}{p}}$$

and for a positive continuous function ρ on $[\mathbf{c},\mathbf{d}]$, we obtain

$$\begin{aligned}\int_{\mathbf{a^i}}^{\mathbf{b^i}} v_i(\mathbf{x},t)^p d\mathbf{x^i} \geqslant (\mathbf{d^i}-\mathbf{c^i})^p \left(\frac{2}{\omega_{n+1}}\right)^p \left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-np} \left(\int_{\mathbf{c}}^{\mathbf{d}} \rho(\mathbf{z}) d\mathbf{z}\right)^{p-1} \\ \int_{\mathbf{c}}^{\mathbf{d}} \frac{F^p(\mathbf{z})\rho(\mathbf{z})(x_i-z_i)^p}{[t^2+\alpha^2+(x_i-z_i)^2]^{\frac{p(n+1)}{2}}} d\mathbf{z}.\end{aligned} \tag{3.56}$$

Moreover, if the function F satisfies

$$\frac{\left[t^2+\alpha^2+(b_i-c_i)^2\right]^{\frac{n+1}{2}}}{a_i-d_i}m^{\frac{1}{p}} \leqslant F(\mathbf{z}) \leqslant \frac{\left[t^2+(a_i-d_i)^2\right]^{\frac{n+1}{2}}}{b_i-c_i}M^{\frac{1}{p}}$$

and for a positive continuous function ρ on $[\mathbf{c},\mathbf{d}]$, we obtain

$$\begin{aligned}\int_{\mathbf{a}}^{\mathbf{b}} v_i(\mathbf{x},t)^p d\mathbf{x} \geqslant \left(\frac{2}{\omega_{n+1}}\right)^p \frac{(\mathbf{b}-\mathbf{a})(a_i-d_i)^p}{\left[t^2+\alpha^2+(b_i-c_i)^2\right]^{\frac{p(n+1)}{2}}} \left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-np} \\ \left(\int_{\mathbf{c}}^{\mathbf{d}} \rho(\mathbf{z})d\mathbf{z}\right)^{p-1} \int_{\mathbf{c}}^{\mathbf{d}} F^p(\mathbf{z})\rho(\mathbf{z})d\mathbf{z}.\end{aligned} \tag{3.57}$$

Consider now the integral transform

$$u(\mathbf{x}) = \frac{-1}{(n-2)\omega_n}\int_{\mathbb{R}^n}\frac{F(\mathbf{z})\rho(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^{n-2}}d\mathbf{z} \tag{3.58}$$

and this is the solution of the Poisson equation

$$\Delta_n u(\mathbf{x}) = F(\mathbf{x})\rho(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^n \ (n\geqslant 3). \tag{3.59}$$

Take $G(\mathbf{z}) = \frac{1}{|\mathbf{z}|^{n-2}}$. Let

$$\mathbf{a}\leqslant\mathbf{x}\leqslant\mathbf{b}, \quad \mathbf{c}\leqslant\mathbf{z}\leqslant\mathbf{d}, \quad (\mathbf{b}<\mathbf{c} \text{ or } \mathbf{d}<\mathbf{a}), \quad \frac{\alpha}{\beta}\leqslant\frac{1}{p(n-2)}\log\frac{M}{m}.$$

Here

$$\alpha = \max\{|\mathbf{a}-\mathbf{d}|, |\mathbf{b}-\mathbf{c}|\}, \quad \beta = \min\{|\mathbf{a}-\mathbf{d}|, |\mathbf{b}-\mathbf{c}|\}.$$

Then, we have

$$\frac{1}{\alpha^{n-2}} \leqslant \frac{F(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^{n-2}} \leqslant \frac{1}{\beta^{n-2}}.$$

Hence, for a function F satisfying

$$\alpha^{n-2}m^{\frac{1}{p}} \leqslant F(\mathbf{z}) \leqslant \beta^{n-2}M^{\frac{1}{p}}$$

and for a positive continuous function ρ on $[\mathbf{c},\mathbf{d}]$, we obtain

$$\begin{aligned}\int_{\mathbf{a}}^{\mathbf{b}} |u(\mathbf{x})|^p d\mathbf{x} \geqslant \left(\frac{1}{(n-2)\omega_n}\right)^p \frac{\mathbf{d}-\mathbf{c}}{\alpha^{\mathbf{p}(\mathbf{n}-\mathbf{2})}} \left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-np} \\ \left(\int_{\mathbf{c}}^{\mathbf{d}} \rho(\mathbf{z})d\mathbf{z}\right)^{p-1} \int_{\mathbf{c}}^{\mathbf{d}} F^p(\mathbf{z})\rho(\mathbf{z})d\mathbf{z}.\end{aligned} \tag{3.60}$$

3.5. Biharmonic equation

The solution of the biharmonic equation

$$\Delta_{n+1}^2 u(\mathbf{x},t) = 0, \quad (\mathbf{x},t) \in \mathbb{R}^n \times \mathbb{R}_+, \quad \Delta_{n+1}^2 = \Delta_{n+1}(\Delta_{n+1}) \tag{3.61}$$

with the boundary conditions

$$u(\mathbf{x},0) = F(\mathbf{x})\rho(\mathbf{x}), \quad u_t(\mathbf{x},0) = 0 \tag{3.62}$$

is given by

$$u(\mathbf{x},t) = \frac{2(n+1)t^3}{\omega_{n+1}} \int_{\mathbb{R}^n} \frac{F(\mathbf{z})\rho(\mathbf{z})}{(t^2+|\mathbf{x}-\mathbf{z}|^2)^{\frac{n+3}{2}}} d\mathbf{z}. \tag{3.63}$$

Take

$$G(\mathbf{z}) = \frac{1}{[t^2+|\mathbf{z}|^2]^{\frac{n+3}{2}}}.$$

Let

$$\mathbf{a} \leqslant \mathbf{x} \leqslant \mathbf{b}, \quad \mathbf{c} \leqslant \mathbf{z} \leqslant \mathbf{d}.$$

Denote

$$\alpha = \max\{|\mathbf{a}^\mathbf{i}-\mathbf{c}^\mathbf{i}|, |\mathbf{a}^\mathbf{i}-\mathbf{d}^\mathbf{i}|, |\mathbf{b}^\mathbf{i}-\mathbf{c}^\mathbf{i}|, |\mathbf{b}^\mathbf{i}-\mathbf{d}^\mathbf{i}|\},$$
$$\beta = \max\{|\mathbf{a}-\mathbf{c}|, |\mathbf{a}-\mathbf{d}|, |\mathbf{b}-\mathbf{c}|, |\mathbf{b}-\mathbf{d}|\}.$$

Then, we have

$$\frac{1}{[t^2+\alpha^2+(x_i-z_i)^2]^{\frac{n+3}{2}}} \leqslant \frac{1}{[t^2+|\mathbf{x}-\mathbf{z}|^2]^{\frac{n+3}{2}}} \leqslant \frac{1}{[t^2+(x_i-z_i)^2]^{\frac{n+3}{2}}},$$

and

$$\frac{1}{[t^2+\beta^2]^{\frac{n+3}{2}}} \leqslant \frac{1}{[t^2+|\mathbf{x}-\mathbf{z}|^2]^{\frac{n+3}{2}}} \leqslant \frac{1}{t^{n+3}}.$$

Hence, for a function F satisfying

$$\left[t^2+\alpha^2+(x_i-z_i)^2\right]^{\frac{n+3}{2}} m^{\frac{1}{p}} \leqslant F(\mathbf{z}) \leqslant \left[t^2+(x_i-z_i)^2\right]^{\frac{n+3}{2}} M^{\frac{1}{p}}$$

and for a positive function ρ on $[\mathbf{c},\mathbf{d}]$, we obtain

$$\int_{\mathbf{a}^\mathbf{i}}^{\mathbf{b}^\mathbf{i}} u(\mathbf{x},t)^p d\mathbf{x}^\mathbf{i} \geqslant (\mathbf{b}^\mathbf{i}-\mathbf{a}^\mathbf{i}) \left(\frac{2(n+1)t^3}{\omega_{n+1}}\right)^p \left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-np} \left(\int_{\mathbf{c}}^{\mathbf{d}} \rho(\mathbf{z})d\mathbf{z}\right)^{p-1}$$
$$\int_{\mathbf{c}}^{\mathbf{d}} \frac{F^p(\mathbf{z})\rho(\mathbf{z})}{[t^2+\alpha^2+(x_i-z_i)^2]^{\frac{p(n+3)}{2}}} d\mathbf{z}. \tag{3.64}$$

Moreover, if the function F satisfies

$$\left[t^2+\beta^2\right]^{\frac{n+3}{2}} m^{\frac{1}{p}} \leqslant F(\mathbf{z}) \leqslant t^{n+3} M^{\frac{1}{p}}$$

and for a positive continuous function ρ on $[\mathbf{c},\mathbf{d}]$, we obtain

$$\int_{\mathbf{a}}^{\mathbf{b}} u(\mathbf{x},t)^p d\mathbf{x} \geqslant \left(\frac{2(n+1)t^3}{\omega_{n+1}}\right)^p \frac{(\mathbf{b}-\mathbf{a})}{[t^2+\beta^2]^{\frac{p(n+3)}{2}}} \left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-np} I, \tag{3.65}$$

where

$$I = \left(\int_{\mathbf{c}}^{\mathbf{d}} \rho(\mathbf{z})d\mathbf{z}\right)^{p-1} \int_{\mathbf{c}}^{\mathbf{d}} F^p(\mathbf{z})\rho(\mathbf{z})d\mathbf{z}.$$

Acknowledgments. The authors wish to express their deep thanks to Professor S. Saitoh for his valuable advice and suggestions.

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(Received October 15, 2007)

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