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ON LOGARITHMIC CONVEXITY FOR DIFFERENCES OF POWER MEANS AND RELATED RESULTS

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Abstract. We give some further considerations about logarithmic convexity for differences of power Means for positive linear functionals as well as some related results.

1. Introduction

Let $\{x_1, x_2, ..., x_n\}$ and $\{p_1, p_2, ..., p_n\}$ denote two sequences of positive real numbers with $\sum_{i=1}^n p_i = 1$. The well known Jensen's Inequality states for $t < 0$ or $t > 1$,

$$\sum_{i=1}^n p_i x_i^t \geqslant \left(\sum_{i=1}^n p_i x_i\right)^t \tag{1.1}$$

with reversed sign for $0 < t < 1$ (see e.g. [2], [3]). S. Simić [6] has consider the difference of the expression of both sides of (1.1), and stated the following theorem.

THEOREM 1. ([6], Theorem 2.2.) *Let x_i, p_i, $i = 1, ..., n$, $P_n = \sum_{i=1}^n p_i = 1$ be positive real numbers and let $-\infty < r < s < t < \infty$. Then*

$$(\lambda_s)^{t-r} \leqslant (\lambda_r)^{t-s} (\lambda_t)^{s-r} \tag{1.2}$$

where

$$\lambda_t = \begin{cases} \frac{1}{t(t-1)}\left[\sum_{i=1}^n p_i x_i^t - \left(\sum_{i=1}^n p_i x_i\right)^t\right], & t \in \mathbb{R} \setminus \{0,1\}; \\ \log\left(\sum_{i=1}^n p_i x_i\right) - \sum_{i=1}^n p_i \log x_i, & t = 0; \\ \sum_{i=1}^n p_i x_i \log x_i - \left(\sum_{i=1}^n p_i x_i\right) \log \sum_{i=1}^n p_i x_i, & t = 1. \end{cases} \tag{1.3}$$

Let $p > 1$ and q is defined by $\frac{1}{p} + \frac{1}{q} = 1$, then the well known Holder's inequality is,

$$\sum_{i=1}^n |x_i y_i| \leqslant \left[\sum_{i=1}^n |x_i|^p\right]^{\frac{1}{p}} \left[\sum_{i=1}^n |y_i|^q\right]^{\frac{1}{q}}. \tag{1.4}$$

By using Theorem 1, S. Simić [6] proved the following converse of (1.4).

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THEOREM 2. *Let* a_i, b_i, $i = 1,2....$ *be arbitrary sequences of positive real numbers and* $\frac{1}{p}+\frac{1}{q}=1$, $p>1$. *Then*

$$pq\Big[(\sum a_i^p)^{\frac{1}{p}}(\sum b_i^q)^{\frac{1}{q}} - \sum a_ib_i\Big] \leqslant \Big(\sum a_i^p \log\frac{a_i^p}{b_i^q} - (\sum a_i^p)\log\frac{\sum a_i^p}{\sum b_i^q}\Big)^{\frac{1}{p}}$$
$$\Big(\sum b_i^q \log\frac{b_i^q}{a_i^p} - (\sum b_i^q)\log\frac{\sum b_i^q}{\sum a_i^q}\Big)^{\frac{1}{q}}. \quad (1.5)$$

Integral version of (1.2) is also obtained but only in the case $0<r<s<t$, r, s, $t\neq 1$.

However, there is a lack in the proof of Theorem 2.2. in [6], the log-convexity of the functions λ_s is not justified.

In this paper we shall give a correct proof of Theorem 1, introducing family of functions φ_s. Also, we shall give extension of these results to the case of positive linear functionals, and inequalities different from power-mean type inequalities.

2. Main Results

Let E be a nonempty set and L be a linear class of real valued functions $f : E \to R$ having the properties:

$$f,g \in L \implies (af+bg)\in L \quad \forall a,b\in\mathbb{R}, \quad \text{(L1)}$$

$$1\in L, \quad \text{where } 1(t)=1 \text{ for all } t\in E. \quad \text{(L2)}$$

A positive linear functional is a mapping $A : L \longrightarrow R$ with properties

$$A(af+bg) = aA(f)+bA(g) \quad \text{for} \quad f,g\in L,\ a,b\in\mathbb{R}, \quad \text{(A1)}$$

$$f\in L, f(t)\geqslant 0 \quad \text{on} \quad E \implies A(f)\geqslant 0. \quad \text{(A2)}$$

If $A(1)=1$ we say that A is a normalized functional. Jessen (see [4], p–47) gave the following generalization of Jensen's inequality for convex functions.

THEOREM 3. *Let* L *satisfy L1, L2 on a nonempty set* E, *and assume that* ϕ *is a continuous convex function on an interval* $I\subset\mathbb{R}$. *If A is a linear positive functional with* $A(1)=1$ *then for all* $f\in L$ *such that* $\phi(f)\in L$ *we have* $A(f)\in I$ *and*

$$\phi(A(f)) \leqslant A(\phi(f)). \quad (2.1)$$

Now we shall give some generalization of Theorem 1.

LEMMA 1. *Let us define the function*

$$\varphi_s(x) = \begin{cases} \frac{x^s}{s(s-1)}, & s\neq 0,1;\\ -\log x, & s=0;\\ x\log x, & s=1.\end{cases} \quad (2.2)$$

Then $\varphi_s'' = x^{s-2}$, *that is* φ_s *is convex for* $x>0$.

THEOREM 4. *Let L satisfy properties L1, L2 on a nonempty set E. Let a positive function $f \in L$ be such that $f^r \in L$ for $r \in I \setminus \{0,1\}$, I is an interval from $\mathbb{R}$, $\log f \in L$ if $r = 0$ and $f \log f \in L$ if $r = 1$. Let us define*

$$\Lambda_t = A(\varphi_t(f)) - \varphi_t(A(f)), \tag{2.3}$$

and let Λ_t be positive.

(1) *For all $s,t \in I$ we have*

$$\Lambda^2_{\frac{s+t}{2}} \leqslant \Lambda_s \Lambda_t \tag{2.4}$$

that is Λ_t is log-convex in Jensen sense.

(2) *If Λ_t is continuous on I, then it is also log-convex. That is, for $r < s < t$ $(r,s,t \in I)$ we have*

$$(\Lambda_s)^{t-r} \leqslant (\Lambda_r)^{t-s}(\Lambda_t)^{s-r}. \tag{2.5}$$

Proof. (1) We shall use the idea from [6, Theorem 2.2]. Let us consider the function defined by

$$f(x) = u^2\varphi_s(x) + 2uw\varphi_r(x) + w^2\varphi_t(x),$$

where $r = \frac{s+t}{2}$, $u,w \in \mathbb{R}$ and φ_s is given by (2.2). We have

$$f''(x) = u^2x^{s-2} + 2uwx^{r-2} + w^2x^{t-2} = (ux^{\frac{s}{2}-1} + wx^{\frac{t}{2}-1})^2 \geqslant 0,\ x > 0.$$

Therefore f is convex for $x > 0$. Inequality (2.1) gives

$$u^2\varphi_s(A(f)) + 2uw\varphi_r(A(f)) + w^2\varphi_t(A(f)) \\ \leqslant u^2A(\varphi_s(f)) + 2uwA(\varphi_r(f)) + w^2A(\varphi_t(f))$$

i.e.

$$u^2\Lambda_s + 2uw\Lambda_r + w^2\Lambda_t \geqslant 0$$

therefore we get (2.4).

(2) Since Λ_t is log-convex in Jensen-sense, if it is continuous it is also log-convex. Therefore (2.5) is valid, too. □

REMARK 1. In applications of Theorem 4 (as well as other similar results throughout the paper) we shall assume that similar conditions about positivity of Λ_t are satisfied and all expressions obtained from Λ_t are positive as well.

In the next corollary and two theorems we shall suppose that functional A is such that continuity property for Theorem 4 (2) is satisfied on appropriate interval.

COROLLARY 1. *We have*

(1) *For $s > 3$*

$$A(f^s) \geqslant (A(f))^s + \binom{s}{2}\left(\frac{d_3}{3d_2}\right)^{s-2} d_2; \tag{2.6}$$

(2) *for* $0 < s < 1$

$$A(f^s) \leqslant (A(f))^s - \frac{s(1-s)}{2}\left(\frac{3d_2}{d_3}\right)^{2-s} d_2 \tag{2.7}$$

where $d_k = A(f^k) - (A(f))^k,\ k = 2,3$.

Proof. Applying Theorem 4 (2) with $2 < 3 < s$ and $0 < s < 1 < 2 < 3$, respectively. □

THEOREM 5. *Let* L *satisfy conditions L1, L2, and A satisfy conditions A1, A2 on a base set* E. *Let* $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, *if* $f,g > 0$ *and* f^p, g^q, fg, $f^p \log f^p$, $f^p \log g^q$, $g^q \log g^q$, $g^q \log f^p \in L$ *then we have*

$$\begin{aligned} pq\left[(A(f^p))^{\frac{1}{p}}(A(g^q))^{\frac{1}{q}} - A(fg)\right] \ \leqslant\ & \left(A\left(f^p \log\frac{f^p}{g^q}\right) - A(f^p)\log\frac{A(f^p)}{A(g^q)}\right)^{\frac{1}{p}} \\ & \left(A\left(g^q \log\frac{g^q}{f^p}\right) - A(g^q)\log\frac{A(g^q)}{A(f^p)}\right)^{\frac{1}{q}}. \end{aligned} \tag{2.8}$$

For $0 < p < 1$ *the inequality (2.8) is reversed.*

Proof. As in [6], from Theorem 4, for $r = 0$, $s = s$, $t = 1$, we get

$$\Lambda_s \leqslant (\Lambda_0)^{1-s}(\Lambda_1)^s.$$

Set $A(f) = \frac{A(wf)}{A(w)}$ we will get

$$\begin{aligned} \frac{1}{s(1-s)}\left[\left(\frac{A(wf)}{A(w)}\right)^s - \frac{A(wf^s)}{A(w)}\right] \ \leqslant\ & \left[\log\left(\frac{A(wf)}{A(w)}\right) - \frac{A(w\log f)}{A(w)}\right]^{1-s} \\ & \left[\frac{A(wf\log f)}{A(w)} - \left(\frac{A(wf)}{A(w)}\right)\log\left(\frac{A(wf)}{A(w)}\right)\right]^s. \end{aligned}$$

Putting $s = \frac{1}{p}$, $1-s = \frac{1}{q}$; $w \to \frac{g^q}{A(g^q)}$, $f \to \frac{f^p}{g^q}$. After some calculation we obtain the inequality (2.8). □

Moreover we can give another version of converse of Holder's inequality, that is the following theorem.

THEOREM 6. *Let* L *satisfy conditions L1, L2, and A satisfy conditions A1, A2 on a base set* E. *Let* $1 < p < 2$, $\frac{1}{p}+\frac{1}{q} = 1$, *if* $f,g > 0$ *and* f^p, g^q, fg, $fg\log fg^{1-q}$, $f^2g^{2-q} \in L$ *then we have*

$$\begin{aligned} &\frac{1}{p(p-1)}\left(\left((A(f^p))^{\frac{1}{p}}(A(g^q))^{\frac{1}{q}}\right)^p - A(fg)^p\right) \\ &\qquad \leqslant \frac{1}{2^{p-1}}\left(A(f^2g^{2-q})A(g^q) - A(fg)^2\right)^{p-1} \\ &\qquad\quad \left(A(fg\log fg^{1-q}) - A(fg)\log\left(\frac{A(fg)}{A(g^q)}\right)\right)^{2-p}. \end{aligned} \tag{2.9}$$

For $p > 2$ *the above inequality is reversed.We have equality for* $p = 2$.

Proof. As in [6], from Theorem 4, for $r = 1$, $s = p$, $t = 2$, we get

$$\Lambda_p \leqslant (\Lambda_2)^{p-1}(\Lambda_1)^{2-p}.$$

Set $A(f) = \frac{A(wf)}{A(w)}$ we will get

$$\frac{1}{p(p-1)}\Big(\frac{A(wf^p)}{A(w)} - \Big(\frac{A(wf)}{A(w)}\Big)^p\Big) \leqslant \frac{1}{2^{p-1}}\Big(\frac{A(wf^2)}{A(w)} - \Big(\frac{A(wf)}{A(w)}\Big)^2\Big)^{p-1} \Big(\frac{A(wf\log f)}{A(w)} - \frac{A(wf)}{A(w)}\log\frac{A(wf)}{A(w)}\Big)^{2-p}.$$

Putting $w = g^q$, $f = f\,g^{1-q}$. After some calculation we obtain the inequality (2.9). □

REMARK 2. This result is the conversion of Theorem 4.12 in [4], p–113.

Let us note that the well known Jensen-Steffensen inequality is valid (see, for example [4], pp. 57–58).

THEOREM 7. *If $f : I \to \mathbb{R}$ is a convex function, $(x_1,...,x_n)$ is a real monotonic n-tuple such that $x_i \in I$ $(i = 1,...,n)$, and $(p_1,...,p_n)$ is a real n-tuple such that $P_k = \sum_{i=1}^{k} p_i$ for $1 \leqslant k \leqslant n$ and*

$$0 \leqslant P_k \leqslant P_n = 1 \quad (k = 1,..,n) \tag{2.10}$$

is satisfied. Then,

$$f\Big(\sum_{i=1}^{n} p_i x_i\Big) \leqslant \sum_{i=1}^{n} p_i f(x_i). \tag{2.11}$$

As in proof of Theorem 4 we can get.

THEOREM 8. *Let $(x_1,...,x_n)$ be monotonic n-tuple of positive numbers, $p_i \in \mathbb{R}$ such that (2.11) be valid and let $-\infty < r < s < t < +\infty$. Then (1.2) is still valid.*

Moreover, we can also use related integral analogues of Jensen-Steffensen inequality and generalizations (see Jensen-Steffensen's, Jensen-Boas and Jensen-Brunk inequalities as well as Theorem 2.26 from [4], pp. 59–65).

LEMMA 2. *Let us define the function*

$$\phi_t(x) = \begin{cases} \frac{1}{t^2}e^{tx}, & t \neq 0, \\ \frac{1}{2}x^2, & t = 0. \end{cases}$$

Then $\phi_t''(x) = e^{tx}$, that is $\phi_t(x)$ is a convex.

THEOREM 9. *Theorem 4 and 8 are still valid if we set $\varphi_s = \phi_s$.*

Proof. As in proof of Theorem 4 we consider the function,

$$f(x) = u^2\phi_s(x) + 2uw\phi_r(x) + w^2\phi_t(x),$$

where $r = \frac{s+t}{2}$, $u, w \in \mathbb{R}$. We have $f''(x) = (ue^{\frac{s}{2}x} + we^{\frac{t}{2}x})^2 > 0$ so that f is convex. Therefore by using (2.1) we get required results. □

REMARK 3. In fact we can use substitution $f = \log g$ in previous theorem. So we have that

$$\Lambda_t = \begin{cases} \frac{1}{t^2}\Big[M_t^t(g,A) - M_0^t(g,A)\Big], & t \neq 0 \\ \\ \frac{1}{2}\Big[M_2^2(\log g, A) - (M_1(\log g, A))^2\Big], & t = 0 \end{cases}$$

where,

$$M_t(g,A) = \begin{cases} (A(g^t))^{\frac{1}{t}}, & t \neq 0 \\ \exp A(\log g), & t = 0 \end{cases}$$

is the power mean of g with respect to positive linear functional A and Λ_t be positive, is also log-convex.

3. Some Results of Aczél's type

The following version of Jensen's inequality is valid ([4], p. 124–125).

THEOREM 10. *Let L satisfy conditions L1, L2 and A satisfy conditions A1, A2 on a base set E suppose that $w \in L$ with $w \geqslant 0$ on E and $0 < A(w) < u \in \mathbb{R}$, $\frac{(ua - A(wf))}{u - A(w)} \in I$, $a \in I$. When I is an interval $I \subset \mathbb{R}$ and f is an arbitrary real function defined on E such that $wf \in L$. Suppose that ψ is a continuous convex function on I and $w\psi(f) \in L$. Then*

$$\psi\Big(\frac{ua - A(wf)}{u - A(w)}\Big) \geqslant \frac{u\psi(a) - A(w\psi(f))}{u - A(w)}. \tag{3.1}$$

Similarly as in the proof of Theorem 4 we can prove:

THEOREM 11. *Let the conditions of Theorem 10 be satisfied for an interval $I \subseteq (0, +\infty)$ for function $\psi = \varphi_s$ (as is defined by (2.2)) for $s \in J$ is some interval in $\mathbb{R}$. Let us define*

$$\Omega_t = \varphi_t\Big(\frac{ua - A(wf)}{u - A(w)}\Big) - \frac{u\varphi_t(a) - A(w\varphi_t(f))}{u - A(w)}.$$

(1) *For all $s, t \in J$ we have*

$$\Omega^2_{\frac{s+t}{2}} \leqslant \Omega_s \Omega_t$$

that is Ω_t is log-convex in Jensen-sense.

(2) *If Ω_t is continuous on J, then it is also log-convex, i.e for $r < s < t$ $(r, s, t \in J)$ we have*

$$(\Omega_s)^{t-r} \leqslant (\Omega_r)^{t-s}(\Omega_t)^{s-r}.$$

In next two theorems we shall suppose that functional A is such that continuity property for Theorem 11 (2) is satisfied on appropriate interval.

THEOREM 12. *Let L satisfy conditions L1, L2, and A satisfy conditions A1, A2 on a base set E. Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, if $f,g>0$ and f^p, g^q, fg, $f^p\log f^p$, $f^p\log g^q$, $g^q\log g^q$, $g^q\log f^p\in L$ and f_0, g_0 are positive numbers such that $g_0^q-A(g^q)>0$, $f_0^p-A(f^p)>0$ then we have*

$$\begin{aligned}
&pq\Big[(f_0g_0-A(fg))-(f_0^p-A(f^p))^{\frac{1}{p}}(g_0^q-A(g^q))^{\frac{1}{q}}\Big]\\
&\quad\leqslant\Big((f_0-A(f^p))\log\Big(\frac{f_0^p-A(f^p)}{g_0^q-A(g^q)}\Big)-\Big(f_0^p\log\frac{f_0^p}{g_0^q}-A\Big(f^p\log\frac{f^p}{g^q}\Big)\Big)\Big)^{\frac{1}{p}}\\
&\quad\quad\Big((g_0-A(g^q))\log\Big(\frac{g_0^q-A(g^q)}{f_0^p-A(f^p)}\Big)-\Big(g_0^q\log\frac{g_0^q}{f_0^p}-A\Big(g^q\log\frac{g^q}{f^p}\Big)\Big)\Big)^{\frac{1}{q}}. \qquad (3.2)
\end{aligned}$$

Proof. As in [6], from Theorem 11, for $r=0$, $s=s$, $t=1$, we get

$$\Omega_s\leqslant(\Omega_0)^{1-s}(\Omega_1)^s.$$

Set $A(f)=\frac{ua-A(wf)}{u-A(w)}$ we will get

$$\begin{aligned}
&\frac{1}{s(1-s)}\Big(\frac{ua^s-A(wf^s)}{u-A(w)}-\Big(\frac{ua-A(wf)}{u-A(w)}\Big)^s\Big)\\
&\quad\leqslant\Big(\frac{u\log a-A(w\log f)}{u-A(w)}-\log\Big(\frac{ua-A(wf)}{u-A(w)}\Big)\Big)^{1-s}\\
&\quad\quad\Big(\Big(\frac{ua-A(wf)}{u-A(w)}\Big)\log\Big(\frac{ua-A(wf)}{u-A(w)}\Big)-\frac{ua\log a-A(wf\log f)}{u-A(w)}\Big)^s.
\end{aligned}$$

Putting $s=\frac{1}{p}$, $1-s=\frac{1}{q}$; $w\to\frac{g^q}{g_0^q-A(g^q)}$, $f\to\frac{f^p}{g^q}$, $u=\frac{g_0^q}{g_0^q-A(g^q)}$, $a=\frac{f_0^p}{g_0^q}$. After some calculation we obtain the inequality (3.2). □

THEOREM 13. *Let L satisfy conditions L1, L2, and A satisfy conditions A1, A2 on a base set E. Let $1<p<2$, $\frac{1}{p}+\frac{1}{q}=1$, if $f,g>0$ and f^p, g^q, fg, $fg\log fg^{1-q}$, $f^2g^{2-q}\in L$ then we have*

$$\begin{aligned}
&\frac{1}{p(p-1)}\Big((f_0g_0-A(fg))^p-((f_0^p-A(f^p))^{\frac{1}{p}}(g_0^q-A(g^q))^{\frac{1}{q}})^p\Big)\\
&\leqslant\frac{1}{2^{p-1}}\Big((f_0g_0-A(fg))^2-(f_0^2-A(f^2))(g_0^q-A(g^q))\Big)^{p-1} \qquad (3.3)\\
&\quad\Big((f_0g_0-A(fg))\log\Big(\frac{f_0g_0-A(fg)}{g_0^q-A(g^q)}\Big)-\Big(f_0g_0\log(f_0g_0^{1-q})-A(fg)\log(f_0g_0^{1-q})\Big)\Big)^{2-p}.
\end{aligned}$$

For $p>2$ the above inequality is reversed.We have equality for $p=2$.

Proof. As in [6], from Theorem 11, for $r = 1$, $s = p$, $t = 2$, we get

$$\Omega_p \leqslant (\Omega_2)^{p-1}(\Omega_1)^{2-p}.$$

Set $A(f) = \frac{ua-A(wf)}{u-A(w)}$ we will get

$$\frac{1}{p(p-1)}\Big(\Big(\frac{ua-A(wf)}{u-A(w)}\Big)^p - \frac{ua^p - A(wf^p)}{u-A(w)}\Big)$$
$$\leqslant \frac{1}{2^{p-1}}\Big(\Big(\frac{ua-A(wf)}{u-A(w)}\Big)^2 - \frac{ua^2 - A(wf^2)}{u-A(w)}\Big)^{p-1}$$
$$\Big(\frac{ua-A(wf)}{u-A(w)}\log\Big(\frac{ua-A(wf)}{u-A(w)}\Big) - \frac{ua\log a - A(wf\,\log f)}{u-A(w)}\Big)^{2-p}.$$

Putting $w \to \frac{g^q}{g_0^q - A(g^q)}, f \to f g^{1-q}$, $u = \frac{g_0^q}{g_0^q - A(g^q)}$, $a = f_0 g_0^{1-q}$. After some calculation we obtain the inequality (3.3). □

Let us note that the following converse of Jensen-Steffensen's inequality was given by J. E. Pečarić ([5], see also [4], pp. 83–84).

THEOREM 14. *Let* $(x_1, x_2, ..., x_n)$ *and* $(p_1, p_2, ..., p_n)$ *be the real n-tuples such that* $x_i \in I$ $(1 \leqslant i \leqslant n,$ *I is an interval in* $\mathbb{R})$ $P_n = 1$, $\sum_{i=1}^n p_i x_i \in I$, *x is monotonic, and there exist an* $m \in \{1, 2, ..., n\}$ *such that*

$$P_k \leqslant 0 \ \ (k < m), \ \ 1 \leqslant P_{k-1} \ \ (k > m). \tag{3.4}$$

If $f : I \to \mathbb{R}$ *is a convex function, then*

$$f\Big(\sum_{i=1}^n p_i x_i\Big) \geqslant \sum_{i=1}^n p_i f(x_i). \tag{3.5}$$

We can use Theorem 14 in similar way for a proof of the following result.

THEOREM 15. *Let* $(x_1, x_2, ..., x_n)$ *be monotonic n-tuple of positive numbers,* $p_i \in \mathbb{R}$ *such that (3.4) and* $\sum_{i=1}^n p_i x_i \in I$ *are valid. Denote*

$$\tilde{\lambda}_t = \varphi_t\Big(\sum_{i=1}^n p_i x_i\Big) - \sum_{i=1}^n p_i \varphi_t(x_i), \tag{3.6}$$

and let $\tilde{\lambda}_t$ *be positive. If* $-\infty < r < s < t < \infty$, *then*

$$(\tilde{\lambda}_s)^{t-r} \leqslant (\tilde{\lambda}_r)^{t-s}(\tilde{\lambda}_t)^{s-r}. \tag{3.7}$$

Moreover we can also use related integral analogues of Jensen-Steffensen inequality (see for example [4], pp. 84–87).

4. Some Results of Mercer's type

The following version of Jessen's inequality is valid see [1].

THEOREM 16. *Let L satisfy conditions L1, L2 on a nonempty set E, and let φ be a convex function on an interval $I = [m,M]$ $(-\infty < m < M < \infty)$. if A is a positive linear functional on L with $A(1) = 1$, then for all $g \in L$ such that $\varphi(g), \varphi(m+M-g) \in L$ (so that $m \leqslant g(t) \leqslant M$ for all $t \in E$), we have the following variant of Jessen'n inequality*

$$\varphi(m+M-A(g)) < \varphi(m) + \varphi(M) - A(\varphi(g)). \tag{4.1}$$

As previously we can prove the following two theorems:

THEOREM 17. *Let the conditions of Theorem 16 be satisfied for an interval $I = [m,M]$ for function $\varphi = \varphi_s$ (as is defined by (2.2)) for $s \in J$ is some interval in R. Let us define*

$$\tilde{\Omega}_t = \varphi_t(m) + \varphi_t(M) - A(\varphi_t(g)) - \varphi_t(m+M-A(g)),$$

and let $\tilde{\Omega}_t$ be positive.

(1) *For all $s,t \in J$ we have*

$$\tilde{\Omega}^2_{\frac{s+t}{2}} \leqslant \tilde{\Omega}_s \tilde{\Omega}_t \tag{4.2}$$

that is $\tilde{\Omega}_t$ is log-convex in Jensen-sense.

(2) *If $\tilde{\Omega}_t$ is continuous on J, then it is also log-convex, i.e is for $r < s < t$ $(r,s,t \in J)$ we have*

$$(\tilde{\Omega}_s)^{t-r} \leqslant (\tilde{\Omega}_r)^{t-s} (\tilde{\Omega}_t)^{s-r}. \tag{4.3}$$

THEOREM 18. *Let the conditions of Theorem 16 be satisfied for an interval $I = [m,M]$ for function $\varphi = \phi_s$ (as is defined Lemma 2) for $s \in J$ is some interval in R. Let us define*

$$\hat{\Omega}_t = \phi_t(m) + \phi_t(M) - A(\phi_t(g)) - \phi_t(m+M-A(g)),$$

and let $\hat{\Omega}_t$ be positive.

(1) *For all $s,t \in J$ we have (4.2) that is $\hat{\Omega}_t$ is log-convex in Jensen-sense.*

(2) *If $\hat{\Omega}_t$ is continuous on J, then it is also log-convex, i.e is for $r < s < t$ $(r,s,t \in J)$ we have (4.3).*

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