Mathematical
Inequalities
& Applications

Volume 12, Number 1 (2009), 99–113

SOME INEQUALITIES OF DIFFERENTIAL POLYNOMIALS

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(*communicated by H. Srivastava*)

Abstract. In this paper, we consider the value distribution of the differential polynomials $f^nf^{(k)}-1$ where $n(\geqslant 2), k$ are positive integers, and obtain some estimates only by the reduced counting function.

1. Introduction and results

Let $\mathbb{C}$ be the open complex plane and $\mathscr{D}\in\mathbb{C}$ be a domain. Let f be a meromorphic function in the complex plane. We assumed that the reader is familiar with the notations of Nevanlinna theory (see, e.g., [7, 16, 15]).

DEFINITION 1.1. Let k be a positive integer, for any a in the complex plane. We denote by $N_{k)}(r,1/(f-a))$ the counting function of a-points of f with multiplicity $\leqslant k$, by $N_{(k}(r,1/(f-a))$ the counting function of a-points of f with multiplicity $\geqslant k$, by $N_k(r,1/(f-a))$ the counting function of a-points of f with multiplicity of k. and denote the reduced counting function by $\overline{N}_{k)}(r,1/(f-a))$, $\overline{N}_{(k}(r,1/(f-a))$ and $\overline{N}_k(r,1/(f-a))$, respectively.

Zhang and Li ([17]) proved the following theorem:

THEOREM A. *Let f be transcendental meromorphic in the complex plane, $L[f]=a_kf^{(k)}+a_{k-1}f^{(k-1)}+\cdots+a_0f$, where $a_0,a_1,\cdots,a_k(\not\equiv 0)$ are small functions, for $c\neq 0,\infty$, let $F=f^nL[f]-c$, where n is a positive integer. Then for $n\geqslant 2$, $F=f^nL[f]-c$ has infinitely many zeros.*

Recently, Huang and Gu ([9]) have obtained a quantitative result in the case of $n=2$.

THEOREM B. *Let f be transcendental meromorphic in the complex plane and k be a positive integer, then*

$$T(r,f)\leqslant 6N\left(r,\frac{1}{f^2f^{(k)}-1}\right)+S(r,f).$$

Mathematics subject classification (2000): 30D35, 26D10.

Keywords and phrases: Meromorphic function, differential polynomials, Nevanlinna theory, value distribution.

As we all known, the second fundamental theorem in Nevanlinna's theory of value distribution use the reduced counting function to estimate the Nevanlinna characteristic function (cf. [13]). Naturally, we can pose the following important question whether one can give some quantitative estimates on the generally differential polynomials by the reduced counting function. Here we give some affirmative answers.

THEOREM 1.2. *Let f be transcendental meromorphic function, $L[f]=a_kf^{(k)}+a_{k-2}f^{(k-2)}+\cdots+a_0f$, where $a_0,a_1,\cdots,a_k(\not\equiv 0)$ are small functions, for $c\neq 0,\infty$, let $F=f^nL[f]-c$, where n is a positive integer, then we have*

(1) *if $n>2$,*

$$T(r,f)\leqslant 6\overline{N}\left(r,\frac{1}{F}\right)+S(r,f)$$

(2) *if $n=2$, there exists a constant $M>0$, which does not depend on f, we have*

$$T(r,f)\leqslant M\overline{N}\left(r,\frac{1}{F}\right)+S(r,f).$$

REMARKS 1. If $n\geqslant 2$, we know F has infinitely many zeros, and in (2), we know the constant M at least is 6 from Theorem B. But the method of Theorem 1.2 can't give the certain coefficient. Hence, we want to get the more precise estimates for the coefficient in (2). In the following, for transcendental meromorphic function which has few simple zeros, we take the different method by constructing the auxiliary function and obtain.

THEOREM 1.3. *Let f be transcendental meromorphic function, and let k be a positive integer. Then if $N_1\left(r,\frac{1}{f}\right)=S(r,f)$;*

$$T(r,f)\leqslant 2\overline{N}\left(r,\frac{1}{f^2f^{(k)}-1}\right)+S(r,f). \tag{1.1}$$

COROLLARY 1.4. *Let f be transcendental meromorphic function all of whose zeros are multiple, and let k be a positive integer. Then we have the inequality (1.1) holds.*

REMARKS 2. If $n=1$, we know there are some better estimates on the differential monomial $ff^{(k)}$ by W. Hennekemper [8], C. Yang and P. Hu [14], A. Alotaibi [1], J. Wang [11], Xu and Zhang [12]. In fact, these estimates hold all in the condition of restricted zeros. Hence our condition in Theorem 1.3 and Corollary 1.4 is natural.

2. Some Lemmas

If the coefficients of differential polynomials $M[f]$ are $a_j, j=0,1,\cdots,n$, which satisfy $m(r,a_j)=S(r,f)$, then differential polynomials $M[f]$ is called a quasi-differential polynomials in f. The following Lemma is nothing but an easy variant of standard Clunie lemma [4], Lemma 1.

LEMMA 2.1. *Let f be a non-constant meromorphic in the complex plane, $Q_1[f]$, $Q_2[f]$ are quasi-differential polynomials in f, satisfy $f^nQ_1[f] = Q_2[f]$, if the total degree of $Q_2 \leqslant n$, then*

$$m(r,Q_1[f]) = S(r,f).$$

The following lemma, due to G. Valiron and A. Mohon'ko, is of essential importance in the theory of complex differential equation and so on. The proof below can be found in [15, 10].

LEMMA 2.2. *Let $P(z,w) = \sum_{k=0}^{p} a_k(z)w^k, w(z)$ is algebroid function, then*

$$T(r,P(z,w)) = pT(r,w) + O\{\sum_{k=0}^{p} T(r,a_k)\}.$$

LEMMA 2.3. *Let f be transcendental meromorphic function, and let k be a positive integer. Then*

$$\begin{aligned} 3T(r,f) \;\leqslant\; & \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right) + N_{k)}\left(r,\frac{1}{f}\right) + k\overline{N}_{(k+1}\left(r,\frac{1}{f}\right) \\ & +\overline{N}\left(r,\frac{1}{f^2f^{(k)}-1}\right) - N_0(r,\frac{1}{(f^2f^{(k)})'}) + S(r,f). \end{aligned} \tag{2.1}$$

where $N_0(r,\frac{1}{(f^2f^{(k)})'})$ denotes the counting function of the zeros of $(f^2f^{(k)})'$, not of $f(f^2f^{(k)}-1)$.

Proof. We first claim $f^2f^{(k)} \not\equiv$ constant. If $f^2f^{(k)} \not\equiv C$, where C is a complex constant. Obviously, $C \neq 0$. Hence f has no zero and $\frac{1}{f^3} = \frac{1}{C}\frac{f^{(k)}}{f}$. Therefore,

$$\begin{aligned} 3T(r,f) &= m\left(r,\frac{1}{f^3}\right) + N\left(r,\frac{1}{f^3}\right) + O(1) \\ &= m\left(r,\frac{f^{(k)}}{f}\right) + O(1) = S(r,f). \end{aligned}$$

It is a contradiction. Hence $f^2f^{(k)} \not\equiv$ constant. Let

$$\frac{1}{f^3} \equiv \frac{f^2f^{(k)}}{f^3} - \frac{(f^2f^{(k)})'}{f^3}\frac{f^2f^{(k)}-1}{(f^2f^{(k)})'},$$

we have

$$
\begin{aligned}
3m\left(r,\frac{1}{f}\right) &= m\left(r,\frac{1}{f^3}\right)\\
&\leqslant m\left(r,\frac{f^2f^{(k)}-1}{(f^2f^{(k)})'}\right)+m\left(r,\frac{f^{(k)}}{f}\right)+m\left(r,\frac{(f^2f^{(k)})'}{f^3}\right)+O(1)\\
&\leqslant N\left(r,\frac{(f^2f^{(k)})'}{f^2f^{(k)}-1}\right)-N\left(r,\frac{f^2f^{(k)}-1}{(f^2f^{(k)})'}\right)+S(r,f)\\
&= N(r,(f^2f^{(k)})')+N\left(r,\frac{1}{f^2f^{(k)}-1}\right)-N\left(r,\frac{1}{(f^2f^{(k)})'}\right)\\
&\quad -N(r,f^2f^{(k)}-1)+S(r,f)\\
&= \overline{N}(r,f)+N\left(r,\frac{1}{f^2f^{(k)}-1}\right)-N\left(r,\frac{1}{(f^2f^{(k)})'}\right)+S(r,f).
\end{aligned}
$$

Hence

$$
\begin{aligned}
3T(r,f) &= 3m\left(r,\frac{1}{f}\right)+3N\left(r,\frac{1}{f}\right)+O(1)\\
&= \overline{N}(r,f)+3N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^2f^{(k)}-1}\right)-N\left(r,\frac{1}{(f^2f^{(k)})'}\right)+S(r,f).
\end{aligned}
\tag{2.2}
$$

Let

$$
N\left(r,\frac{1}{(f^2f^{(k)})'}\right)=N_{000}\left(r,\frac{1}{(f^2f^{(k)})'}\right)+N_{00}\left(r,\frac{1}{(f^2f^{(k)})'}\right)+N_{0}\left(r,\frac{1}{(f^2f^{(k)})'}\right)
\tag{2.3}
$$

where $N_{000}(r,\frac{1}{(f^2f^{(k)})'})$ denotes the counting function of the zeros of $(f^2f^{(k)}-1)'$, which come from the zeros of $f^2f^{(k)}-1$, $N_{00}(r,\frac{1}{(f^2f^{(k)})'})$ denotes the counting function of the zeros of $(f^2f^{(k)}-1)'$, which come from the zeros of f. Hence we have

$$
N\left(r,\frac{1}{f^2f^{(k)}-1}\right)-N_{000}\left(r,\frac{1}{(f^2f^{(k)})'}\right)=\overline{N}\left(r,\frac{1}{f^2f^{(k)}-1}\right).
\tag{2.4}
$$

Supposed that z_0 is a zero of f with multiplicity q, if $q\leqslant k$, then z_0 is a zero of $(f^2f^{(k)})'$ with multiplicity at least $2q-1$; if $q\geqslant k+1$, then z_0 is a zero of $(f^2f^{(k)})'$ with multiplicity at least $3q-(k+1)$. Hence we have

$$
\begin{aligned}
3N\left(r,\frac{1}{f}\right)-N_{00}\left(r,\frac{1}{(f^2f^{(k)})'}\right) &\leqslant N_{k)}\left(r,\frac{1}{f}\right)+\overline{N}_{k)}\left(r,\frac{1}{f}\right)+(k+1)\overline{N}_{(k+1}\left(r,\frac{1}{f}\right)\\
&= N_{k)}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f}\right)+k\overline{N}_{(k+1}\left(r,\frac{1}{f}\right).
\end{aligned}
\tag{2.5}
$$

Combining (2.2)–(2.5), we have

$$3T(r,f) \leqslant \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+N_{k)}\left(r,\frac{1}{f}\right)+k\overline{N}_{(k+1}\left(r,\frac{1}{f}\right)$$
$$+\overline{N}\left(r,\frac{1}{f^2f^{(k)}-1}\right)-N_0\left(r,\frac{1}{(f^2f^{(k)})'}\right)+S(r,f).$$

This completes the proof of the lemma. □

In the following, we first construct an auxiliary function to remove the restriction of the pole in (2.1).

LEMMA 2.4. *Let f be transcendental meromorphic function, and let k be a positive integer. Then*

$$H=4(k+3)\frac{f'}{f}-(k^2+5k+8)\frac{(f^2f^{(k)})'}{f^2f^{(k)}-1}+(k+1)(k+3)\frac{(f^2f^{(k)})''}{(f^2f^{(k)})'}, \tag{2.6}$$

then we have (1) $H(z)\not\equiv 0$; (2) The simple poles of $f(z)$ are the zeros of $H(z)$.

Proof. (1). If $H(z)\equiv 0$. We fist prove $f(z)$ has no zeros. If $f(z)$ has the zero z_1, let z_1 be the zero of $f(z)$ with multiplicity $p(\geqslant 1)$ and the zero of $(f^2f^{(k)})'$ with multiplicity $q(\geqslant 0)$. Then $F(z)$ can be expanded at point z_1, and the coefficient of $(z-z_1)^{-1}$ in the expansion is $4(k+3)p+(k+1)(k+3)q>0$. Hence z_1 is the pole of $F(z)$. It contradicts with $H(z)\equiv 0$.

By integrating both of sides of the equality (2.6), we have

$$f^{4(k+3)}[(f^2f^{(k)})']^{(k+1)(k+3)} \equiv C(f^2f^{(k)}-1)^{k^2+5k+8},$$

where C is a nonzero complex constant. Therefore,

$$\begin{aligned}\frac{1}{C}\left[\frac{(f^2f^{(k)})'}{f^2f^{(k)}-1}\right]^{(k+1)(k+3)} &= \frac{(f^2f^{(k)}-1)^{k+5}}{f^{4(k+3)}}\\ &= \frac{1}{f^{k-3}}\left(\frac{f^{(k)}}{f}-\frac{1}{f^3}\right)^{k+5}\\ &= \frac{1}{f^{k-3}}\sum_{i=0}^{k+5}(-1)^{k+5-i}C_{k+5}^{i}\left(\frac{f^{(k)}}{f}\right)^{i}\frac{1}{f^{3(k+5-i)}}\\ &= \frac{(-1)^{k+5}}{f^{4(k+3)}}+\frac{1}{f^{4k+11}}\sum_{i=1}^{k+5}(-1)^{k+5-i}C_{k+5}^{i}f^{3i-1}\left(\frac{f^{(k)}}{f}\right)^{i}.\end{aligned}$$

This is,

$$\frac{(-1)^{k+5}}{f^{4(k+3)}} \equiv \frac{1}{C}\left[\frac{\left(f^2f^{(k)}\right)'}{f^2f^{(k)}-1}\right]^{(k+1)(k+3)}-\frac{1}{f^{4k+11}}\sum_{i=1}^{k+5}(-1)^{k+4-i}C_{k+5}^{i}f^{3i-1}\left(\frac{f^{(k)}}{f}\right)^{i}.$$

Let $E_r = \{\theta \,||f(z)| < 1;\ z = re^{i\theta}, \theta \in [0, 2\pi]\}$. If $|f(z)| < 1$, we have

$$\left|\frac{1}{f^{4(k+3)}}\right| \leqslant \left|\frac{1}{C}\right| \left| \left[\frac{(f^2 f^{(k)})'}{f^2 f^{(k)} - 1}\right]^{(k+1)(k+3)} \right| + \left|\frac{1}{f^{4k+11}}\right| \left| \sum_{i=1}^{k+5} (-1)^{k+5-i} C_{k+5}^{i} \left(\frac{f^{(k)}}{f}\right)^{i} \right|.$$

By Nevanlinna first fundamental theorem and noting $f(z)$ has no zeros, we get

$$\begin{aligned}
4(k+3)T(r,f) &= 4(k+3)T\left(r, \frac{1}{f}\right) + O(1) = 4(k+3)m\left(r, \frac{1}{f}\right) + O(1) \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{1}{f^{4(k+3)(re^{i\theta})}}\right| + O(1) \\
&\leqslant \frac{1}{2\pi}\int_{E_r} \log^+ \left|\frac{(f^2(re^{i\theta})f^{(k)}(re^{i\theta}))'}{f^2(re^{i\theta})f^{(k)}(re^{i\theta}) - 1})\right|^{(k+1)(k+3)} d\theta \\
&\quad + \frac{1}{2\pi}\int_{E_r} \log^+ \left|\frac{1}{f^{4k+11}(re^{i\theta})}\right| d\theta \\
&\quad + \frac{1}{2\pi}\sum_{i=1}^{k+5}\int_{E_r} \log^+ \left|\frac{f^{(k)}(re^{i\theta})}{f}\right|^{i} d\theta + O(1) \\
&\leqslant (k+1)(k+3)m\left(r, \frac{(f^2 f^{(k)})'}{f^2 f^{(k)} - 1}\right) + (4k+11)m\left(r, \frac{1}{f}\right) \\
&\quad + \sum_{i=1}^{k+5} m\left(r, \frac{f^{(k)}}{f}\right) + O(1) \\
&\leqslant (4k+11)T(r,f) + S(r,f).
\end{aligned}$$

This is, $T(r,f) = S(r,f)$, it is a contradiction. Hence we have $H(z) \not\equiv 0$.

(2) Let z_2 be a simple pole of $f(z)$, then $f(z)$ has the following expansions in a neighborhood of z_2:

$$f(z) = \frac{a}{z - z_2}\{1 + b(z - z_2) + O((z - z_2)^2)\}, \quad (a \neq 0).$$

By taking the derivatives on both sides of the above equality, we have

$$f^{(k)}(z) = \frac{(-1)^k k! a}{(z - z_2)^{k+1}}\{1 + O((z - z_2)^{k+1})\}.$$

We can easily obtain

$$\frac{f'}{f} = \frac{-1}{z - z_2}\{1 - b(z - z_2) + O((z - z_2)^2)\}, \tag{2.7}$$

$$\frac{(f^2 f^{(k)})'}{f^2 f^{(k)} - 1} = \frac{-1}{z - z_2}\{(k+3) - 2b(z - z_2) + O((z - z_2)^2)\}, \tag{2.8}$$

$$\frac{(f^2 f^{(k)})''}{(f^2 f^{(k)})'} = \frac{-1}{z - z_2}\{(k+4) - \frac{2k+4}{k+3}b(z - z_2) + O((z - z_2)^2)\}, \tag{2.9}$$

By substituting (2.7)–(2.9) into (2.6), we can obtain

$$H(z) = O((z - z_2)).$$

Hence z_2 is the zero of $H(z)$. We complete the proof of the lemma. □

LEMMA 2.5. *Let f be transcendental meromorphic function, and let k be a positive integer. Then*

$$N_{1)}(r,f) \leqslant \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}_{(2}(r,f) + \overline{N}\left(r,\frac{1}{f^2f^{(k)}-1}\right) + N_0\left(r,\frac{1}{(f^2f^{(k)})'}\right) + S(r,f), \tag{2.10}$$

where $\overline{N}_0(r,\frac{1}{(f^2f^{(k)})'})$ *denotes the reduced counting function of* $N_0(r,\frac{1}{(f^2f^{(k)})'})$.

Proof. By Lemma 2.4, we have

$$N_{1)}(r,f) \leqslant N\left(r,\frac{1}{F}\right) \leqslant T(r,F) + O(1) \leqslant N(r,F) + S(r,f).$$

From (2.6), the poles of $F(z)$ can only occur at the multiple poles of $f(z)$, the zeros of $f(z)$, the zeros of $f^2f^{(k)} - 1$, or the zeros of $(f^2f^{(k)})'$ and all these poles or zeros are the simple pole. Hence

$$N_{1)}(r,f) \leqslant \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}_{(2}(r,f) + \overline{N}\left(r,\frac{1}{f^2f^{(k)}-1}\right) + N_0\left(r,\frac{1}{(f^2f^{(k)})'}\right) + S(r,f).$$

This completes the proof of the lemma. □

We now state the main lemma of the paper which is interesting by itself.

LEMMA 2.6. *Let f be transcendental meromorphic function, and let k be a positive integer. Then*

$$\begin{aligned} 7T(r,f) \;\leqslant\; & 4\overline{N}\left(r,\frac{1}{f}\right) + 4\overline{N}\left(r,\frac{1}{f^2f^{(k)}-1}\right) \\ & +3N_{k)}\left(r,\frac{1}{f}\right) + 3k\overline{N}_{(k+1}\left(r,\frac{1}{f}\right) + S(r,f), \end{aligned} \tag{2.11}$$

where $\overline{N}_0(r,\frac{1}{(f^2f^{(k)})'})$ *denotes the reduced counting function of* $N_0(r,\frac{1}{(f^2f^{(k)})'})$.

Proof. From (2.1)–(2.10), and $\overline{N}(r,f) = N_{1)}(r,f) + \overline{N}_{(2}(r,f)$, we have

$$\begin{aligned} 3T(r,f) \;\leqslant\; & 2\overline{N}_{(2}(r,f) + 2\overline{N}\left(r,\frac{1}{f}\right) + 2\overline{N}\left(r,\frac{1}{f^2f^{(k)}-1}\right) \\ & +N_{k)}\left(r,\frac{1}{f}\right) + k\overline{N}_{(k+1}\left(r,\frac{1}{f}\right) + S(r,f), \end{aligned}$$

Hence

$$\begin{aligned}7T(r,f) \leqslant\ & 2(\overline{N}_{(2}(r,f)+2T(r,f))+2\overline{N}\left(r,\frac{1}{f}\right)+2\overline{N}\left(r,\frac{1}{f^2f^{(k)}-1}\right)\\ & +N_{k)}\left(r,\frac{1}{f}\right)+k\overline{N}_{(k+1}\left(r,\frac{1}{f}\right)+S(r,f),\end{aligned} \tag{2.12}$$

By (2.1), we have

$$\begin{aligned}\overline{N}_{(2}(r,f)+2T(r,f) \leqslant\ & 2T(r,f)+N(r,f)-\overline{N}(r,f) \leqslant 3T(r,f)-\overline{N}(r,f)\\ \leqslant\ & \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f^2f^{(k)}-1}\right)+N_{k)}\left(r,\frac{1}{f}\right)\\ & +k\overline{N}_{(k+1}\left(r,\frac{1}{f}\right)-N_0\left(r,\frac{1}{(f^2f^{(k)})'}\right)+S(r,f),\end{aligned} \tag{2.13}$$

Substituting (2.13) into (2.12), we have (2.11). □

3. Proof of Theorem 1.2

Proof of (1). Without loss of generality, let $c=1$, then

$$F=f^nL[f]-1, \tag{3.1}$$

where $n \geqslant 1$ is a positive integer. From (3.1) we obtain

$$T(r,F)=O(T(r,f)). \tag{3.2}$$

By differentiating the equation (3.1), we get

$$f\beta=-\frac{F'}{F}, \tag{3.3}$$

where

$$\beta=nf^{n-2}f'L[f]+f^{n-1}L'[f]-f^{n-1}L[f]\frac{F'}{F}, \tag{3.4}$$

obviously $F \not\equiv$ constant, $\beta \not\equiv 0$. Applying Lemma 2.1 to (3.3) and noting (3.2), we have

$$m(r,\beta)=S(r,f). \tag{3.5}$$

Let z_0 be a pole of f of order q, and not any poles and zeros of the coefficients of $L[f]$, then z_0 is the simple pole of $\frac{F'}{F}$, hence the poles of f of order $q(\geqslant 2)$ are the zeros of β of order $q-1$ from (3.3), the simple pole of f is the non-zero analytic point of β. Thus we have

$$N_{(2}(r,f) \leqslant N\left(r,\frac{1}{\beta}\right)+\overline{N}\left(r,\frac{1}{\beta}\right)+S(r,f) \leqslant 2N\left(r,\frac{1}{\beta}\right)+S(r,f). \tag{3.6}$$

For any poles of f are not the poles of β with (3.3), if $n > 2$ we have

$$N(r,\beta) \leqslant \overline{N}\left(r,\frac{1}{F}\right) + S(r,f), \tag{3.7}$$

From (3.5) and (3.7), we have

$$T(r,\beta) \leqslant \overline{N}\left(r,\frac{1}{F}\right) + S(r,f), \tag{3.8}$$

Next with (3.7), we have

$$N_{(2}(r,f) \leqslant 2\overline{N}\left(r,\frac{1}{F}\right) + S(r,f), \tag{3.9}$$

By using (3.3), we obtain

$$m(r,f) \leqslant m\left(r,\frac{1}{\beta}\right) + m\left(r,\frac{F'}{F}\right) \leqslant T(r,\beta) + S(r,f).$$

Combining these with (3.8), we have

$$m(r,f) \leqslant \overline{N}\left(r,\frac{1}{F}\right) + S(r,f), \tag{3.10}$$

If f only have finite simple poles, we get Case (1) in Theorem 1.2 by (3.9)–(3.10).

In the sequel, we suppose that f have infinity simple poles. Let z_0 be any simple pole of f, then z_0 is the non-zero analytic of β, so near z_0, we have

$$f(z) = \frac{d_1}{z-z_0} + d_0 + O(z-z_0) \tag{3.11}$$

and

$$\begin{aligned} \beta(z) &= \beta(z_0) + \beta'(z_0)(z-z_0) + O((z-z_0)^2) \\ a_k(z) &= a_k(z_0) + a_k'(z_0)(z-z_0) + O((z-z_0)^2) \end{aligned} \tag{3.12}$$

where $d_1 \neq 0$, $\beta(z_0) \neq 0$, $a_k(z_0) \neq 0$. By taking the derivatives on both sides of (3.11), we get

$$f^{(j)}(z) = (-1)^j \frac{j!d_1}{(z-z_0)^{j+1}} + O(1), \qquad j = 1,2,\cdots,k. \tag{3.13}$$

with (3.3) and (3.4) we have

$$f\beta = nf^{n-1}f'L[f] + f^nL'[f] + f^{n+1}L[f]\beta. \tag{3.14}$$

Substituting (3.11)–(3.13) into (3.14), we obtain that the coefficients have the form

$$d_1 = \frac{k+n+1}{\beta(z_0)}, \tag{3.15}$$

$$d_0 = -\frac{k+n+1}{k+2n+1}\left(\frac{a_k'(z_o)}{a_k(z_o)} + (k+n+1)\frac{\beta'(z_0)}{\beta(z_0)}\right)\frac{1}{\beta(z_0)}, \tag{3.16}$$

so that

$$\frac{d_0}{d_1} = -\frac{1}{k+2n+1}\frac{a_k'(z_o)}{a_k(z_o)} - \frac{k+n+1}{k+2n+1}\frac{\beta'(z_0)}{\beta(z_0)}. \tag{3.17}$$

Through the calculating from (3.11) and (3.13), we get

$$\frac{f'}{f} = -\frac{1}{z-z_0} + \frac{d_0}{d_1} + O(z-z_0), \tag{3.18}$$

$$\frac{F'}{F} = -\frac{k+n+1}{z-z_0} + n\frac{d_0}{d_1} + \frac{a_k'(z)}{a_k(z)} + O(z-z_0). \tag{3.19}$$

Let

$$h(z) = \frac{F'}{F} - (k+n+1)\frac{f'}{f} - \frac{2k+2n+2}{k+2n+1}\frac{a_k'(z)}{a_k(z)} - \frac{(k+1)(k+n+1)}{k+2n+1}\frac{\beta'(z)}{\beta(z)}. \tag{3.20}$$

Then from (3.17)–(3.20), it can be seen that $h(z_0)=0$. Hence the simple poles of f must be the zeros of $h(z)$. From (3.20), we have

$$m(r,h) = S(r,f). \tag{3.21}$$

We assert that $h(z)\not\equiv 0$, otherwise $h(z)\equiv 0$, then

$$\frac{F'}{F} = (k+n+1)\frac{f'}{f} + \frac{2k+2n+2}{k+2n+1}\frac{a_k'(z)}{a_k(z)} + \frac{(k+1)(k+n+1)}{k+2n+1}\frac{\beta'(z)}{\beta(z)}.$$

By integration

$$F^{k+2n+1} = Cf^{(k+n+1)(k+2n+1)}a_k^{2k+2n+2}\beta^{(k+1)(k+n+1)}, \tag{3.22}$$

where $C\neq 0$ is a constant. From (3.1), we know any zeros of f are not the zeros and poles of F. Suppose that z_0 is the zero of f with multiplicity q, we have $F(z_0)\neq 0,\infty$, therefore, z_0 is a pole of β from (3.2). But z_0 is a zero of β by (3.4) for $n>2$. This is a contradiction, hence $h(z)\not\equiv 0$.

Since $h(z)\not\equiv 0$, and the simple pole of f is the zeros of h, thus if $n>2$, the zeros of f of order q at least is the zeros of β of order q by (3.4), and the multiply poles of f of order p are the zeros of β of order $p-1$, and $h(z)$ just has the simple pole , by (3.6),(3.8) and (3.20), we have

$$\begin{aligned} N(r,h) &\leqslant \overline{N}\left(r,\tfrac{1}{F}\right) + \overline{N}(r,\beta) + \overline{N}\left(r,\tfrac{1}{\beta}\right) + \overline{N}(r,a_k) + \overline{N}\left(r,\tfrac{1}{a_k}\right) \\ &\leqslant \overline{N}\left(r,\tfrac{1}{F}\right) + 2T(r,\beta) + O(1) \leqslant 3\overline{N}\left(r,\tfrac{1}{F}\right) + S(r,f). \end{aligned}$$

Noting (3.21), we obtain

$$N_{1)}(r,f) \leqslant N\left(r,\frac{1}{h}\right) \leqslant N(r,h) + S(r,f) \leqslant 3\overline{N}\left(r,\frac{1}{F}\right) + S(r,f). \tag{3.23}$$

Hence the Theorem 1 holds by (3.9), (3.10) and (3.23).

□

Proof of (2). As the above, the simple pole of f near z_0 have (3.11)–(3.12) and (3.15), (3.16). Let

$$d_1(z)=\frac{k+3}{\beta(z)},$$

$$d_0(z)=-\frac{k+3}{k+5}\left(\frac{a_k'(z)}{a_k(z)}+(k+3)\frac{\beta'(z)}{\beta(z)}\right)\frac{1}{\beta(z)},$$

then

$$d_1=d_1(z_0),\qquad d_0=d_0(z_0),$$

$$\frac{d_0(z)}{d_1(z)}=-\frac{1}{k+5}\cdot\frac{a_k'(z)}{a_k(z)}-\frac{k+3}{k+5}\cdot\frac{\beta'(z)}{\beta(z)}.$$

By (3.11)–(3.13) and through the calculating, we deduce

$$A_0(z)=f'(z)+\frac{1}{d_1(z)}f^2(z)-\frac{2d_0(z)-d_1'(z)}{d_1(z)}f(z)$$

is analytic near z_0. Thus let

$$A_1(z)=\frac{2d_0(z)-d_1'(z)}{d_1(z)},\quad A_2(z)=-\frac{1}{d_1(z)},$$

then f satisfies Riccati equation

$$w'=A_0(z)+A_1(z)w+A_2(z)w^2, \tag{3.24}$$

where $A_2(z)\not\equiv 0$. From the proof of (1), if $n=2$ we still have (3.5), (3.6) and (3.10). Hence we have

$$T(r,A_1)\leqslant T(r,\beta)+T\left(r,\frac{1}{\beta}\right)+S(r,f)\leqslant 2\overline{N}\left(r,\frac{1}{F}\right)+S(r,f),$$

$$T(r,A_2)\leqslant T(r,\beta)\leqslant \overline{N}\left(r,\frac{1}{F}\right)+S(r,f),$$

$$m(r,A_0)\leqslant 4m(r,f)+T(r,\beta)+S(r,f)\leqslant 5\overline{N}\left(r,\frac{1}{F}\right)+S(r,f).$$

Since the simple pole of f is not the pole of A_0, from (3.9) and (3.10) we have

$$N(r,A_0)\leqslant 2N_{(2}(r,f)+3T(r,\beta)+S(r,f)\leqslant 7\overline{N}\left(r,\frac{1}{F}\right)+S(r,f).$$

It follows that,

$$T(r,A_0)\leqslant 12\overline{N}\left(r,\frac{1}{F}\right)+S(r,f).$$

Using (3.24) over and over again, we deduce that $f^{(j)}$ is the $(j+1)$-th polynomial of f. Thus, F have the form

$$F=f^2L[f]-1=f^2(k!A_2^k(z)a_kf^{k+1}+\cdots)-1.$$

We define the function of z, w

$$G(z,w) = w^2(k!A_2^k(z)a_k w^{k+1} + \cdots) - 1.$$

This is a polynomial in w, and $G(z,w) \equiv F$. Obviously $w = w(z)$ is the solution of Riccati equation (3.24), it must satisfy

$$G(z,w(z)) = w(z)^2 L[w(z)] - 1.$$

In the sequel, we consider the function equation

$$G(z,w) = 0, \tag{3.25}$$

its solutions are the algebroidal function, satisfy (by Lemma 2.2)

$$T(r,w(z)) = O(\sum_{j=0}^{2} T(r,A_j)) \leqslant M_1\overline{N}\left(r,\frac{1}{F}\right) + S(r,f) \tag{3.26}$$

where $M_1 > 0$ is a constant, rewrite (3.3) to $\beta(z)fF + F' \equiv 0$.

Note $F \equiv G(z,f(z))$ and f satisfies Riccati equation (3.24), we have

$$\beta(z)fG(z,f) + G_z(z,f) + G_f(z,f)(A_0(z) + A_1(z)f + A_2(z)f^2) \equiv 0.$$

Thus, let $H(z,w) = \beta(z)wG(z,w) + G_z(z,w) + G_w(z,w)(A_0(z) + A_1(z)w + A_2(z)w^2)$, then $H(z,f) \equiv 0$. Since $H(z,f)$ is a polynomial in f with the coefficients of $A_0(z), A_1(z), A_2(z)$ and its derivatives, then $H(z,f) \equiv 0$ means that all the coefficients are identically zero or at least two terms not identically zero, for the latter, we have

$$T(r,f) = O(\sum_{j=0}^{2} T(r,A_j)) \leqslant M_2\overline{N}\left(r,\frac{1}{F}\right) + S(r,f), \tag{3.27}$$

where $M_2 > 0$ is a constant. If all the coefficients of $H(z,f)$ are identically zero, then

$$\beta(z)wG(z,w) + G_z(z,w) + G_w(z,w)(A_0(z) + A_1(z)w + A_2(z)w^2) \equiv 0$$

for arbitrary z and w.

If let $w = p(z)$ be the solution of (3.25), then it satisfies (3.24), therefore there is a unique positive integer λ such that

$$G(z,w) = (w - p(z))^{\lambda} G^*(z,w), G^*(z,p) \not\equiv 0.$$

Substituting the above into $H(z,w) \equiv 0$, we know $w = p(z)$ satisfies Riccati equation (3.24) and $p^{(j)} = j!A_2^j(z)p^{j+1} + \cdots$, $j = 1,2,\cdots$. Thus $w = p(z)$ satisfies equation $w^2L[w] - 1 = 0$.

We claim the equation (3.25) cannot only be a solution $w = p(z)$. Otherwise, $G(z,w) = k!A_2^k(z)a_k(w - p(z))^{k+3}$, but $G(z,w)$ have not 1-th term, we get a contradiction. The Eq. (3.25) also can't have two different solutions $p_1(z), p_2(z)$, otherwise $G(z,w) = k!A_2^k(z)(w - p_1(z))^{\lambda_1}(w - p_2(z))^{\lambda_2}, \lambda_1, \lambda_2$ are the positive integers, also $G(z,w)$ has not 1-th term, we get

$$\lambda_1 p_1(z) + \lambda_2 p_2(z) \equiv 0.$$

Hence, the two solutions of equation (3.25) are $p_1(z)$ and $-\frac{\lambda_2}{\lambda_1}p_1(z)$, substituting them into the equation(3.25) respectively, we deduce

$$p_1^2(z)L[p_1(z)]-1=0,$$
$$-\left(\frac{\lambda_2}{\lambda_1}\right)^3 p_1^2(z)L[p_1(z)]-1=0.$$

Therefore $1+(\frac{\lambda_2}{\lambda_1})^3=0$, it is impossible. Hence, the equation (3.25) at least has three different solutions p_1,p_2,p_3, and they all are the solutions of Riccati equation(3.24), and f also satisfies (3.25), this is to say, Riccati equation (3.24) at least have four different solutions f,p_1,p_2,p_3. Since arbitrary four solution's cross ratio of Riccati equation is a constant (see. [6]), f can be written in the rational functions of p_1,p_2,p_3, hence

$$T(r,f)\leqslant M_3(T(r,p_1)+T(r,p_2)+T(r,p_3)),\quad M_3>0$$

where M_3 is a constant. Because p_1,p_2,p_3 all are the solutions of (3.25), they satisfy (3.26),

$$T(r,f)\leqslant 3M_1\times M_3\overline{N}\left(r,\frac{1}{F}\right)+S(r,f)=M\overline{N}\left(r,\frac{1}{F}\right)+S(r,f),\tag{3.28}$$

where $M>0$ is a constant, thus we completes the proof of Theorem 1.2 by (3.27) and (3.28).

□

4. Proof of Theorem 1.3

Proof. We shall divide our argument into two cases:

Case (i). $k=1$

By assumption, we have $\overline{N}\left(r,\frac{1}{f}\right)=\overline{N}_{(2}\left(r,\frac{1}{f}\right)+S(r,f)\leqslant\frac{1}{2}N\left(r,\frac{1}{f}\right)+S(r,f)$. From this and (2.11), we obtain

$$\begin{aligned}7T(r,f) &< 2N\left(r,\frac{1}{f}\right)+4\overline{N}\left(r,\frac{1}{f^2f'-1}\right)\\ &\quad +3N_{1)}\left(r,\frac{1}{f}\right)+3N_{(2}\left(r,\frac{1}{f}\right)+S(r,f)\\ &= 2N\left(r,\frac{1}{f}\right)+4\overline{N}\left(r,\frac{1}{f^2f'-1}\right)+\frac{3}{2}N\left(r,\frac{1}{f}\right)+S(r,f)\\ &\leqslant \frac{7}{2}N\left(r,\frac{1}{f}\right)+4\overline{N}\left(r,\frac{1}{f^2f'-1}\right)+S(r,f),\end{aligned}$$

Hence we have

$$T(r,f)<\frac{8}{7}\overline{N}\left(r,\frac{1}{f^2f^{(k)}-1}\right)+S(r,f).\tag{4.1}$$

Case (ii). $k \geqslant 2$.

By assumption, we have $\overline{N}\left(r,\frac{1}{f}\right)=\overline{N}_{(2}\left(r,\frac{1}{f}\right)+S(r,f)\leqslant\frac{1}{2}N\left(r,\frac{1}{f}\right)+S(r,f)$. From this and (2.11), we obtain

$$\begin{aligned}7T(r,f) &< 2N\left(r,\frac{1}{f}\right)+4\overline{N}\left(r,\frac{1}{f^2f^{(k)}-1}\right)\\ &+ 3N_{k)}\left(r,\frac{1}{f}\right)+3N_{(k+1}\left(r,\frac{1}{f}\right)+S(r,f)\\ &= 2N\left(r,\frac{1}{f}\right)+4\overline{N}\left(r,\frac{1}{f^2f^{(k)}-1}\right)+3N\left(r,\frac{1}{f}\right)+S(r,f)\\ &\leqslant 5N\left(r,\frac{1}{f}\right)+4\overline{N}\left(r,\frac{1}{f^2f^{(k)}-1}\right)+S(r,f),\end{aligned}$$

Hence we have

$$T(r,f)<2\overline{N}\left(r,\frac{1}{f^2f^{(k)}-1}\right)+S(r,f). \tag{4.2}$$

We complete the proof of the theorem. □

5. Final Remark

Though Theorem 1.3 has the better coefficient 2, form Theorem 1.2 we know the condition of the simple zero is not necessary. Hence how to remove the restricted condition is an interesting question.

Acknowledgement. Supported by Specialized Research Fund for the Doctoral Program of Higher Education (20060422049) and NSF of China(10771121).

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(Received January 15, 2007)

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