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DIFFERENTIAL SUBORDINATION AND SUPERORDINATION OF ANALYTIC FUNCTIONS DEFINED BY THE MULTIPLIER TRANSFORMATION

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Abstract. Differential subordination and superordination results are obtained for analytic functions in the open unit disk which are associated with the multiplier transformation. These results are obtained by investigating appropriate classes of admissible functions. Sandwich-type results are also obtained.

1. Introduction

Let $\mathcal{H}(U)$ be the class of functions analytic in $U := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots$, with $\mathcal{H}_0 \equiv \mathcal{H}[0,1]$ and $\mathcal{H} \equiv \mathcal{H}[1,1]$. Let $\mathcal{A}_p$ denote the class of all analytic functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (z \in U) \tag{1.1}$$

and let $\mathcal{A}_1 := \mathcal{A}$. Let f and F be members of $\mathcal{H}(U)$. The function $f(z)$ is said to be *subordinate* to $F(z)$, or $F(z)$ is said to be *superordinate* to $f(z)$, if there exists a function $w(z)$ analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f(z) = F(w(z))$. In such a case we write $f(z) \prec F(z)$. If F is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$. For two functions $f(z)$ given by (1.1) and $g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k$, the Hadamard product (or convolution) of f and g is defined by

$$(f*g)(z) := z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k =: (g*f)(z). \tag{1.2}$$

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Motivated by the multiplier transformation on $\mathscr{A}$, we define the operator $I_p(n,\lambda)$ on $\mathscr{A}_p$ by the following infinite series

$$I_p(n,\lambda)f(z) := z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k \quad (\lambda > -p). \tag{1.3}$$

The operator $I_p(n,\lambda)$ is closely related to the Sălăgean derivative operators [11]. The operator $I_\lambda^n := I_1(n,\lambda)$ was studied recently by Cho and Srivastava [6] and Cho and Kim [7]. The operator $I_n := I_1(n,1)$ was studied by Uralegaddi and Somanatha [13].

To prove our results, we need the following definitions and theorems.

Denote by $\mathscr{Q}$ the set of all functions $q(z)$ that are analytic and injective on $\overline{U}\backslash E(q)$ where

$$E(q) = \{\zeta \in \partial U : \lim_{z\to\zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$. Further let the subclass of $\mathscr{Q}$ for which $q(0) = a$ be denoted by $\mathscr{Q}(a)$, $\mathscr{Q}(0) \equiv \mathscr{Q}_0$ and $\mathscr{Q}(1) \equiv \mathscr{Q}_1$.

DEFINITION 1.1. [9, Definition 2.3a, p. 27] Let Ω be a set in $\mathbb{C}$, $q \in \mathscr{Q}$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega,q]$ consists of those functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition $\psi(r,s,t;z) \notin \Omega$ whenever $r = q(\zeta)$, $s = k\zeta q'(\zeta)$, and

$$\Re\left\{\frac{t}{s}+1\right\} \geqslant k\,\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right\},$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ and $k \geqslant n$. We write $\Psi_1[\Omega,q]$ as $\Psi[\Omega,q]$.

In particular when $q(z) = M\frac{Mz+a}{M+\bar{a}z}$, with $M > 0$ and $|a| < M$, then $q(U) = U_M := \{w : |w| < M\}$, $q(0) = a$, $E(q) = \emptyset$ and $q \in \mathscr{Q}$. In this case, we set $\Psi_n[\Omega,M,a] := \Psi_n[\Omega,q]$, and in the special case when the set $\Omega = U_M$, the class is simply denoted by $\Psi_n[M,a]$.

DEFINITION 1.2. [10, Definition 3, p. 817] Let Ω be a set in $\mathbb{C}$, $q(z) \in \mathscr{H}[a,n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi_n'[\Omega,q]$ consists of those functions $\psi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition $\psi(r,s,t;\zeta) \in \Omega$ whenever $r = q(z), s = \frac{zq'(z)}{m}$, and

$$\Re\left\{\frac{t}{s}+1\right\} \leqslant \frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)}+1\right\},$$

$z \in U$, $\zeta \in \partial U$ and $m \geqslant n \geqslant 1$. In particular, we write $\Psi_1'[\Omega,q]$ as $\Psi'[\Omega,q]$.

THEOREM 1.1. *[9, Theorem 2.3b, p. 28] Let $\psi \in \Psi_n[\Omega,q]$ with $q(0) = a$. If the analytic function $p(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots$ satisfies*

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega,$$

then $p(z) \prec q(z)$.

THEOREM 1.2. *[10, Theorem 1, p. 818] Let* $\psi \in \Psi'_n[\Omega, q]$ *with* $q(0) = a$. *If* $p(z) \in \mathscr{Q}(a)$ *and* $\psi(p(z), zp'(z), z^2p''(z); z)$ *is univalent in* U, *then*

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z) : z \in U\}$$

implies $q(z) \prec p(z)$.

In the present investigation, the differential subordination result of Miller and Mocanu [9, Theorem 2.3b, p. 28] is extended for functions associated with the multiplier transformation $I_p(n,\lambda)$, and we obtain certain other related results. A similar problem for analytic functions defined by Dizok-Srivastava linear operator was considered by Ali *et al.* [4] (see also [1], [2], [3], [5]). Additionally, the corresponding differential superordination problem is investigated, and several sandwich-type results are obtained.

2. Subordination Results involving the Multiplier Transformation

DEFINITION 2.1. Let Ω be a set in $\mathbb{C}$ and $q(z) \in \mathscr{Q}_0 \cap \mathscr{H}[0,p]$. The class of admissible functions $\Phi_I[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; z) \notin \Omega$$

whenever

$$u = q(\zeta), \quad v = \frac{k\zeta q'(\zeta) + \lambda q(\zeta)}{\lambda + p},$$

$$\Re\left\{\frac{(\lambda+p)^2 w - \lambda^2 u}{(\lambda+p)v - \lambda u} - 2\lambda\right\} \geqslant k\,\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ and $k \geqslant p$.

THEOREM 2.1. *Let* $\phi \in \Phi_I[\Omega, q]$. *If* $f(z) \in \mathscr{A}_p$ *satisfies*

$$\{\phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z); z\right) : z \in U\} \subset \Omega, \tag{2.1}$$

then

$$I_p(n,\lambda)f(z) \prec q(z).$$

Proof. Define the analytic function $p(z)$ in U by

$$p(z) := I_p(n,\lambda)f(z). \tag{2.2}$$

In view of the relation

$$(p+\lambda)I_p(n+1,\lambda)f(z) = z[I_p(n,\lambda)f(z)]' + \lambda I_p(n,\lambda)f(z), \tag{2.3}$$

from (2.2), we get

$$I_p(n+1,\lambda)f(z) = \frac{zp'(z) + \lambda p(z)}{\lambda + p}. \tag{2.4}$$

Further computations show that

$$I_p(n+2,\lambda)f(z) = \frac{z^2p''(z) + (2\lambda+1)zp'(z) + \lambda^2 p(z)}{(\lambda+p)^2}. \tag{2.5}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r,\ v = \frac{s+\lambda r}{\lambda + p},\ w = \frac{t+(2\lambda+1)s+\lambda^2 r}{(\lambda+p)^2}. \tag{2.6}$$

Let

$$\psi(r,s,t;z) = \phi(u,v,w;z) = \phi\left(r, \frac{s+\lambda r}{\lambda+p}, \frac{t+(2\lambda+1)s+\lambda^2 r}{(\lambda+p)^2}; z\right). \tag{2.7}$$

The proof shall make use of Theorem 1.1. Using equations (2.2), (2.4) and (2.5), from (2.7), we obtain

$$\psi(p(z), zp'(z), z^2p''(z); z) = \phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z); z\right). \tag{2.8}$$

Hence (2.1) becomes

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_I[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.1. Note that

$$\frac{t}{s} + 1 = \frac{(\lambda+p)^2 w - \lambda^2 u}{(\lambda+p)v - \lambda u} - 2\lambda,$$

and hence $\psi \in \Psi_p[\Omega, q]$. By Theorem 1.1, $p(z) \prec q(z)$ or

$$I_p(n,\lambda)f(z) \prec q(z). \qquad \square$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case the class $\Phi_I[h(U), q]$ is written as $\Phi_I[h, q]$. The following result is an immediate consequence of Theorem 2.1.

THEOREM 2.2. *Let* $\phi \in \Phi_I[h, q]$. *If* $f(z) \in \mathscr{A}_p$ *satisfies*

$$\phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z); z\right) \prec h(z), \tag{2.9}$$

then

$$I_p(n,\lambda)f(z) \prec q(z).$$

Our next result is an extension of Theorem 2.2 to the case where the behavior of $q(z)$ on ∂U is not known.

COROLLARY 2.1. *Let* $\Omega \subset \mathbb{C}$ *and let* $q(z)$ *be univalent in* U, $q(0) = 0$. *Let* $\phi \in \Phi_I[\Omega, q_\rho]$ *for some* $\rho \in (0,1)$ *where* $q_\rho(z) = q(\rho z)$. *If* $f(z) \in \mathscr{A}_p$ *and*

$$\phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z); z\right) \in \Omega,$$

then

$$I_p(n,\lambda)f(z) \prec q(z).$$

Proof. Theorem 2.1 yields $I_p(n,\lambda)f(z) \prec q_\rho(z)$. The result is now deduced from $q_\rho(z) \prec q(z)$. □

THEOREM 2.3. *Let $h(z)$ and $q(z)$ be univalent in U, with $q(0) = 0$ and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ satisfy one of the following conditions:*

(1) $\phi \in \Phi_I[h, q_\rho]$, *for some* $\rho \in (0,1)$, *or*

(2) *there exists* $\rho_0 \in (0,1)$ *such that* $\phi \in \Phi_I[h_\rho, q_\rho]$, *for all* $\rho \in (\rho_0, 1)$.

If $f(z) \in \mathscr{A}_p$ satisfies (2.9), then

$$I_p(n,\lambda)f(z) \prec q(z).$$

Proof. The proof is similar to the proof of [9, Theorem 2.3d, p. 30] and is therefore omitted. □

The next theorem yields the best dominant of the differential subordination (2.9).

THEOREM 2.4. *Let $h(z)$ be univalent in U. Let $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$. Suppose that the differential equation*

$$\phi\left(q(z), \frac{zq'(z)+\lambda q(z)}{\lambda+p}, \frac{z^2q''(z)+(2\lambda+1)zq'(z)+\lambda^2 q(z)}{(\lambda+p)^2}; z\right) = h(z) \qquad (2.10)$$

has a solution $q(z)$ with $q(0) = 0$ and satisfy one of the following conditions:

(1) $q(z) \in \mathscr{Q}_0$ *and* $\phi \in \Phi_I[h, q]$,

(2) $q(z)$ *is univalent in* U *and* $\phi \in \Phi_I[h, q_\rho]$, *for some* $\rho \in (0,1)$, *or*

(3) $q(z)$ *is univalent in* U *and there exists* $\rho_0 \in (0,1)$ *such that* $\phi \in \Phi_I[h_\rho, q_\rho]$, *for all* $\rho \in (\rho_0, 1)$.

If $f(z) \in \mathscr{A}_p$ satisfies (2.9), then

$$I_p(n,\lambda)f(z) \prec q(z),$$

and $q(z)$ is the best dominant.

Proof. Following the same arguments in [9, Theorem 2.3e, p. 31], we deduce that $q(z)$ is a dominant from Theorems 2.2 and 2.3. Since $q(z)$ satisfies (2.10) it is also a solution of (2.9) and therefore $q(z)$ will be dominated by all dominants. Hence $q(z)$ is the best dominant. □

In the particular case $q(z) = Mz$, $M > 0$, and in view of the Definition 2.1, the class of admissible functions $\Phi_I[\Omega, q]$, denoted by $\Phi_I[\Omega, M]$, is described below.

DEFINITION 2.2. Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_I[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ such that

$$\phi\left(Me^{i\theta}, \frac{k+\lambda}{\lambda+p}Me^{i\theta}, \frac{L+((2\lambda+1)k+\lambda^2)Me^{i\theta}}{(\lambda+p)^2}; z\right) \notin \Omega \qquad (2.11)$$

whenever $z \in U$, $\theta \in \mathbb{R}$, $\Re(Le^{-i\theta}) \geqslant (k-1)kM$ for all real θ and $k \geqslant p$.

COROLLARY 2.2. *Let* $\phi \in \Phi_I[\Omega, M]$. *If* $f(z) \in \mathscr{A}_p$ *satisfies*

$$\phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z); z\right) \in \Omega,$$

then

$$|I_p(n,\lambda)f(z)| < M.$$

In the special case $\Omega = q(U) = \{\omega : |\omega| < M\}$, the class $\Phi_I[\Omega, M]$ is simply denoted by $\Phi_I[M]$.

COROLLARY 2.3. *Let* $\phi \in \Phi_I[M]$. *If* $f(z) \in \mathscr{A}_p$ *satisfies*

$$\left|\phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z); z\right)\right| < M,$$

then

$$|I_p(n,\lambda)f(z)| < M.$$

REMARK 2.1. When $\Omega = U$ and $M = 1$, Corollary 2.2 reduces to [1, Theorem 2, p. 271]. When $\Omega = U$, $\lambda = a - 1$ $(a > 0)$, $p = 1$ and $M = 1$, Corollary 2.2 reduces to [8, Theorem 2, p. 231]. When $\Omega = U$, $\lambda = 1$, $p = 1$ and $M = 1$, Corollary 2.2 reduces to [5, Theorem 1, p. 477].

COROLLARY 2.4. *If* $M > 0$ *and* $f(z) \in \mathscr{A}_p$ *satisfies*

$$\begin{aligned}&\left|(\lambda+p)^2 I_p(n+2,\lambda)f(z) - (\lambda+p)I_p(n+1,\lambda)f(z) - \lambda^2 I_p(n,\lambda)f(z)\right| \\ &\quad < [(2p-1)\lambda + p(p-1)]M, \quad \textit{then} \quad |I_p(n,\lambda)f(z)| < M. \end{aligned} \tag{2.12}$$

Proof. This follows from Corollary 2.2 by taking $\phi(u,v,w;z) = (\lambda+p)^2 w - (\lambda+p)v - \lambda^2 u$ and $\Omega = h(U)$ where $h(z) = [(2p-1)\lambda + p(p-1)]Mz$, $M > 0$. To use Corollary 2.2, we need to show that $\phi \in \Phi_I[\Omega, M]$, that is, the admissible condition (2.11) is satisfied. This follows since

$$\begin{aligned}&\left|\phi\left(Me^{i\theta}, \frac{k+\lambda}{\lambda+p}Me^{i\theta}, \frac{L + ((2\lambda+1)k + \lambda^2)Me^{i\theta}}{(\lambda+p)^2}; z\right)\right| \\ &\quad = \left|L + ((2\lambda+1)k+\lambda^2)Me^{i\theta} - (k+\lambda)Me^{i\theta} - \lambda^2 Me^{i\theta}\right| \\ &\quad = \left|L + (2k-1)\lambda Me^{i\theta}\right| \\ &\quad \geqslant (2k-1)\lambda M + \Re(Le^{-i\theta}) \\ &\quad \geqslant (2k-1)\lambda M + k(k-1)M \geqslant [(2p-1)\lambda + p(p-1)]M\end{aligned}$$

$z \in U$, $\theta \in \mathbb{R}$, $\Re(Le^{-i\theta}) \geqslant k(k-1)M$ and $k \geqslant p$. Hence by Corollary 2.2, we deduce the required result. □

DEFINITION 2.3. Let Ω be a set in $\mathbb{C}$ and $q(z) \in \mathscr{Q}_0 \cap \mathscr{H}_0$. The class of admissible functions $\Phi_{I,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u,v,w;z) \notin \Omega$$

whenever

$$u = q(\zeta), \quad v = \frac{k\zeta q'(\zeta) + (\lambda + p - 1)q(\zeta)}{\lambda + p},$$

$$\Re\left\{\frac{(\lambda+p)^2 w - (\lambda+p-1)^2 u}{(\lambda+p)v - (\lambda+p-1)u} - 2(\lambda+p-1)\right\} \geqslant k\,\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ and $k \geqslant 1$.

THEOREM 2.5. *Let* $\phi \in \Phi_{I,1}[\Omega, q]$. *If* $f(z) \in \mathscr{A}_p$ *satisfies*

$$\left\{\phi\left(\frac{I_p(n,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+1,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+2,\lambda)f(z)}{z^{p-1}}; z\right) : z \in U\right\} \subset \Omega, \tag{2.13}$$

then

$$\frac{I_p(n,\lambda)f(z)}{z^{p-1}} \prec q(z).$$

Proof. Define an analytic function $p(z)$ in U by

$$p(z) := \frac{I_p(n,\lambda)f(z)}{z^{p-1}}. \tag{2.14}$$

By making use of (2.3), we get,

$$\frac{I_p(n+1,\lambda)f(z)}{z^{p-1}} = \frac{zp'(z) + (\lambda+p-1)p(z)}{\lambda+p}. \tag{2.15}$$

Further computations show that

$$\frac{I_p(n+2,\lambda)f(z)}{z^{p-1}} = \frac{z^2p''(z) + [2(\lambda+p)-1]zp'(z) + (\lambda+p-1)^2 p(z)}{(\lambda+p)^2}. \tag{2.16}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r,\ v = \frac{s + (\lambda+p-1)r}{\lambda+p},\ w = \frac{t + [2(\lambda+p)-1]s + (\lambda+p-1)^2 r}{(\lambda+p)^2}. \tag{2.17}$$

Let

$$\begin{aligned} \psi(r,s,t;z) &= \phi(u,v,w;z) \qquad (2.18) \\ &= \phi\left(r, \frac{s+(\lambda+p-1)r}{\lambda+p}, \frac{t+[2(\lambda+p)-1]s+(\lambda+p-1)^2 r}{(\lambda+p)^2}; z\right). \end{aligned}$$

The proof shall make use of Theorem 1.1. Using equations (2.14), (2.15) and (2.16), from (2.18), we obtain

$$\psi(p(z), zp'(z), z^2p''(z); z) = \phi\left(\frac{I_p(n,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+1,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+2,\lambda)f(z)}{z^{p-1}}; z\right). \tag{2.19}$$

Hence (2.13) becomes

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_{I,1}[\Omega,q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.1. Note that

$$\frac{t}{s}+1=\frac{(\lambda+p)^2w-(\lambda+p-1)^2u}{(\lambda+p)v-(\lambda+p-1)u}-2(\lambda+p-1),$$

and hence $\psi \in \Psi[\Omega,q]$. By Theorem 1.1, $p(z) \prec q(z)$ or

$$\frac{I_p(n,\lambda)f(z)}{z^{p-1}} \prec q(z).$$

□

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$, for some conformal mapping $h(z)$ of U onto Ω. In this case the class $\Phi_{I,1}[h(U),q]$ is written as $\Phi_{I,1}[h,q]$. In the particular case $q(z)=Mz,\ M>0$, the class of admissible functions $\Phi_{I,1}[\Omega,q]$, denoted by $\Phi_{I,1}[\Omega,M]$. The following result is an immediate consequence of Theorem 2.5.

THEOREM 2.6. *Let* $\phi \in \Phi_{I,1}[h,q]$. *If* $f(z) \in \mathscr{A}_p$ *satisfies*

$$\phi\left(\frac{I_p(n,\lambda)f(z)}{z^{p-1}},\frac{I_p(n+1,\lambda)f(z)}{z^{p-1}},\frac{I_p(n+2,\lambda)f(z)}{z^{p-1}};z\right) \prec h(z), \tag{2.20}$$

then

$$\frac{I_p(n,\lambda)f(z)}{z^{p-1}} \prec q(z).$$

DEFINITION 2.4. Let Ω be a set in $\mathbb{C}$ and $M>0$. The class of admissible functions $\Phi_{I,1}[\Omega,M]$ consists of those functions $\phi:\mathbb{C}^3\times U \to \mathbb{C}$ such that

$$\phi\left(Me^{i\theta},\frac{k+\lambda+p-1}{\lambda+p}Me^{i\theta},\frac{L+[(2(\lambda+p)-1)k+(\lambda+p-1)^2]Me^{i\theta}}{(\lambda+p)^2};z\right) \notin \Omega \tag{2.21}$$

whenever $z\in U,\ \theta\in\mathbb{R}$, $\Re(Le^{-i\theta}) \geqslant (k-1)kM$ for all real θ and $k \geqslant 1$.

COROLLARY 2.5. *Let* $\phi \in \Phi_{I,1}[\Omega,M]$. *If* $f(z)\in\mathscr{A}_p$ *satisfies*

$$\phi\left(\frac{I_p(n,\lambda)f(z)}{z^{p-1}},\frac{I_p(n+1,\lambda)f(z)}{z^{p-1}},\frac{I_p(n+2,\lambda)f(z)}{z^{p-1}};z\right) \in \Omega,$$

then

$$\left|\frac{I_p(n,\lambda)f(z)}{z^{p-1}}\right| < M.$$

In the special case $\Omega = q(U) = \{\omega : |\omega| < M\}$, the class $\Phi_{I,1}[\Omega,M]$ is simply denoted by $\Phi_{I,1}[M]$.

COROLLARY 2.6. *Let* $\phi \in \Phi_{I,1}[M]$. *If* $f(z)\in\mathscr{A}_p$ *satisfies*

$$\left|\phi\left(\frac{I_p(n,\lambda)f(z)}{z^{p-1}},\frac{I_p(n+1,\lambda)f(z)}{z^{p-1}},\frac{I_p(n+2,\lambda)f(z)}{z^{p-1}};z\right)\right| < M,$$

then

$$\left|\frac{I_p(n,\lambda)f(z)}{z^{p-1}}\right| < M.$$

REMARK 2.2. When $\Omega = U$, $\lambda = a-1$ $(a>0)$, $p=1$ and $M=1$, Corollary 2.5 reduces to [8, Theorem 2, p. 231]. When $\Omega = U$, $\lambda = 1$, $p=1$ and $M=1$, Corollary 2.5 reduces to [5, Theorem 1, p. 477].

COROLLARY 2.7. *If $f(z) \in \mathscr{A}_p$, then,*

$$\left|\frac{I_p(n+1,\lambda)f(z)}{z^{p-1}}\right| < M \Rightarrow \left|\frac{I_p(n,\lambda)f(z)}{z^{p-1}}\right| < M.$$

This follows from Corollary 2.6 by taking $\phi(u,v,w;z) = v$.

COROLLARY 2.8. *If $M>0$ and $f(z) \in \mathscr{A}_p$ satisfies*

$$\left|(\lambda+p)^2\frac{I_p(n+2,\lambda)f(z)}{z^{p-1}} + (\lambda+p)\frac{I_p(n+1,\lambda)f(z)}{z^{p-1}} - (\lambda+p-1)^2\frac{I_p(n,\lambda)f(z)}{z^{p-1}}\right|$$

$$< [3(\lambda+p)-1]M, \quad \textit{then} \quad \left|\frac{I_p(n,\lambda)f(z)}{z^{p-1}}\right| < M. \tag{2.22}$$

Proof. This follows from Corollary 2.5 by taking $\phi(u,v,w;z) = (\lambda+p)^2 w + (\lambda+p)v - (\lambda+p-1)^2 u$ and $\Omega = h(U)$ where $h(z) = (3(\lambda+p)-1)Mz$, $M>0$. To use Corollary 2.5, we need to show that $\phi \in \Phi_{I,1}[\Omega, M]$, that is, the admissible condition (2.21) is satisfied. This follows since

$$\begin{aligned}
&\left|\phi\left(Me^{i\theta}, \frac{k+\lambda+p-1}{\lambda+p}Me^{i\theta}, \frac{L+[(2(\lambda+p)-1)k+(\lambda+p-1)^2]Me^{i\theta}}{(\lambda+p)^2}; z\right)\right| \\
&= \left|L+[(2(\lambda+p)-1)k+(\lambda+p-1)^2]Me^{i\theta}+(k+\lambda+p-1)Me^{i\theta}-(\lambda+p-1)^2Me^{i\theta}\right| \\
&= \left|L+[(2k+1)(\lambda+p)-1]Me^{i\theta}\right| \geqslant [(2k+1)(\lambda+p)-1]M + \Re(Le^{-i\theta}) \\
&\geqslant [(2k+1)(\lambda+p)-1]M + k(k-1)M \geqslant (3(\lambda+p)-1)M
\end{aligned}$$

$z \in U$, $\theta \in \mathbb{R}$, $\Re(Le^{-i\theta}) \geqslant k(k-1)M$ and $k \geqslant 1$. Hence by Corollary 2.5, we deduce the required result. □

DEFINITION 2.5. Let Ω be a set in $\mathbb{C}$ and $q(z) \in \mathscr{Q}_1 \cap \mathscr{H}$. The class of admissible functions $\Phi_{I,2}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u,v,w;z) \notin \Omega$$

whenever

$$u = q(\zeta),\ v = \frac{1}{\lambda+p}\left((\lambda+p)q(\zeta) + \frac{k\zeta q'(\zeta)}{q(\zeta)}\right) \qquad (q(\zeta) \neq 0),$$

$$\Re\left\{\frac{(\lambda+p)v(w-v)}{v-u} - (\lambda+p)(2u-v)\right\} \geqslant k\,\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ and $k \geqslant 1$.

THEOREM 2.7. *Let* $\phi \in \Phi_{I,2}[\Omega, q]$ *and* $I_p(n,\lambda)f(z) \neq 0$. *If* $f(z) \in \mathscr{A}_p$ *satisfies*

$$\left\{\phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}; z\right) : z \in U\right\} \subset \Omega, \tag{2.23}$$

then

$$\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \prec q(z).$$

Proof. Define an analytic function $p(z)$ in U by

$$p(z) := \frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}. \tag{2.24}$$

By making use of (2.3) and (2.24), we get

$$\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)} = \frac{1}{\lambda+p}\left[(\lambda+p)p(z) + \frac{zp'(z)}{p(z)}\right]. \tag{2.25}$$

Further computations show that

$$\frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)} = p(z)+\frac{1}{\lambda+p}\left[\frac{zp'(z)}{p(z)}+\frac{(\lambda+p)zp'(z)+\frac{zp'(z)}{p(z)}-\left(\frac{zp'(z)}{p(z)}\right)^2+\frac{z^2p''(z)}{p(z)}}{(\lambda+p)p(z)+\frac{zp'(z)}{p(z)}}\right]. \tag{2.26}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r,\ v = r+\frac{1}{\lambda+p}\left(\frac{s}{r}\right),\ w = r+\frac{1}{\lambda+p}\left[\frac{s}{r}+\frac{(\lambda+p)s+\frac{s}{r}-(\frac{s}{r})^2+\frac{t}{r}}{(\lambda+p)r+\frac{s}{r}}\right]. \tag{2.27}$$

Let

$$\begin{aligned}\psi(r,s,t;z) &= \phi(u,v,w;z) \qquad (2.28)\\ &= \phi\left(r, \frac{1}{\lambda+p}\left[(\lambda+p)r+\frac{s}{r}\right], \frac{1}{\lambda+p}\left[(\lambda+p)r+\frac{s}{r}+\frac{(\lambda+p)s+\frac{s}{r}-(\frac{s}{r})^2+\frac{t}{r}}{(\lambda+p)r+\frac{s}{r}}\right]; z\right).\end{aligned}$$

The proof shall make use of Theorem 1.1. Using equations (2.24), (2.25) and (2.26), from (2.28), we obtain

$$\psi(p(z), zp'(z), z^2p''(z); z) = \phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}; z\right). \tag{2.29}$$

Hence (2.23) becomes

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_{I,2}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.1. Note that

$$\frac{t}{s}+1 = \frac{(\lambda+p)v(w-v)}{v-u} - (\lambda+p)(2u-v),$$

and hence $\psi \in \Psi[\Omega, q]$. By Theorem 1.1, $p(z) \prec q(z)$ or

$$\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \prec q(z).$$

□

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$, for some conformal mapping $h(z)$ of U onto Ω. In this case the class $\Phi_{I,2}[h(U), q]$ is written as $\Phi_{I,2}[h, q]$. In the particular case $q(z) = 1+Mz,\ M > 0$, the class of admissible functions $\Phi_{I,2}[\Omega, q]$ becomes the class $\Phi_{I,2}[\Omega, M]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 2.7.

THEOREM 2.8. *Let* $\phi \in \Phi_{I,2}[h, q]$. *If* $f(z) \in \mathscr{A}_p$ *satisfies*

$$\phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}; z\right) \prec h(z), \tag{2.30}$$

then

$$\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \prec q(z).$$

DEFINITION 2.6. Let Ω be a set in $\mathbb{C}$. The class of admissible functions $\Phi_{I,2}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ such that

$$\phi\Bigg(1+Me^{i\theta},\ 1+\frac{k+(\lambda+p)(1+Me^{i\theta})}{(\lambda+p)(1+Me^{i\theta})}Me^{i\theta},\ 1+\frac{k+(\lambda+p)(1+Me^{i\theta})}{(\lambda+p)(1+Me^{i\theta})}Me^{i\theta} + \frac{(M+e^{-i\theta})[Le^{-i\theta}+[\lambda+p+1]kM+(\lambda+p)kM^2e^{i\theta}]-k^2M^2}{(\lambda+p)(M+e^{-i\theta})[(\lambda+p)e^{-i\theta}+(2(\lambda+p)+k)M+(\lambda+p)M^2e^{i\theta}]}; z\Bigg) \notin \Omega \tag{2.31}$$

$z \in U,\ \theta \in \mathbb{R},\ \Re(Le^{-i\theta}) \geqslant (k-1)kM$ for all real θ and $k \geqslant 1$.

COROLLARY 2.9. *Let* $\phi \in \Phi_{I,2}[\Omega, M]$. *If* $f(z) \in \mathscr{A}_p$ *satisfies*

$$\phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}; z\right) \in \Omega,$$

then

$$\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \prec 1+Mz.$$

When $\Omega = \{\omega : |\ \omega - 1| < M\} = q(U)$, the class $\Phi_{I,2}[\Omega, M]$ is denoted by $\Phi_{I,2}[M]$

COROLLARY 2.10. *Let* $\phi \in \Phi_{I,2}[M]$. *If* $f(z) \in \mathscr{A}_p$ *satisfies*

$$\left|\phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}; z\right) - 1\right| < M,$$

then

$$\left|\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} - 1\right| < M.$$

COROLLARY 2.11. *If* $M > 0$ *and* $f(z) \in \mathscr{A}_p$ *satisfies*

$$\left|\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)} - \frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right| < \frac{M}{(\lambda+p)(1+M)},$$

then

$$\left|\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} - 1\right| < M.$$

This follows from Corollary 2.9 by taking $\phi(u,v,w;z) = v-u$ and $\Omega = h(U)$ where $h(z) = \frac{M}{(\lambda+p)(1+M)}z$.

3. Superordination of the Multiplier Transformation

The dual problem of differential subordination, that is, differential superordination of the multiplier transformation is investigated in this section. For this purpose the class of admissible functions is given in the following definition.

DEFINITION 3.1. Let Ω be a set in $\mathbb{C}$ and $q(z) \in \mathscr{H}[0,p]$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_I[\Omega,q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u,v,w;\zeta) \in \Omega$$

whenever

$$u = q(z), \quad v = \frac{(zq'(z)/m) + \lambda q(z)}{\lambda + p},$$

$$\Re\left\{\frac{(\lambda+p)^2 w - \lambda^2 u}{(\lambda+p)v - \lambda u} - 2\lambda\right\} \leqslant \frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U$ and $m \geqslant p$

THEOREM 3.1. *Let* $\phi \in \Phi'_I[\Omega,q]$. *If* $f(z) \in \mathscr{A}_p$, $I_p(n,\lambda)f(z) \in \mathscr{Q}_0$ *and*

$$\phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z); z\right)$$

is univalent in U, *then*

$$\Omega \subset \{\phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z); z\right) : z \in U\} \tag{3.1}$$

implies

$$q(z) \prec I_p(n,\lambda)f(z).$$

Proof. From (2.8) and (3.1), we have

$$\Omega \subset \left\{\psi\left(p(z), zp'(z), z^2p''(z); z\right) : z \in U\right\}.$$

From (2.6), we see that the admissibility condition for $\phi \in \Phi'_I[\Omega,q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.2. Hence $\psi \in \Psi'_p[\Omega,q]$, and by Theorem 1.2, $q(z) \prec p(z)$ or

$$q(z) \prec I_p(n,\lambda)f(z).$$

□

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case the class $\Phi_I'[h(U),q]$ is written as $\Phi_I'[h,q]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.1.

THEOREM 3.2. *Let $q(z) \in \mathscr{H}[0,p]$, h(z) is analytic on U and $\phi \in \Phi_I'[h,q]$. If $f(z) \in \mathscr{A}_p$, $I_p(n,\lambda)f(z) \in \mathscr{Q}_0$ and $\phi\,(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z);z)$ is univalent in U, then*

$$h(z) \prec \phi\,(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z);z) \tag{3.2}$$

implies

$$q(z) \prec I_p(n,\lambda)f(z).$$

Theorem 3.1 and 3.2 can only be used to obtain subordinants of differential superordination of the form (3.1) or (3.2). The following theorem proves the existence of the best subordinant of (3.2) for certain ϕ.

THEOREM 3.3. *Let $h(z)$ be analytic in U and $\phi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$. Suppose that the differential equation*

$$\phi\left(q(z), \frac{zq'(z)+\lambda q(z)}{\lambda+p}, \frac{z^2q''(z)+(2\lambda+1)zq'(z)+\lambda^2 q(z)}{(\lambda+p)^2}; z\right) = h(z) \tag{3.3}$$

has a solution $q(z) \in \mathscr{Q}_0$. If $\phi \in \Phi_I'[h,q]$, $f(z) \in \mathscr{A}_p$, $I_p(n,\lambda)f(z) \in \mathscr{Q}_0$ and

$$\phi\,(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z);z)$$

is univalent in U, then

$$h(z) \prec \phi\,(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z);z)$$

implies

$$q(z) \prec I_p(n,\lambda)f(z)$$

and $q(z)$ is the best subordinant.

Proof. The proof is similar to the proof of Theorem 2.4 and is therefore omitted.

□

Combining Theorems 2.2 and 3.2, we obtain the following sandwich theorem.

COROLLARY 3.1. *Let $h_1(z)$ and $q_1(z)$ be analytic functions in U, $h_2(z)$ be univalent function in U, $q_2(z) \in \mathscr{Q}_0$ with $q_1(0) = q_2(0) = 0$ and $\phi \in \Phi_I[h_2,q_2] \cap \Phi_I'[h_1,q_1]$. If $f(z) \in \mathscr{A}_p$, $I_p(n,\lambda)f(z) \in \mathscr{H}[0,p] \cap \mathscr{Q}_0$ and*

$$\phi\,(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z);z)$$

is univalent in U, then

$$h_1(z) \prec \phi\,(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z);z) \prec h_2(z),$$

implies

$$q_1(z) \prec I_p(n,\lambda)f(z) \prec q_2(z).$$

DEFINITION 3.2. Let Ω be a set in $\mathbb{C}$ and $q(z) \in \mathscr{H}_0$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_{I,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; \zeta) \in \Omega$$

whenever

$$u = q(z), \quad v = \frac{(zq'(z)/m) + (\lambda + p - 1)q(z)}{\lambda + p},$$

$$\Re\left\{\frac{(\lambda+p)^2 w - (\lambda+p-1)^2 u}{(\lambda+p)v - (\lambda+p-1)u} - 2(\lambda+p-1)\right\} \leqslant \frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U$ and $m \geqslant 1$.

Now we will give the dual result of Theorem 2.5 for differential superordination.

THEOREM 3.4. *Let $\phi \in \Phi'_{I,1}[\Omega, q]$. If $f(z) \in \mathscr{A}_p$, $\frac{I_p(n,\lambda)f(z)}{z^{p-1}} \in \mathscr{Q}_0$ and*

$$\phi\left(\frac{I_p(n,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+1,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+2,\lambda)f(z)}{z^{p-1}}; z\right)$$

is univalent in U, then

$$\Omega \subset \left\{\phi\left(\frac{I_p(n,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+1,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+2,\lambda)f(z)}{z^{p-1}}; z\right) : z \in U\right\} \tag{3.4}$$

implies

$$q(z) \prec \frac{I_p(n,\lambda)f(z)}{z^{p-1}}.$$

Proof. From (2.19) and (3.4), we have

$$\Omega \subset \left\{\psi\left(p(z), zp'(z), z^2p''(z); z\right) : z \in U\right\}.$$

From (2.17), we see that the admissibility condition for $\phi \in \Phi'_{I,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.2. Hence $\psi \in \Psi'[\Omega, q]$, and by Theorem 1.2, $q(z) \prec p(z)$ or

$$q(z) \prec \frac{I_p(n,\lambda)f(z)}{z^{p-1}}. \qquad \square$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, and $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω and the class $\Phi'_{I,1}[h(U), q]$ is written as $\Phi'_{I,1}[h, q]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.4.

THEOREM 3.5. *Let $q(z) \in \mathscr{H}_0$, h(z) is analytic on U and $\phi \in \Phi'_{I,1}[h, q]$. If $f(z) \in \mathscr{A}_p$, $I_p(n,\lambda)f(z) \in \mathscr{Q}_0$ and $\phi\left(\frac{I_p(n,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+1,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+2,\lambda)f(z)}{z^{p-1}}; z\right)$ is univalent in U, then*

$$h(z) \prec \phi\left(\frac{I_p(n,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+1,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+2,\lambda)f(z)}{z^{p-1}}; z\right) \tag{3.5}$$

implies

$$q(z) \prec \frac{I_p(n,\lambda)f(z)}{z^{p-1}}.$$

Combining Theorems 2.6 and 3.5, we obtain the following sandwich theorem.

COROLLARY 3.2. *Let $h_1(z)$ and $q_1(z)$ be analytic functions in U, $h_2(z)$ be univalent function in U, $q_2(z) \in \mathcal{Q}_0$ with $q_1(0) = q_2(0) = 0$ and $\phi \in \Phi_{I,1}[h_2,q_2] \cap \Phi'_{I,1}[h_1,q_1]$. If $f(z) \in \mathcal{A}_p$, $\frac{I_p(n,\lambda)f(z)}{z^{p-1}} \in \mathcal{H}_0 \cap \mathcal{Q}_0$ and*

$$\phi\left(\frac{I_p(n,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+1,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+2,\lambda)f(z)}{z^{p-1}}; z\right)$$

is univalent in U, then

$$h_1(z) \prec \phi\left(\frac{I_p(n,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+1,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+2,\lambda)f(z)}{z^{p-1}}; z\right) \prec h_2(z),$$

implies

$$q_1(z) \prec \frac{I_p(n,\lambda)f(z)}{z^{p-1}} \prec q_2(z).$$

Now we will give the dual result of Theorem 2.7 for the differential superordination.

DEFINITION 3.3. Let Ω be a set in $\mathbb{C}$, $q(z) \neq 0$, $zq'(z) \neq 0$ and $q(z) \in \mathcal{H}$. The class of admissible functions $\Phi'_{I,2}[\Omega,q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u,v,w;\zeta) \in \Omega$$

whenever

$$u = q(z),\ v = \frac{1}{\lambda+p}\left((\lambda+p)q(z) + \frac{zq'(z)}{mq(z)}\right),$$

$$\Re\left\{\frac{(\lambda+p)v(w-v)}{v-u} - (\lambda+p)(2u-v)\right\} \leqslant \frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U$ and $m \geqslant 1$.

THEOREM 3.6. *Let $\phi \in \Phi'_{I,2}[\Omega,q]$. If $f(z) \in \mathcal{A}_p$, $\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \in \mathcal{Q}_1$ and*

$$\phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}; z\right)$$

is univalent in U, then

$$\Omega \subset \left\{\phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}; z\right) : z \in U\right\} \quad (3.6)$$

implies

$$q(z) \prec \frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}.$$

Proof. From (2.29) and (3.6), we have

$$\Omega \subset \left\{\psi\left(p(z), zp'(z), z^2p''(z); z\right) : z \in U\right\}.$$

From (2.28), we see that the admissibility condition for $\phi \in \Phi'_{I,2}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.2. Hence $\psi \in \Psi'[\Omega, q]$, and by Theorem 1.2, $q(z) \prec p(z)$ or

$$q(z) \prec \frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}.$$

□

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case the class $\Phi'_{I,2}[h(U), q]$ is written as $\Phi'_{I,2}[h, q]$. The following result is an immediate consequence of Theorem 3.6.

THEOREM 3.7. *Let $h(z)$ be analytic in U and $\phi \in \Phi'_{I,2}[h, q]$. If $f(z) \in \mathscr{A}_p$, $\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \in \mathscr{Q}_1$, and $\phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}; z\right)$ is univalent in U, then*

$$h(z) \prec \phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}; z\right), \tag{3.7}$$

implies

$$q(z) \prec \frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}.$$

Combining Theorems 2.8 and 3.7, we obtain the following sandwich theorem.

COROLLARY 3.3. *Let $h_1(z)$ and $q_1(z)$ be analytic functions in U, $h_2(z)$ be univalent function in U, $q_2(z) \in \mathscr{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_{I,2}[h_2, q_2] \cap \Phi'_{I,2}[h_1, q_1]$. If $f(z) \in \mathscr{A}_p$, $\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \in \mathscr{H} \cap \mathscr{Q}_1$, $I_p(n,\lambda)f(z) \neq 0$ and*

$$\phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}; z\right)$$

is univalent in U, then

$$h_1(z) \prec \phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}; z\right) \prec h_2(z),$$

implies

$$q_1(z) \prec \frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \prec q_2(z).$$

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