Mathematical
Inequalities
& Applications

Volume 12, Number 1 (2009), 209–215

VARIATION OF PARAMETERS FOR DIFFERENTIAL EQUATIONS WITH CAUSAL MAPS

Z. DRICI, F. A. MCRAE AND J. VASUNDHARA DEVI

(*communicated by V. Lakshmikantham*)

Abstract. In this paper we obtain variation of parameters formula for differential equations with causal maps.

1. Introduction

The concept of causal operator allows a unified treatment of a wide array of dynamic systems described by functional equations, such as ordinary differential equations, integrodifferential equations, differential equations with finite or infinite delay, Volterra equations, and neutral functional equations, to name only a few. See Corduneanu [1] for a recent monograph on the subject.

Because of its generality, the causal operator has attracted much attention, and the theory of dynamic systems with such operators has already been the subject of a number of investigations [2, 3, 4, 5]. In this paper, we obtain nonlinear variation of parameters formula in the general set-up of differential equations with causal operators, and as a prerequisite we obtain the continuity and differentiability of solutions with respect to initial values.

2. Preliminaries

Let $E = C[J, \mathbb{R}^n]$, where $J = [t_0, t_0 + T]$.

DEFINITION 2.1. Suppose $Q \in C[E, E]$, then Q is said to be a causal map or a nonanticipative map if $u(s) = v(s)$, $t_0 \leqslant s \leqslant t$, where u, $v \in E$ implies $(Qu)(s) = (Qv)(s)$, $t_0 \leqslant s \leqslant t$.

Consider the differential equation

$$\begin{cases} u'(t) = (Qu)(t) \\ u(t_0) = u_0 \end{cases} \tag{2.1}$$

where $Q \in C[D, E]$ is a causal operator and D is an open set in E.

Mathematics subject classification (2000): 34D20.
Keywords and phrases: Causal operators, variation of parameters.

Paper MIA-12-17

We begin with the basic comparison theorem, a necessary tool in our work.

THEOREM 2.2. *Assume that*

(a) $v, w \in C^1[J, \mathbb{R}^n]$, $Q \in C[E, E]$,

$$\begin{aligned} v'(t) &\leqslant (Qv)(t) \\ w'(t) &\geqslant (Qw)(t), \qquad t \in J, \end{aligned}$$

and for any $u, v \in E$ *such that* $u \geqslant v$, $t \in J$, $(Qu)(t) - (Qv)(t) \leqslant L(u - v)(t)$ *for some* $L > 0$;

(b) *for* $t_1 \in J$, $v(t) \leqslant u(t)$, $t_0 \leqslant t \leqslant t_1$ *implies* $(Qv)(t_1) \leqslant (Qu)(t_1)$. *Then,* $v(t_0) \leqslant w(t_0)$ *implies* $v(t) \leqslant w(t)$, $t \geqslant t_0$, $t \in J$.

Proof. We shall first prove the result for strict inequalities and later extend it to nonstrict inequalities. Suppose the result does not hold. Then there exists a $t_1 \in J$, $t_1 > t_0$ such that $v(t) < w(t)$, $t_0 < t < t_1$ and $v(t_1) = w(t_1)$.

This implies from assumption (b) that $(Qv)(t_1) \leqslant (Qw)(t_1)$.

Also, for $h < 0$, $v(t_1 + h) < w(t_1 + h)$. Hence, we obtain

$$D_{-}v(t_1) \geqslant D_{-}w(t_1).$$

Further, we have $v'(t_1) \leqslant (Qv)(t_1) \leqslant (Qw)(t_1) < w'(t_1)$, which is a contradiction to the earlier statement. Hence, the conclusion holds. For the proof of nonstrict inequalities, set

$$\hat{w}(t) = w(t) + \epsilon e^{2Lt}, \qquad \epsilon > 0.$$

Then,

$$\hat{w}'(t) > (Q\hat{w})(t)$$

and

$$\hat{w}(t_0) > w(t_0) \geqslant v(t_0).$$

Thus, we have

$$\begin{aligned} \hat{w}'(t) &> (Q\hat{w})(t), \\ v'(t) &\leqslant (Qv)(t), \qquad t \in J, \\ v(t_0) &< \hat{w}(t_0). \end{aligned}$$

Hence, we conclude that

$$v(t) < \hat{w}(t) = w(t) + \epsilon e^{2Lt}.$$

Letting $\epsilon \to 0$, we have

$$v(t) \leqslant w(t), \quad t \in J,$$

and hence the proof. □

We first define the norm

$$\|u - v\|_0(t) = \max_{t_0 \leqslant s \leqslant t} \|u(s) - v(s)\|,$$

and next list some known results which are needed later. We begin with an existence and uniqueness result, which is a special case of Theorem 3.5 in [2].

THEOREM 2.3. *Assume that*

(a) $Q \in C[B,E]$ *is a causal map where*

$$B = B[u_0, b] = \{u \in E : \|u - u_0\|_0(t) \leqslant b\}$$

and $\|Qu\|_0(t) \leqslant M_1$, *on* B*;*

(b) $g \in C[J \times [0,2b], \mathbb{R}_+]$ *where* $J = [t_0, t_0 + T]$, $g(t,w) \leqslant M_2$ *on* $J \times [0,2b]$, $g(t,0) \equiv 0$, $g(t,w)$ *is nondecreasing in* w *for each* $t \in J$ *and* $w(t) \equiv 0$ *is the only solution of the scalar differential equation*

$$w' = g(t,w),\ w(t_0) = 0 \text{ on } J; \tag{2.2}$$

(c) $\|(Qu)(t) - (Qv)(t)\| \leqslant g(t, \|u - v\|_0(t))$ *on* B.

Then, there exists a unique solution $u(t,t_0,u_0)$ *of (2.1) on* $J_0 = [t_0, t_0 + \eta]$ *where* $\eta = \min[T, \frac{b}{M}]$, *and* $M = max\{M_1, M_2\}$.

The next result, on the continuous dependence of the solutions of (2.1) with respect to the initial conditions (t_0, u_0) is a special case of Theorem 3.8 in [2].

THEOREM 2.4. *Assume that*

(a) *the assumptions of Theorem 2.3 hold;*

(b) *the solutions* $w(t,t_0,w_0)$ *of (2.2) through every point* (t_0,w_0) *are continuous with respect to* (t_0,w_0).

Then, the solutions $u(t,t_0,u_0)$ *of (2.1) are continuous with respect to* (t_0,u_0).

Next, we state a result on the dependence of solutions of (2.1) with respect to a parameter.

THEOREM 2.5. *Let* $Q \in C[E \times \mathbb{R}, E]$ *and* $\lim_{\mu \to \mu_0} (Q(u,\mu))(t) = (Q(u,\mu_0))(t)$ *uniformly in* t *and* u. *Suppose further that*

$$\|(Q(u,\mu))(t) - (Q(v,\mu))(t)\| \leqslant g(t, \|u - v\|_0(t))$$

for (u,μ), $(v,\mu) \in E \times \mathbb{R}$ *and* $t \in J$, *where* g *satisfies condition (b) of Theorem 2.3. Let* $\mu_0 \in \mathbb{R}$. *Then, given* $\epsilon > 0$, *there exists a* $\delta = \delta(\epsilon) > 0$ *such that for every* μ *with* $|\mu - \mu_0| < \delta$, *the IVP*

$$u'(t) = (Q(u,\mu))(t),\ u(t_0) = u_0 \in \mathbb{R}^n$$

admits a unique solution $u(t,t_0,u_0,\mu_0)$ *satisfying*

$$\|u(t,t_0,u_0,\mu) - u(t,t_0,u_0,\mu_0)\| < \epsilon,\ t \in J_0.$$

The proof of this result can be constructed on the basis of the corresponding theorem for differential systems in [5]. We omit the details.

For an operator $Q \in C[E,E]$, which is Frechet differentiable, the integral mean value theorem can be expressed by the following result, which is useful in our work.

THEOREM 2.6. *Let* $Q \in C[B,E]$. *Assume that the Frechet derivative* $(Qu)_u$ *exists and is continuous on* B. *Then, for* $u_1, u_2 \in B$ *and* $t \in J_0$,

$$(Qu_1)(t) - (Qu_2)(t) = \int_0^1 [Q(\lambda u_1 + (1-\lambda)u_2)]_u(t)\ (u_2 - u_1)(t) d\lambda.$$

3. Differentiability

We begin with the following theorem, which establishes the continuity and differentiability of the solutions with respect to initial values.

For convenience, we rewrite equations (2.1) as

$$\begin{cases} u'(t) &= (Q_{t_0}u)(t) \\ u(t_0) &= u_0 \end{cases} \tag{3.1}$$

where $Q_{t_0} \in C[D,E]$ denotes a causal operator.

THEOREM 3.1. *Let $u(t,t_0,u_0)$ be the unique solution of (2.1), existing on some interval $J_0 = [t_0, t_0+\eta]$. Assume that the Frechet derivative $(Q_{t_0}u)_u \equiv \mathscr{L}(t,t_0,u_0)$ exists and is continuous on D. Then,*

(a) $\Phi(t,t_0,u_0) = \frac{\partial u(t,t_0,u_0)}{\partial u_0}$ *exists and is a solution of*

$$z'(t) = \mathscr{L}(t,t_0,u_0)(z) \text{ such that } \Phi(t_0,t_0,u_0) = I, \tag{3.2}$$

where $\mathscr{L}(t,t_0,u_0)$ is a linear operator and I is the identity matrix;

(b) $\Psi(t,t_0,u_0) = \frac{\partial u(t,t_0,u_0)}{\partial t_0}$ *exists and is a solution of*

$$\begin{aligned} y'(t) &= \mathscr{L}(t,t_0,u_0)(y) - (\hat{Q}_{t_0}u)(t), \\ y(t_0) &= -(Q_{t_0}u)(t_0) \end{aligned} \tag{3.3}$$

where $(\hat{Q}_{t_0})(t)$ is the term in $(Qu)(t)$ that depends on the initial time t_0;

(c) *the functions $\Phi(t,t_0,u_0)$ and $\Psi(t,t_0,u_0)$ satisfy the relation*

$$\Psi(t,t_0,u_0) + \Phi(t,t_0,u_0)(Q_{t_0}u)(t_0) = \int_{t_0}^{t} R(t,s;t_0,u_0)(\hat{Q}_{t_0}u)(s)ds. \tag{3.4}$$

where $R(t,s;t_0,u_0)$ is the solution of the IVP

$$\frac{\partial R(t,s;t_0,u_0)}{\partial s} + \mathscr{L}(t,t_0,u_0)(R(t,s;t_0,u_0)) = 0, \tag{3.5}$$

$$R(t,t;t_0,u_0) = I,\ t_0 \leqslant s \leqslant t \text{ and } R(t,t_0;t_0,u_0) = \Phi(t,t_0,u_0).$$

Proof. Under the assumptions on Q, it is clear that solutions $u(t,t_0,u_0)$ of (3.1) exist, are unique and continuous in t, t_0, and u_0 on some interval. Consequently, the operator $\mathscr{L}(t,t_0,u_0)$ is continuous in t, t_0, and u_0 on that interval. Therefore, the solutions of the linear initial-value problems (3.2) and (3.3) exist and are unique over the same interval.

To prove (a), let $e_k = (e_k^1, e_k^2, \ldots, e_k^n)$ be the vector such that $e_k^j = 0$ if $j \neq k$ and $e_k^k = 1$. Then for some $k, \tilde{u}(t,h) = u(t,t_0,u_0+e_kh)$ is defined on J_0 and $\lim_{h\to 0} \tilde{u}(t,h) =$

$u(t,t_0,u_0)$ uniformly on J_0. Let $u(t) = u(t,t_0,u_0)$ and $u(t,h) = \tilde{u}(t,h) - u(t)$. Then differentiating $u(t,h)$ with respect to t and using Theorem 2.6, it follows that

$$\begin{aligned} \frac{d}{dt}u(t,h) &= \tilde{u}'(t,h) - u'(t) \\ &= (Q_{t_0}\tilde{u} - Q_{t_0}u)(t) \\ &= \int_0^1 [Q_{t_0}(\lambda\tilde{u} + (1-\lambda)u)]_u(t)d\lambda(\tilde{u}(t,h) - u(t)) \\ &\equiv \mathscr{L}(t,t_0,u_0,h)(\tilde{u}(t+h) - u(t)) \end{aligned}$$

Dividing by $h, h \neq 0$,

$$\frac{u'(t,h)}{h} = \mathscr{L}(t,t_0,u_0,h)\frac{u(t,h)}{h},$$

and since

$$\frac{u(t_0,h)}{h} = \frac{u(t_0,t_0,u_0+e_kh) - u(t_0,t_0,u_0)}{h} = e_k,$$

it is clear that $\frac{u(t,h)}{h}$ is a solution of the following initial-value problem

$$\begin{cases} z'(t) = \mathscr{L}(t,t_0,u_0,h)z, \\ z(t_0) = e_k, \end{cases} \tag{3.6}$$

where $\mathscr{L}(t,t_0,u_0,h) = \int_0^1 [Q_{t_0}(\lambda\tilde{u} + (1-\lambda)u)]_u(t)d\lambda$. Since $\lim_{h\to 0}\tilde{u}(t,h) = u(t)$ uniformly on J_0, continuity of $(Q_{t_0}u)_u$ implies that $\lim_{h\to 0}\mathscr{L}(t,t_0,u_0,h) = \mathscr{L}(t,t_0,u_0)$ uniformly on J_0. Also observe that $\mathscr{L}(t,t_0,u_0,h)$ satisfies the second condition of Theorem 2.5 with $g(t, ||\tilde{u} - u||_0(t)) = ||\mathscr{L}(t,t_0,u_0,h)||\ \ ||\tilde{u} - u||_0(t)$. Hence, by Theorem 2.5, we conclude that (3.6) admits a unique solution, which is continuous with respect to h for fixed t, t_0, u_0.

Next, consider the family of initial-value problems defined by (3.6), with a small parameter h, for $k = 1, 2, \cdots, n$. Since the solutions corresponding to this family of initial-value problems are all continuous functions of h for fixed t, t_0, u_0, it follows that, $\lim_{h\to 0}\frac{u(t,h)}{h} = \frac{\partial}{\partial u_0}u(t,t_0,u_0)$, which is the solution of (3.2) with $\frac{\partial}{\partial u_0}u(t_0,t_0,u_0) = I$. Also, in view of the assumptions on $\mathscr{L}(t,t_0,u_0)$, it is clear that $\frac{\partial}{\partial u_0}u(t,t_0,u_0)$ is also continuous with respect to its arguments.

To prove (b), define $\hat{u}(t,h) = u(t,t_0+h,u_0)$. Then, differentiating with respect to t we have

$$\begin{aligned} u(t,h) &= \hat{u}(t,h) - u(t) \\ u'(t,h) &= (Q_{t_0+h}\hat{u})(t) - (Q_{t_0}u)(t) \\ &= (Q_{t_0+h}\hat{u})(t) - (Q_{t_0+h}u)(t) - (Q_{t_0}u)(t_0+h) \\ &= \int_0^1 [Q_{t_0+h}(\lambda\hat{u} + (1-\lambda)u)]_u(t)d\lambda](\hat{u}(t,h) - u(t)) - (Q_{t_0}u)(t_0+h) \\ &\equiv \hat{\mathscr{L}}(t,t_0,u_0,h)(\hat{u}(t,h) - u(t)) - (\hat{Q}_{t_0}u)(t) \end{aligned}$$

It is clear that $\frac{u(t,h)}{h}$ is solution of the following initial-value problem

$$\begin{cases} y'(t) = \hat{\mathscr{L}}(t,t_0,u_0,h)(z) - (\hat{Q}_{t_0}y)(t) \\ y(t_0+h) = \frac{u(t_0+h,h)}{h} = -\frac{1}{h}\int_{t_0}^{t_0+h}(Q_{t_0}u)(s)ds. \end{cases} \tag{3.7}$$

where $\hat{\mathscr{L}}(t,t_0,u_0,h) = \int_0^1 [Q_{t_0+h}(\lambda\hat{u} + (1-\lambda)u)]_u(t)d\lambda$.

Noting that $\lim\limits_{h\to 0}\frac{1}{h}\int_{t_0}^{t_0+h}(Q_{t_0}u)(s)ds = (Q_{t_0}u)(t_0)$ and using an argument similar the argument used in the proof of (a), we see that $\frac{\partial}{\partial t_0}u(t,t_0,u_0)$ exists, is continuous in its arguments, and is a solution of (3.3).

The result in (c) follows from the fact that $\Phi(t,t_0,u_0)$ and $\Psi(t,t_0,u_0)$ are solutions of (3.2) and (3.3), respectively, and the fact that $R(t,s;t_0,u_0)$ is the solution of the IVP (3.5), which is a linear equation. Observe that (3.2) is the homogeneous linear equation corresponding to (3.3). □

4. Variation of Parameters

Having established the continuity and differentiability of the solutions of (2.1) with respect to initial values, we now proceed to obtain the nonlinear variation of parameters formula for solutions $r(t,t_0,u_0)$ of the perturbed system

$$\begin{cases} r'(t) = (Q_{t_0}r)(t) + P_{t_0}r)(t) \\ r(t_0) = u_0. \end{cases} \tag{4.1}$$

where $P_{t_0} \in C[D,E]$.

THEOREM 4.1. *Suppose the hypotheses of Theorem 3.1 hold. Let* $r(t,t_0,u_0)$ *be any solution of (4.1) existing on* J_0. *Then* $r(t,t_0,u_0)$ *satisfies the integral equation*

$$\begin{aligned} r(t,t_0,u_0) &= u(t,t_0,u_0) + \int_{t_0}^t\int_s^t R(s,t;t_0,u_0)(\hat{Q}_s u)(\sigma)d\sigma ds \\ &+ \int_{t_0}^t \Phi(t,s,r(s))(P_{t_0}r)(s)ds \end{aligned} \tag{4.2}$$

where $R(s,t;t_0,u_0)$ *is the solution of the IVP (3.5).*

Proof. Setting $p(s) = u(t,s,r(s))$ where $r(s) = r(s,t_0,u_0)$, we have

$$\begin{aligned} p'(s) &= \frac{\partial u(t,s,r(s))}{\partial t_0} + \frac{\partial u(t,s,r(s))}{\partial u_0}r'(s) \\ &= \Psi(t,s,r(s)) + \Phi(t,s,r(s))[(Q_{t_0}r)(s) + (P_{t_0}r)(s)]. \end{aligned}$$

Integrating from t_0 to t, we have

$$\begin{aligned} p(t) - p(t_0) &= \int_{t_0}^t[\Psi(t,s,r(s)) + \Phi(t,s,r(s))(Q_{t_0}r)(s)]ds \\ &\quad + \int_{t_0}^t \Phi(t,s,r(s))(P_{t_0}r)(s)ds. \\ &= \int_{t_0}^t\int_s^t R(t,s;t_0,u_0)(\hat{Q}_s u)(\sigma)d\sigma ds \\ &\quad + \int_{t_0}^t \Phi(t,s,r(s))(P_{t_0}r)(s)ds. \end{aligned}$$

Thus, using the fact that

$$u(t,t,r(t)) = r(t,t_0,u_0) \quad \text{and} \quad u(t,t_0,r(t_0)) = u(t,t_0,u_0),$$

we have

$$\begin{aligned} r(t,t_0,u_0) &= u(t,t_0,u_0) + \int_{t_0}^{t}\int_{s}^{t} R(s,t;t_0,u_0)(\hat{Q}_s u)(\sigma)d\sigma ds \\ &\quad + \int_{t_0}^{t} \Phi(t,s,r(s))(P_{t_0} r)(s)ds, \end{aligned}$$

completing the proof. □

REFERENCES

[1] CORDUNEANU, C., *Functional Equations with Causal Operators, Stability and Control: Theory, Methods and Applications*, Taylor and Francis, London, 2002.

[2] DRICI, Z., MCRAE, F. A., AND VASUNDHARA DEVI, J., *Set Differential Equations with Causal Operators*, Mathematical Problems in Engineering, 2005, no. 2, 185–294.

[3] DRICI, Z., MCRAE, F. A., AND VASUNDHARA DEVI, J., *Set Differential Equations with Causal Operators in a Banach Space*, Nonlinear Analysis TMA, Series A, **62** (2005), no. 2, 301–313.

[4] DRICI, Z., MCRAE, F. A., AND VASUNDHARA DEVI, J., *Stability Results for Set Differential Equations with Causal Maps*, Dynamic Systems and Applications, **15**, (2006), no. 3-4, 451–463.

[5] DRICI, Z., MCRAE, F. A., AND VASUNDHARA DEVI, J., *Monotone Iterative Technique for Periodic Boundary Value Problems with Causal Operators*, *Nonlinear Analysis TMA*, **64** (2006), no. 6, 1153–1400.

[6] LAKSHMIKANTHAM, V., LEELA S., *Differential and Integral Inequalities, Volume I*, Academic Press, New York 1969.

[7] LAKSHMIKANTHAM, V., RAMA MOHANA RAO, M., *Theory of Integro-Differential Equations*, Gordon and Breach Science Publishers, Amsterdam, 1995.

(Received June 18, 2006)

Z. Drici
Department of Mathematics and Computer Science
Illinois Wesleyan University
Bloomington, IL 61702-2900
USA

F. A. McRae
Department of Mathematics
Catholic University of America
Washington, DC 20064
USA

J. Vasundhara Devi
Department of Mathematics
Gayatri Vidya Parıshad Institute for Advanced studies Visakhapatnam
India