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A NEW REFINED JORDAN'S INEQUALITY AND ITS APPLICATION

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Abstract. New refined lower and upper bound forms of Jordan's inequality are proved. As an application, the lower bound form is shown to improve L. Yang's inequality that plays a pivotal role in the theory of distribution of values of functions. Some numerical results are included.

1. Introduction

The celebrated Jordan's inequality [2] states that if $x \in (0, \pi/2]$, then

$$\frac{2}{\pi} \leqslant \frac{\sin x}{x} < 1, \tag{1.1}$$

where the left-hand side inequality becomes equality if and only if $x = \pi/2$.

Jordan's inequality and its subsequent refinements are useful in several mathematical areas such as calculus and trigonometry, where specifically the applications of the theory of limits [5] are involved. These are important tools in approximating Riemann zeta function $\zeta(x)$ [2], in improving Yang Le's inequality [6] and its generalization which play an important role in the theory of distribution of values of functions [6, 9, 10].

During the past few years many authors [1, 4, 5, 7, 8] established several refined forms of Jordan's inequality. One of these forms was due to Özban in 2006 [4]. He obtained a new lower bound for the function $\frac{\sin x}{x}$. He showed that

$$\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) + \frac{4(\pi-3)}{\pi^3}\left(x - \frac{\pi}{2}\right)^2 \leqslant \frac{\sin x}{x}, \quad x \in \left(0, \frac{\pi}{2}\right] \tag{1.2}$$

where the equality holds if and only if $x = \pi/2$.

Almost at the same time, a new interesting refined form of Jordan's inequality was established by Zhu [7]. He proved the following theorem (Theorem 1.1).

THEOREM 1.1. *If* $0 < x \leqslant \pi/2$, *then*

$$\frac{2}{\pi}+\frac{\pi^2-4x^2}{\pi^3}+\frac{4(\pi-3)}{\pi^3}\left(x-\frac{\pi}{2}\right)^2 \leqslant \frac{\sin x}{x} \leqslant \frac{2}{\pi}+\frac{\pi^2-4x^2}{\pi^3}+\frac{12-\pi^2}{\pi^3}\left(x-\frac{\pi}{2}\right)^2, \tag{1.3}$$

where both the inequalities become equalities if and only if $x = \pi/2$ *Furthermore,* $\frac{4(\pi-3)}{\pi^3}$ *and* $\frac{12-\pi^2}{\pi^3}$ *are the best constants in (1.3).*

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The main aim of the present paper is to establish one more new refined lower bound form as well as one more new upper bound form of Jordan's inequality. These forms are illustrated to demonstrate their usefulness in the background of most recent contributions.

2. Main results

Our main result, viz., the new refined inequality, follows from Theorem 2.1 below.

THEOREM 2.1. *If* $0 < x \leqslant \pi/2$, *then*

$$1 - B_1 x - B_2 x^2 - B_3 x^3 \leqslant \frac{\sin x}{x}, \tag{2.1}$$

where the equality holds if and only if $x = \pi/2$, *where*

$$B_1 = \frac{4}{\pi^2}(-66 + 43\pi - 7\pi^2),$$
$$B_2 = \frac{4}{\pi^3}(124 - 83\pi + 14\pi^2),$$
$$B_3 = \frac{4}{\pi^4}(12 - 4\pi).$$

Proof. Define a function $f : (0, \pi/2] \to R$ by

$$f(x) = \frac{x - \sin x}{x^2} - B_1 - B_2 x - B_3 x^2. \tag{2.2}$$

Then, we have

$$f^{(4)}(x) = \frac{1}{x^6}(24x + 96x\cos x + 36x^2 \sin x - 8x^3 \cos x - x^4 \sin x - 120 \sin x). \tag{2.3}$$

Now consider the function $g : [0, \pi/2] \to R$ defined by $g(x) = x^6 f^{(4)}(x)$. Then, we get

$$g''(x) = x^4 \sin x.$$

Clearly, $g''(x) > 0$ for all $x \in (0, \pi/2]$, this implies that $g'(x)$ is strictly increasing on $(0, \pi/2]$. Using the equality $g'(0) = 0$, we find $g'(x) > 0$ for all $x \in (0, \pi/2]$. Thus, $g(x)$ is strictly increasing with $g(0) = 0$, it follows that $g(x) > 0$ for all $x \in (0, \pi/2]$. Now combining the function g and the equality (2.3), we obtain $f^4(x) > 0$ for all $x \in (0, \pi/2]$. Hence $f'''(x)$ is strictly increasing and $f'''(x) < 0$ on $x \in (0, \pi/2]$. On the other hand, using the Taylor's formula, we have for $\xi \in (x, \pi/4)$,

$$\begin{aligned} f(x) &= f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{f''(\pi/4)}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{f'''(\xi)}{3!}\left(x - \frac{\pi}{4}\right)^3 \\ &\leqslant f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{f''(\pi/4)}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{\lim_{\xi \to 0} f'''(\xi)}{3!}\left(x - \frac{\pi}{4}\right)^3 \\ &= f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{f''(\pi/4)}{2}\left(x - \frac{\pi}{4}\right)^2 - \frac{(x - \pi/4)^3}{120}, \end{aligned} \tag{2.4}$$

where $x \in (0, \pi/4)$. Now define an auxiliary function $p_1 : [0, \pi/4] \to R$ by

$$p_1(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{f''(\pi/4)}{2}\left(x - \frac{\pi}{4}\right)^2 - \frac{(x - \pi/4)^3}{120}.$$

It is obvious that $f(x) \leqslant p_1(x)$ and $p_1(x)$ is continuous with $p_1'''(x) = -1/20 < 0, x \in [0, \pi/4]$. This implies that $p_1''(x)$ is strictly decreasing on $[0, \pi/4]$. Hence $p_1''(x) > 0$ for all $x \in [0, \pi/4]$ with $p_1''(\pi/4) > 0$. Therefore, $p_1'(x)$ is strictly increasing with $p_1'(\pi/4) > 0$ and $p_1'(0) < 0$. Now consider the fixed point $x_1 \in [0, \pi/4]$ such that $p_1'(x_1) = 0$. Notice that $p_1'(x) > 0$ for all $x \in (x_1, \pi/4]$, this implies that $p_1(x)$ is strictly increasing on $(x_1, \pi/4]$. Since $p_1(\pi/4) = f(\pi/4) < 0$, we have $p_1(x) < 0$ for all $x \in (x_1, \pi/4]$. Notice that $p_1'(x) < 0$ for all $x \in [0, x_1)$. This implies that $p_1(x)$ is strictly decreasing on $x \in [0, x_1)$. Since $p_1(0) < 0$, we have $p_1(x) < 0$ for all $x \in [0, x_1)$. Therefore, $p_1(x) < 0$ for all $x \in [0, \pi/4]$. Thus from the inequality (2.4) and the function $p_1(x)$, we get $f(x) \leqslant 0$ for all $x \in [0, \pi/4]$.

Now we consider the case $x \in [\pi/4, \pi/2]$ below. Using once again the Taylor's formula for $\xi \in (x, \pi/2)$, we obtain

$$\begin{aligned} f(x) &= f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right) + \frac{f''(\pi/2)}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{f'''(\xi)}{3!}\left(x - \frac{\pi}{2}\right)^3 \\ &\leqslant f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right) + \frac{f''(\pi/2)}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{f'''(\pi/4)}{3!}\left(x - \frac{\pi}{2}\right)^3, \quad (2.5) \end{aligned}$$

where $x \in [\pi/4, \pi/2]$. Define an auxiliary function $p_2 : [\pi/4, \pi/2] \to R$ by

$$p_2(x) = f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right) + \frac{f''(\pi/2)}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{f'''(\pi/4)}{3!}\left(x - \frac{\pi}{2}\right)^3.$$

It is obvious that $f(x) \leqslant p_2(x)$ and $p_2(x)$ is continuous with $p_2'''(x) = f'''(\pi/4) < 0$. This implies that $p_2''(x)$ is strictly decreasing on $[\pi/4, \pi/2]$ with $p_2''(\pi/4) > 0$ and $p_2''(\pi/2) < 0$. Let the fixed point $x_2 \in [\pi/4, \pi/2]$ be such that $p_2''(x_2) = 0$.

Notice that $p_2''(x) > 0$ for all $x \in [\pi/4, x_2)$. This implies that $p_2'(x)$ is strictly increasing and $p_2'(x) > p_2'(\pi/4) > 0$ for all $x \in [\pi/4, x_2)$. Therefore $p_2(x)$ is strictly increasing on the semi-open interval - $[\pi/4, x_2)$. From the value

$$\begin{aligned} p_2(x_2) &= f'\left(\frac{\pi}{2}\right)\left(x_2 - \frac{\pi}{2}\right) + \frac{1}{2}\left(x_2 - \frac{\pi}{2}\right)^2 k''(x_2) - \frac{1}{3}\left(x_2 - \frac{\pi}{2}\right)^3 f'''\left(\frac{\pi}{4}\right) \\ &= f'\left(\frac{\pi}{2}\right)\left(x_2 - \frac{\pi}{2}\right) - \frac{1}{3}\left(x_2 - \frac{\pi}{2}\right)^3 f'''\left(\frac{\pi}{4}\right) \\ &< 0, \end{aligned}$$

where

$$x_2 = \frac{\pi}{2} - \frac{f''(\frac{\pi}{2})}{f'''(\frac{\pi}{2})},$$

$$f'\left(\frac{\pi}{2}\right) = -\frac{8(66 - 43\pi + 7\pi^2)}{\pi^3},$$

$$f'''\left(\frac{\pi}{4}\right) = \frac{8(1536\sqrt{2} - 96(2+3\sqrt{2})\pi - 24\sqrt{2}\pi^2 + \sqrt{2}\pi_3}{\pi^5},$$

we have $p_2(x) < 0$ for all $x \in [\pi/4, x_2)$. Observe that $p_2''(x) < 0$ for all $x \in (x_2, \pi/2]$. This implies that $p_2'(x)$ is strictly decreasing and $p_2'(x) > p_2'(\pi/2) > 0$ for all $x \in (x_2, \pi/2)$. Thus $p_2(x)$ is strictly increasing with $p_2(x) < p_2(\pi/2) = 0$. Consequently $p_2(x) \leqslant 0$ for all $x \in [\pi/4, \pi/2]$. Now combining the inequality (2.5) and the function $p_2(x)$, we get $f(x) \leqslant 0$ for all $x \in [\pi/4, \pi/2]$.

Hence, we obtain $f(x) \leqslant 0$ for all $x \in (0, \pi/2]$. Now, multiplying $f(x)$ by x we get the desired result. □

REMARK 2.1. A graph of the distance function $y(x) = h_1(x) - h(x)$, where $h(x) = (\sin x)/x$,

$$h_1(x) = [1 - B_1 x - B_2 x^2 - B_3 x^3], \tag{2.6}$$

is given in Fig. 1.

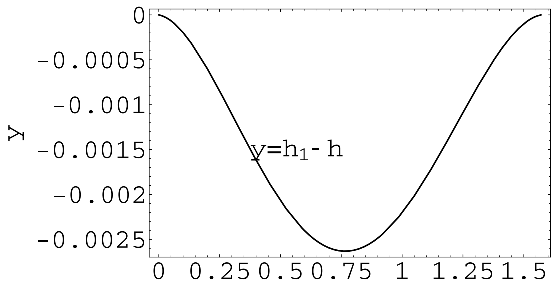

Fig. 1

A graph of the error represented by the function $e(x) = h_1(x) - h_2(x)$, where

$$h_2(x) = 2/\pi + (\pi^2 - 4x^2)/\pi^3 + 4(\pi - 3)(x - \pi/2)^2/\pi^3 \tag{2.7}$$

is given in Fig .2. These imply that $h_2(x) \leqslant h_1(x) \leqslant (\sin x)/x$, where the equalities hold if and only if $x = \pi/2$. Thus, the inequality (2.1) is a new refined form of Jordan's inequality for all $x \in (0, \pi/2]$.

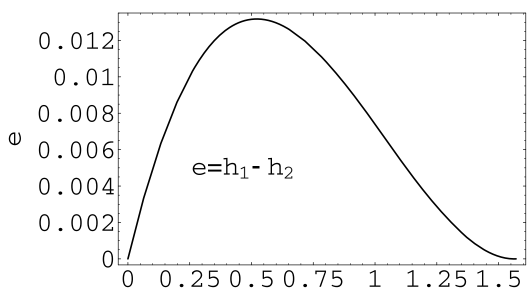

Fig. 2

THEOREM 2.2. *If* $x \in (0, \pi/2]$*, then*

$$\frac{\sin x}{x} \leqslant 1 - C_1 x + C_2 x^2 - B_3 x^3 \tag{2.8}$$

where the equality holds if and only if $x = \pi/2$*, where*

$$C_1 = \frac{4(-75 + 49\pi - 8\pi^2)}{\pi^2},$$
$$C_2 = \frac{4(-142 + 95\pi - 16\pi^2)}{\pi^3},$$

and B_3 *is as defined in Theorem 2.1.*

Proof. Define a function $f : (0, \pi/2] \to R$ by

$$f(x) = \frac{x - \sin x}{x^3} - C_1 \frac{1}{x} + C_2 - B_3(\pi)x. \tag{2.9}$$

Then, we have

$$\begin{aligned} f^{(4)}(x) &= \frac{24x[300x - 196\pi x + \pi^2(5 + 32x)] - 12\pi^2 x(-20 + x^2)\cos x}{\pi^2 x^7} \\ &\quad - \frac{\pi^2(360 - 72x^2 + x^4)\sin x}{\pi^2 x^7}. \end{aligned} \tag{2.10}$$

Now consider a function $g : [0, \pi/2] \to R$ defined by $g(x) = x^7 f^{(4)}(x)$. Then, we get

$$g'''(x) = x^4 \cos x.$$

Clearly, $g'''(x) > 0$ for all $x \in (0, \pi/2]$, this implies that $g''(x)$ is strictly increasing on $(0, \pi/2]$. Using the equality $g''(0) = \frac{192(75-49\pi+8\pi^2)}{\pi^2}$, we find $g''(0) > 0$. This implies that $g''(x) > 0$ for all $x \in (0, \pi/2]$. Hence $g'(x)$ is strictly increasing with $g'(0) = 0$, implying $g'(x) > 0$ for all $x \in (0, \pi/2]$. Therefore, $g(x)$ is strictly increasing with $g(0) = 0$. Thus $g(x) > 0$ for all $x \in (0, \pi/2]$.

Now combining the function g and equality (2.10), we obtain $f^4(x) > 0$ for all $x \in (0, \pi/2]$. Hence $f'''(x)$ is strictly increasing and $f'''(x) < 0$ on $x \in (0, \pi/2]$. On the other hand, using the Taylor's formula for $\xi \in (x, \pi/4)$, we have

$$\begin{aligned} f(x) &= f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{f''(\pi/4)}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{f'''(\xi)}{3!}\left(x - \frac{\pi}{4}\right)^3 \\ &\geqslant f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{f''(\pi/4)}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{f'''(\pi/4)}{3!}\left(x - \frac{\pi}{4}\right)^3, \end{aligned} \tag{2.11}$$

where $x \in (0, \pi/4)$. Now define an auxiliary function $k_1 : [0, \pi/4] \to R$ by

$$k_1(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{f''(\pi/4)}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{f'''(\pi/4)}{3!}\left(x - \frac{\pi}{4}\right)^3.$$

It is obvious that $f(x) \leqslant k_1(x)$ and the function $k_1(x)$ is continuous with $k_1'''(x) = f'''(\pi/4) < 0$, $x \in (0, \pi/4]$, this implies that $k_1''(x)$ is strictly decreasing on $(0, \pi/4]$.

Clearly $k_1''(\pi/4) = f''(\pi/4) > 0$ for all $x \in (0,\pi/4]$. Hence $k_1''(x) > 0$ for all $x \in (0,\pi/4]$. That is, $k_1'(x)$ is strictly increasing with $k_1'(\pi/4) < 0$. So, we have $k_1'(x) < 0$, $x \in (0,\pi/4]$. This implies that $p(x)$ is strictly decreasing on $(0,\pi/4]$. Since $k_1(\pi/4) = f(\pi/4) > 0$, $k_1(x) > 0$ for all $x \in (0,\pi/4]$. Now combining the inequality (2.11) and the function $k_1(x)$, we get $f(x) \geqslant 0$ for all $x \in (0,\pi/4]$.

Consider the case $x \in [\pi/4,\pi/2]$ below. Using once again the Taylor's formula for $\xi \in (x,\pi/2)$, we obtain

$$\begin{aligned} f(x) &= f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\left(x-\frac{\pi}{2}\right) + \frac{f''(\pi/2)}{2}\left(x-\frac{\pi}{2}\right)^2 + \frac{f'''(\xi)}{3!}\left(x-\frac{\pi}{2}\right)^3 \\ &\geqslant f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\left(x-\frac{\pi}{2}\right) + \frac{f''(\pi/2)}{2}\left(x-\frac{\pi}{2}\right)^2 + \frac{f'''(\pi/2)}{3!}\left(x-\frac{\pi}{2}\right)^3, \end{aligned} \tag{2.12}$$

where $x \in [\pi/4,\pi/2]$. Define an auxiliary function $k_2 : [\pi/4,\pi/2] \to R$ by

$$k_2(x) = f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\left(x-\frac{\pi}{2}\right) + \frac{f''(\pi/2)}{2}\left(x-\frac{\pi}{2}\right)^2 + \frac{f'''(\pi/2)}{3!}\left(x-\frac{\pi}{2}\right)^3.$$

Evidently, $f(x) \leqslant k_2(x)$ and $k_2(x)$ is continuous with $k_2'''(x) = f'''(\pi/4) < 0$. This implies that $k_2''(x)$ is strictly decreasing on $[\pi/4,\pi/2]$ with $k_2''(\frac{\pi}{4}) = \frac{4(2664-1888\pi+331\pi^2)}{\pi^5} < 0$. Thus we have $k_2''(x) < 0, x \in [\pi/4,\pi/2]$. Consequently, $k_2'(x)$ is strictly decreasing and we get

$$\begin{aligned} k_2'\left(\frac{\pi}{4}\right) &= -\frac{6168-4216\pi+717\pi^2}{2\pi^4} > 0, \\ k_2'\left(\frac{\pi}{2}\right) &= -\frac{16(75-49\pi+8\pi^2)}{\pi^4} < 0. \end{aligned}$$

Now let the fixed point $x_3 \in [\pi/4,\pi/2]$ be such that $k_2'(x_3) = 0$.

Notice that $k_2'(x) > 0$ for all $x \in [\pi/4,x_3)$, which implies that $k_2(x)$ is strictly increasing and we obtain $k_2(x) > k_2(\pi/4), x \in [\pi/4,x_3)$ where $k_2(\pi/4) = \frac{4024-2704\pi+453\pi^2}{8\pi^3} > 0$. Therefore $k_2(x) > 0$ on $[\pi/4,x_2)$.

Observe that $k_2'(x) < 0$ for all $x \in (x_3,\pi/2]$. This implies that $k_2(x)$ is strictly decreasing and we obtain $k_2(x) > k_2(\pi/2), x \in (x_3,\pi/2]$, where $k_2(\pi/2) = f(\pi/2) = 0$. Therefore $k_2(x) \geqslant 0$ on $x \in (x_3,\pi/2]$.

Consequently, $k_2(x) \geqslant 0, x \in [\pi/4,\pi/2]$. Combining the inequality (2.12) and the function $k_2(x)$, we get $f(x) \geqslant 0$ for all $x \in [\pi/4,\pi/2]$.

Hence, we obtain $f(x) \geqslant 0$ for all $x \in (0,\pi/2]$. Now, multiplying $f(x)$ by x^2 we get the desired result. □

REMARK 2.2. A graph of the distance function $y(x) = h_3(x) - h(x)$, where $h(x) = (\sin x)/x$,

$$h_3(x) = 1 - C_1(\pi)x + C_2(\pi)x^2 - B_3(\pi)x^3, \tag{2.13}$$

is given in Fig. 3.

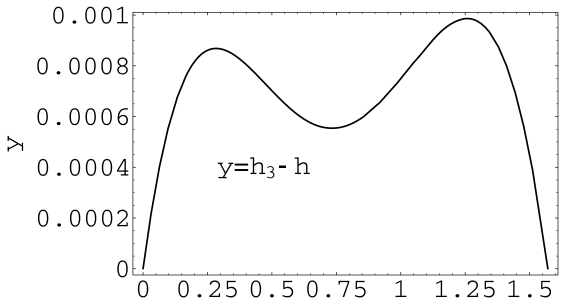

Fig. 3

A graph of the error represented by the function $e(x) = h_3(x) - h_4(x)$, where

$$h_4(x) = \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) + \frac{12 - \pi^2}{\pi^3}\left(x - \frac{\pi}{2}\right)^2, \tag{2.14}$$

is given in Fig. 4. This implies $(\sin x)/x < h_3(x) < h_4(x), x \in (0, 1.2739)$ and $(\sin x)/x \leqslant h_4(x) \leqslant h_3(x), x \in (1.2739, \pi/2)$ where the equalities hold if and only if $x = \pi/2$. Thus, the inequality (2.8) is one refined form of Jordan's inequality for all $x \in (0, \pi/2]$.

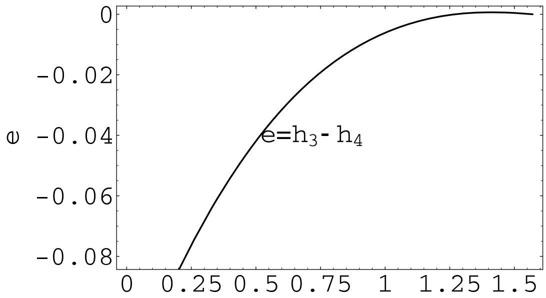

Fig. 4

3. Application

Yang's inequality [6] and its generalization play an important role in the theory of distribution of values of functions [6,9]. In this section, using the inequality (2.1), we show that our result can be used to improve Yang's inequality.

THEOREM 3.1. *Let $n \geqslant 2$ be a natural number, $A_i > 0 (i = 1, 2, ..., n)$ with $\sum_{i=1}^{n} A_i \leqslant \pi$ and $0 \leqslant \lambda \leqslant 1$. Then*

$$U(\lambda) \leqslant \sum_{1 \leqslant i < j \leqslant n} \sin^2 \lambda\pi \leqslant \sum_{1 \leqslant i < j \leqslant n} H_{ij} \leqslant \sum_{1 \leqslant i < j \leqslant n} \lambda^2 \pi^2, \tag{3.1}$$

where $H_{ij} = \cos^2 \lambda A_i + \cos^2 \lambda A_j - 2\cos\lambda A_i \cos\lambda A_j \cos\lambda\pi$, and

$$U(\lambda) = \frac{n(n-1)}{2} \lambda^2 [B(\lambda;\pi)]^2 \cos^2\left(\frac{\lambda}{2}\pi\right)$$

with $B(\lambda;\pi) = \pi + 2(66 - 43\pi + 7\pi^2)\lambda - (124 - 83\pi + 14\pi^2)\lambda^2 + 2(\pi - 3)\lambda^3$.

Proof. Substituting $x = \lambda\pi/2$ in (2.1), the inequality can be written as $\sin\frac{\lambda}{2}\pi \geqslant \frac{\lambda}{2}[B(\lambda;\pi)]$. Hence $\frac{\lambda}{2}[B(\lambda;\pi)] \leqslant \sin\frac{\lambda}{2}\pi \leqslant \frac{\lambda}{2}\pi$ or

$$\frac{\lambda^2}{4}[B(\lambda;\pi)]^2 \leqslant \sin^2\frac{\lambda}{2}\pi \leqslant \frac{\lambda^2}{4}\pi^2. \tag{3.2}$$

On the other hand, using the inequality [9]

$$\sin^2\lambda\pi \leqslant H_{ij} \leqslant 4\sin^2\frac{\lambda}{2}\pi, \tag{3.3}$$

and noting $\sin^2\lambda\pi = 4\sin^2\frac{\lambda}{2}\pi\cos^2\frac{\lambda}{2}\pi$ and the inequality (3.2), we get

$$\lambda^2[B(\lambda;\pi)]^2\cos^2\frac{\lambda}{2}\pi \leqslant \sin^2\lambda\pi \leqslant H_{ij} \leqslant \lambda^2\pi^2. \tag{3.4}$$

Let $1 \leqslant i < j \leqslant n$. Introducing the summation in the inequality (3.4), we obtain

$$U(\lambda) \leqslant \sum_{1 \leqslant i < j \leqslant n} \sin^2 \lambda\pi \leqslant \sum_{1 \leqslant i < j \leqslant n} H_{ij} \leqslant \sum_{1 \leqslant i < j \leqslant n} \lambda^2 \pi^2, \tag{3.5}$$

where

$$U(\lambda) = \sum_{1 \leqslant i < j \leqslant n} \lambda^2[B(\lambda;\pi)]^2\cos^2\left(\frac{\lambda}{2}\pi\right).$$

Hence the theorem. □

REMARK 3.1. It has been shown [4] that $S(\lambda) \leqslant \sum_{1 \leqslant i < j \leqslant n} \sin^2\lambda\pi$, where

$$S(\lambda) = \frac{n(n-1)}{2}\lambda^2[\pi + (6 - 2\pi)\lambda + (\pi - 4)\lambda^2]^2\cos^2\left(\frac{\lambda}{2}\pi\right).$$

On the other hand, for $0 \leqslant \lambda \leqslant 1$ we have

$$(6 - 2\pi)\lambda + (\pi - 4)\lambda^2 \leqslant B(\lambda;\pi). \tag{3.6}$$

From (3.6), it can be readily seen that

$$S(\lambda) \leqslant U(\lambda) \leqslant \sum_{1 \leqslant i<j \leqslant n} \sin^2 \lambda\pi \leqslant \sum_{1 \leqslant i<j \leqslant n} H_{ij} \tag{3.7}$$

which shows that the inequality (2.1) is a strengthened version of the inequality given by Özban in [4] for $\lambda \in [0,1]$. In conclusion, we present Table 1, which enables us to compare the numerical values of lower bounds $S(\lambda)$ and $U(\lambda)$ and those of $\sum_{1 \leqslant i<j \leqslant n} \sin^2 \lambda\pi$ for some values of n and λ.

	λ	$S(\lambda)$	$U(\lambda)$	$\sum_{1 \leqslant i<j \leqslant n} \sin^2 \lambda\pi$
$n=6$	0.25	7.28441	7.47568	7.5
	0.5	14.54708	14.9125	15.00000
	0.624	12.5343	12.7678	12.8365
	0.8	5.13029	5.16792	5.182372
	0.95	0.3667546	0.366948	0.3670761
$n=20$	0.25	92.2692	94.6919	95.000
	0.5	184.2630	188.892	190.0000
	0.624	158.768	161.726	162.596
	0.8	64.98368	65.4604	65.64338
	0.95	4.645558	4.64801	4.649630

Table 1. Comparison $S(\lambda)$, $U(\lambda)$ *and* $\sum_{1 \leqslant i<j \leqslant n} \sin^2 \lambda\pi$

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