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BOUNDS IN SPACES OF MORREY UNDER CHICCO TYPE CONDITIONS

A. CANALE

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Abstract. In the present paper we consider Morrey spaces in unbounded domains and study elliptic equations in nondivergence form with discontinuous coefficients when the class of discontinuities is of Chicco type. In particular we state some local and non local a priori bounds for solutions of Dirichlet problem and study the dependence of the constants in the estimates. The idea is to approximate the principal coefficients by functions with derivatives which belong locally to the space L^s, $2 < s \leqslant n$, while the coefficients of lower terms in the differential operator belong to Morrey spaces. Our results are based on embedding theorems which allow us to require a summability lower than n for the coefficients of the operator L.

1. Introduction

Elliptic equations in non divergence form have been widely studied in bounded open sets. The work of C. Miranda [23] represent a point of reference in the study of Dirichlet problem with discontinuous coefficients belonging to the $W^{1,n}$ spaces. Subsequent results were stated, for example, in [20, 22, 27].

Other results can be found in [2, 13, 15, 16] in wider classes of spaces while different classes of discontinuous operators were studied in [17, 18, 19, 24].

When Ω is an unbounded open set, the problem was studied in more general spaces than L^n spaces in [25], in spaces of Morrey type in [7, 9, 10, 11] and in weighted spaces in [3, 4, 5, 6, 8, 12].

Basic tools for proving existence and, sometimes, uniqueness of solutions of elliptic boundary value problems in Sobolev spaces are a priori bounds.

In this paper we state some a priori bounds for solutions of the problem

$$\begin{cases} Lu = f\,, & f \in L^2(\Omega)\,, \\ u \in W^2(\Omega) \cap W_0^1(\Omega)\,, \end{cases} \tag{1.1}$$

where L is the operator

$$Lu = -\sum_{i,j=1}^{n} a_{ij}\, u_{x_i x_j} + \sum_{i=1}^{n} a_i\, u_{x_i} + a\, u\,. \tag{1.2}$$

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The coefficients a_i and a of the operator L belong to the class of Morrey type spaces $M^{p,\lambda}$ introduced in [26] which are larger than L^n spaces. We observe that, when Ω is a bounded open set, the spaces $M^{p,\lambda}(\Omega)$ are reduced to the classical Morrey space $L^{p,\lambda}(\Omega)$ (see [13, 14]) while, if $\Omega = R^n$, include $L^{p,\lambda}(R^n)$.

It is interesting to remark that we require a lower summability for the coefficients of the operator L when we work with Morrey spaces with respect the other speces. Our results base on embedding theorems proved by C. Fefferman [21], so we do not need to achieve n.

In this paper we consider a wide class of discontinuity: the functions which satisfy Chicco type conditions [17, 19]. We remark that continuous functions, the class of discontinuities considered at first in [23] and of Cordes type belong to this class.

We study the problem under hypotheses on coefficients considerably weakened with respect to the assumptions we can find in the papers until now. The idea is to approximate a_{ij} by some functions e_{ij} 'near' to a_{ij} in bounded open sets and by more regular functions at infinity. The conditions we impose on e_{ij} and on their derivatives are very 'weak', we require only that $(e_{ij})_{x_h} \in L^s_{loc}(\overline{\Omega})$, $2 < s \leqslant n$, and we are able to apply locally some embedding results without further assumptions on $(e_{ij})_{x_h}$.

We remark that an hypothesis of Chicco type as above is not sufficient to get local estimates for $|x|$ large enough without further assumptions.

In this paper we obtain local bounds under different assumptions of Chicco type.

A way is to introduce functions regular 'enough' suitable connected to a_{ij} to obtain the results. Other ways are to assume Chicco condition with a suitable choice of functions e_{ij} or to give an additional assumptions on derivatives of e_{ij} to apply embedding results.

We observe that local a priori estimates allow us to prove a priori bounds for solutions of problem (1.1).

In previous paper [7] we state local a priori bound under Cordes conditions on coefficients of the operator L without to introduce more regular functions close to a_{ij} and without further assumptions. The reason is that Cordes conditions allow us to approximate a_{ij} by means of functions which do not introduce derivatives and, so, further hypotheses on derivatives to use embedding results.

A priori bounds (see Theorem 6.1 and Corollary 6.1 in Section 6) are obtained using embedding theorems and the local a priori bounds stated in Section 5.

2. Notation and function spaces

Let E be a Lebesgue measurable subset of R^n and $\Sigma(E)$ the σ-algebra of Lebesgue measurable subsets of E.

We denote by $\mathscr{D}(A)$ the class of restrictions to $A \in \Sigma(E)$ of functions $\phi \in C_o^\infty(R^n)$ such that supp $\phi \cap \overline{A} \subset A$ and by $L^p_{loc}(A)$ the class of functions $f : A \to C$ such that $\phi f \in L^p(A)$ for any $\phi \in \mathscr{D}(A)$. We set

$$|f|_{p,A} = \|f\|_{L^p(A)}, \qquad 1 \leqslant p \leqslant +\infty .$$

In this paper we define

$$W^{r,2}(\Omega) = W^r(\Omega), \quad r = 1,2, \qquad \text{and} \qquad W_0^{1,2}(\Omega) = W_0^1(\Omega)$$

in order to indicate quadratic integrability of weak derivatives.

Let $B(x,r)$, $x \in R^n$, $r \in R_+$, be the open ball with center in x and radius r.

For $r \in \mathbf{R}_+$, we set $B_r = B(0,r)$ and denote by ζ_r a function of class $C_o^\infty(R^n)$ such that

$$\text{supp}\zeta_r \subset B_{2r}, \qquad 0 \leqslant \zeta_r \leqslant 1, \qquad \zeta_{r|B_r=1}, \qquad (\zeta_r)_x \leqslant \frac{2}{r}.$$

Let Ω be an open subset of R^n and let us consider the spaces $M^{p,\lambda}(\Omega)$, $\tilde{M}^{p,\lambda}(\Omega)$, $M_o^{p,\lambda}(\Omega)$ defined in [26] (we refer also to [10] where we can find many properties of these spaces).

Let us define, for $1 \leqslant p < +\infty$ and $0 \leqslant \lambda < n$, $n \geqslant 2$,

$M^{p,\lambda}(\Omega)$ as the space of functions $g \in L^p_{loc}(\overline{\Omega})$ such that

$$\|g\|_{M^{p,\lambda}(\Omega)} = \sup_{\substack{x\in\Omega \\ 0<\tau\leqslant 1}} \tau^{-\lambda/p} \|g\|_{L^p(\Omega\cap B(x,\tau))} < +\infty\,, \tag{2.1}$$

equipped with the norm defined in (2.1);

$\tilde{M}^{p,\lambda}(\Omega)$ as the closure of $L^\infty(\Omega)$ in $M^{p,\lambda}(\Omega)$;

$M_o^{p,\lambda}(\Omega)$ as the closure of $C_o^\infty(\Omega)$ in $M^{p,\lambda}(\Omega)$.

From the results in [26] we have the following characterizations of the spaces $\tilde{M}^{p,\lambda}(\Omega)$ and $M_o^{p,\lambda}(\Omega)$:

$\tilde{M}^{p,\lambda}(\Omega)$ is the subspace of $M^{p,\lambda}(\Omega)$ of the functions $g \in M^{p,\lambda}(\Omega)$ such that:

$$\forall \epsilon \in R_+ \ \exists \delta_\epsilon \in R_+ \text{ s.t.}$$
$$(E \in \Sigma(\Omega),\ \sup_{x\in\Omega} |E \cap B(x,1)| \leqslant \delta_\epsilon \Rightarrow \|g\chi_E\|_{M^{p,\lambda}(\Omega)} \leqslant \epsilon)\,, \tag{2.2}$$

$M_o^{p,\lambda}(\Omega)$ is the subspace of $M^{p,\lambda}(\Omega)$ of the functions $g \in M^{p,\lambda}(\Omega)$ such that:

$$\forall \epsilon \in R_+ \ \exists h_\epsilon, k_\epsilon \in R_+ \text{ s.t.}$$
$$(E \in \Sigma(\Omega), |E \cap B(0,k_\epsilon)| \leqslant h_\epsilon \ \Rightarrow \ \|g\chi_E\|_{M^{p,\lambda}(\Omega)} \leqslant \epsilon)\,. \tag{2.3}$$

Let us set:

$$M^p(\Omega) = M^{p,0}(\Omega)\,, \quad \tilde{M}^p(\Omega) = \tilde{M}^{p,0}(\Omega)\,, \quad M_o^p(\Omega) = M_o^{p,0}(\Omega).$$

The spaces $M^p(\Omega)$ and $M_o^p(\Omega)$ have been introduced and studied in [25].

It is useful to recall some results about Morrey type spaces introduced above.

We have the embedding:

$$M^{p_o,\lambda_0}(\Omega) \ \hookrightarrow \ M^{p,\lambda}(\Omega)\,, \qquad p \leqslant p_o\,, \qquad \frac{\lambda - n}{p} \leqslant \frac{\lambda_0 - n}{p_o}$$

which implies in particular that:

$$L^\infty(\Omega) \ \hookrightarrow \ M^{p,\lambda}(\Omega)\,.$$

The following inclusions hold:

$$L^n(\Omega) \subset M^{n,0}(\Omega) \subset M^{s,n-s}(\Omega)\,, \qquad s \in]2,n[\,. \tag{2.4}$$

For example the constant functions belong to $M^{n,0}(\Omega)$ but do not belong to $L^n(\Omega)$. Furthermore the function $f(x) = \frac{1}{1+|x|^\alpha} \in M^{p,0}(\Omega)$ if $\alpha > 0$ while belongs to $L^p(\Omega)$ if $\alpha \in [0, \frac{n}{p}[$.

Let us also denote by $VM^{p,\lambda}(\Omega)$ the subspace of $M^{p,\lambda}(\Omega)$ of the functions $g \in M^{p,\lambda}(\Omega)$ such that

$$\lim_{\tau\to 0} \|g\|_{M^{p,\lambda}(\Omega)} = 0\,. \tag{2.5}$$

It is easy to see that

$$L^\infty(\Omega) \subset VM^{p,\lambda}(\Omega)$$

and that

$$\tilde{M}^{p,\lambda}(\Omega) \subset VM^{p,\lambda}(\Omega)\,.$$

In particular we get (see [26])

$$\tilde{M}^{p,\lambda}(\Omega) = VM^{p,\lambda}(\Omega) \cap \tilde{M}^{p}(\Omega)\,. \tag{2.6}$$

We state the following result about $M^{p,\lambda}$ spaces we will use later.

LEMMA 2.1. *If $g \in L^p_{loc}(\overline{\Omega})$, $1 \leqslant p < +\infty$, and $\phi \in \mathscr{D}(\overline{\Omega})$, then $\phi\, g \in \tilde{M}^{p,\lambda}(\Omega)$.*

Proof. The function $\phi\, g \in L^p_{loc}(\overline{\Omega})$ and so there exists a sequence of functions $(g_n)_{n\in N}$, with $g_n \in C^\infty_0(\Omega)$, such that

$$g_n \longrightarrow \phi\, g \qquad \text{in} \quad L^p(\Omega)\,. \tag{2.7}$$

It is easy to see that $\phi\, g \in M^{p,\lambda}(\Omega)$. In fact, using (2.7), we have

$$\|\phi\, g\|_{M^{p,\lambda}(\Omega)} \leqslant \sup_{\substack{x\in\Omega \\ 0<\tau\leqslant 1}} \tau^{-\lambda/p} \Big(\|g_n - \phi\, g\|_{L^p(\Omega\cap B(x,\tau))} + \|g_n\|_{L^p(\Omega\cap B(x,\tau))} \leqslant c_1 \tau^{\frac{n-\lambda}{p}} \Big)\,. \tag{2.8}$$

From (2.8) we deduce also that $\phi\, g \in VM^{p,\lambda}(\Omega)$ taking in mind (2.5).

Now, if we fix $\zeta \in \mathscr{D}(\overline{\Omega})$ with $\zeta_{|supp\,\phi} = 1$, we obtain

$$\zeta\, g_n \longrightarrow \phi\, g \qquad \text{in} \quad L^p(\Omega)\,. \tag{2.9}$$

So we get $\phi\, g \in M^p_0(\Omega)$, then $\phi\, g \in \tilde{M}^p(\Omega)$.

We deduce from (2.6) the result. □

REMARK 2.1. From Lemma 2.1 we can obtain a further information on function $\phi\, g$. In particular we observe that $\phi\, g \in M^{p,\lambda}_0(\Omega)$ since the following relation holds (see [26])

$$M^{p,\lambda}_0(\Omega) = \tilde{M}^{p,\lambda}(\Omega) \cap M^p_0(\Omega)\,.$$

REMARK 2.2. One can proves the function $\phi\, g$ belongs to the space $M^{p,\lambda}_0(\Omega)$ (and then to the space $\tilde{M}^{p,\lambda}(\Omega)$) proceeding as in the proof of Lemma 2.1 and using (2.9) to get the result.

3. Embedding results

Embedding results due to C.Fefferman [21] (see also [14]) allow us to state the following lemma (see [26]).

LEMMA 3.1. *If Ω has the cone property and $g \in M^{s,n-s}(\Omega)$, $s \in]2,n]$, then for any $u \in W^1(\Omega)$ we get $gu \in L^2(\Omega)$ and*

$$|g\,u|_{2,\Omega} \leqslant H\,\|g\|_{M^{s,n-s}(\Omega)}\,\|u\|_{W^1(\Omega)}\,, \tag{3.1}$$

where the constant H, independent of g and u, depends on n and s.

Let us define the modulus of continuity of a function $g \in \tilde{M}^{p,\lambda}(\Omega)$ (see also [9]). If $p \in [1,+\infty[$, $\lambda \in [0,n[$ and $g \in \tilde{M}^{p,\lambda}(\Omega)$, we set

$$\tau^p_\lambda[g](t) = \sup_{\substack{E\in\Sigma(\Omega) \\ \sup_x |E\cap B(x,1)|\leqslant t}} \|g\,\chi_E\|_{M^{p,\lambda}(\Omega)}\,, \qquad t \in R_+\,,$$

where χ_E is the characteristic function of E.

From (2.2) it follows that that $g \in \tilde{M}^{p,\lambda}(\Omega)$ if and only if $g \in M^{p,\lambda}(\Omega)$ and

$$\lim_{t\to 0} \tau^p_\lambda[g](t) = 0\,.$$

We define the modulus of continuity of $g \in \tilde{M}^{p,\lambda}(\Omega)$ as a function $\tau[g] : R_+ \to R_+$ satisfying

$$\tau^p_\lambda[g](t) \leqslant \tau[g](t)\,, \qquad \forall t \in R_+\,, \qquad \lim_{t\to 0} \tau\,[g](t) = 0\,.$$

In the case $g : \Omega \to R$, we put

$$A_r(g) = \{x \in \Omega : |g(x)| \geqslant r\}\,, \qquad r \in R_+\,.$$

If $g \in L^p_{loc}(\overline{\Omega})$, $p \in [1,+\infty[$, we get

$$\lim_{r\to+\infty} |A_r(g) \cap B(x,1)| = 0\,.$$

Let us denote, for all $k \in R_+$, by $r_k = r_k(g)$ a real number such that

$$|A_{r_k}(g) \cap B(x,1)| \leqslant \frac{1}{k+1} \tag{3.2}$$

and by $r[g]$ the function

$$r[g] : k \in R_+ \to r[g](k) = r_k \in R_+\,. \tag{3.3}$$

The following lemma,which we will use later, was stated in [7].

LEMMA 3.2. *In the same hypotheses of Lemma 3.1 and if $g \in \tilde{M}^{s,n-s}(\Omega)$, $s \in]2,n]$, then for any $k \in R_+$ we have*

$$|g\,u|_{2,\Omega} \leqslant H\,\tau[g]\left(\frac{1}{k+1}\right)\,\|u\|_{W^1(\Omega)} + r[g](k)\,\|u\|_{L^2(\Omega)} \qquad \forall\, u \in W^1(\Omega)\,,$$

where H is the constant in (3.1), $\tau[g]$ is the modulus of continuity of g in $\tilde{M}^{s,n-s}(\Omega)$ and $r[g]$ is the function defined by (3.3).

4. Hypotheses

Let us set

$$B_+ = \{x \in B_1 : x_n > 0\}\,, \qquad B_o = \{x \in B_1 : x_n = 0\}\,,$$

and suppose that

(h_1) there are a $d \in R_+$, an open cover $\{U_i\}_{i\in I}$ of $\partial\Omega$ and, for any $i \in I$, a C^2-diffeomorphism $\psi_i : \overline{U}_i \to \overline{B}_1$ such that:

- $\psi_i(U_i \cap \Omega) = B_+\,, \qquad \psi_i(U_i \cap \partial\Omega) = B_o\,;$
- the components of ψ_i and ψ_i^{-1} and of their first and second derivatives are bounded by a constant independent of i;
- for any $x \in \Omega_d$ there exists an $i \in I$ such that $B(x,d) \subset U_i$ and, for any $x \in \Omega \setminus \Omega_d$, we get $B(x,d) \subset \Omega$, where $\Omega_d = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < d\}$.

REMARK 4.1. It is easy to prove that (h_1) holds when Ω has the uniform C^2-regularity property defined in [1].

REMARK 4.2. The condition (h_1) implies that there exists a number $\rho \in R_+$ such that, for any $x \in R^n$, $B(x,\rho) \cap \partial\Omega = \emptyset$ or $B(x,\rho) \cap \partial\Omega \neq \emptyset$ and $B(x,\rho) \subset U_i$ for some $i \in I$.

Let us consider in Ω the second order linear differential operator

$$Lu = -\sum_{i,j=1}^{n} a_{ij}\, u_{x_i x_j} + \sum_{i=1}^{n} a_i\, u_{x_i} + a\, u \tag{4.1}$$

with the following conditions on the coefficients:

(h_2)
$$a_{ij} = a_{ji} \in L^\infty(\Omega)\,, \quad i,j = 1,\dots,n\,,$$

(h_3)
$$a_i \in \tilde{M}^{s,n-s}(\Omega)\,, \quad i = 1,\dots,n\,, \quad a \in \tilde{M}^t(\Omega)\,,$$

where

$$s \in]2,n]\,, \quad t = 2 \quad \text{if} \quad n = 3\,, \quad t > 2 \quad \text{if} \quad n = 4\,, \quad t = \frac{n}{2} \quad \text{if} \quad n > 4\,.$$

Let us denote by $E(\nu,\Omega)$ the class of $n \times n$ real matrix-valued functions (e_{ij}) such that

(h_4)
$$e_{ij} = e_{ji} \in L^\infty(\Omega)\,, \quad i,j = 1,\dots,n$$

$$(e_{ij})_{x_h} \in L^s_{loc}(\overline{\Omega})\,, \quad i,j,h = 1,\dots,n\,,$$

$$\sum_{i,j=1}^{n} e_{ij}\, \xi_i\, \xi_j \geqslant \nu\, |\xi|^2 \quad \forall\, \xi \in R^n\,, \qquad \text{a.e. in } \Omega\,,$$

where ν is a positive constant independent of x and ξ.

Moreover we set

$$\mathscr{G}(\Omega) = \{g \in L^\infty \ : \ \operatorname*{ess\,inf}_{\Omega} g > 0\}\,.$$

and suppose that (a_{ij}) satisfies the following conditions:

(h_5) (*Chicco type condition*)
there exists $\nu \in R_+$, $(e_{ij}) \in E(\nu, \Omega)$ and $g \in \mathscr{G}(\Omega)$ such that

$$\operatorname*{ess\,sup}_{\Omega} \sum_{i,j=1}^{n} \left(e_{ij} - g a_{ij}\right)^2 < \nu^2 .$$

For example one can choose

$$g = \frac{\sum_{i,j=1}^{n} e_{ij}\, a_{ij}}{\sum_{i,j=1}^{n} a_{ij}^2} \in \mathscr{G}(\Omega) . \tag{4.2}$$

Let us set

$$u_x = \left(\sum_{i=1}^{n} u_{x_i}^2\right)^{1/2}, \qquad u_{xx} = \left(\sum_{i,j=1}^{n} u_{x_i x_j}^2\right)^{1/2}.$$

We consider a function $\beta : \Omega \to R_+$ such that the following hypothesis holds:

(h_6) $$\beta \in \tilde{M}^t(\Omega), \quad \exists\, \delta \in \tilde{M}^{s,n-s}(\Omega) \quad \text{such that} \quad \beta_x \leqslant \beta\, \delta .$$

For example, some functions which satisfy the hypothesis (h_6) are given by $\beta = 1$ or $\beta(x) = \frac{1}{(1+|x|^2)^\tau}$, $x \in \Omega$, $\tau > 0$.

(h_7) there exist $(\alpha_{ij}) \in E(\nu, \Omega)$ with

$$(\alpha_{ij})_{x_h} \in \tilde{M}^{s,n-s}(\Omega), \quad i,j,h = 1,\dots,n,$$

and a function $\gamma : R_+ \to R_+$ such that

$$\operatorname*{ess\,sup}_{\Omega \setminus B_k} \sum_{i,j=1}^{n} |\alpha_{ij} - g a_{ij}| \leqslant \gamma(k) \qquad \forall k \in R_+ ,$$

$$\lim_{k \to +\infty} \gamma(k) = 0 .$$

We can suppose in place of (h_7) one of the following assumptions when we state local a priori bounds (see Lemma 5.2 in Section 5).

(h_8) *Chicco type condition* with e_{ij}, for $i,j = 1,\dots,n$, constant functions satisfying (h_4).

(h_9) $$(e_{ij})_{x_h} \in \tilde{M}^{s,n-s}(\Omega), \quad i,j,h = 1,\dots,n .$$

REMARK 4.3. Let us note that (h_4) and (h_5) imply that operator L defined in (4.1) is uniformly elliptic in Ω.

REMARK 4.4. If in (h_5) the functions $e_{ij} = \delta_{ij}$ and $g = \frac{\sum_{i=1}^{n} e_{ij} a_{ij}}{\sum_{i,j=1}^{n} a_{ij}^2}$, as in (4.2), condition of Chicco type reduces to Cordes type conditions (we refer to [7] for some recent results in weighted spaces under Cordes condition). If in (h_4) $e_{ij} = \delta_{ij}$ but

$g \in \mathscr{G}(\Omega)$ is different from (4.2), we have a particular case of Chicco condition (see (h_8)).

REMARK 4.5. One can show that under hypotheses $(h_1)-(h_3)$ and (h_6) it follows that for any $\lambda \in R$ the operator

$$u \in W^2(\Omega) \;\to\; Lu + \lambda\,\beta\,u \in L^2(\Omega)$$

is bounded.

5. Local a priori bounds

Let us set

$$L_o u = -\sum_{i,j=1}^{n} a_{ij}\, u_{x_i x_j}$$

and

$$\tilde{f} = 1 + \sum_{i,j=1}^{n} |e_{ij}|\,\delta + \sum_{i,j=1}^{n} (e_{ij})_x\,, \quad \text{if } (h_5) \text{ holds}$$

or

$$\tilde{f} = \sum_{i,j=1}^{n} |e_{ij}|\,\delta\,, \quad \text{if } (h_8) \text{ holds}\,,$$

where δ is the function defined in (h_6) and e_{ij} are the functions which belong to the class $E(\nu,\Omega)$ (see (h_4)).

Let us fix a bounded open subset V of R^n such that

$$V \subset \Omega \quad \text{or} \quad V \cap \partial\Omega \neq \emptyset \quad \text{and} \quad V \subset U_i \quad \text{for some } i \in I\,.$$

LEMMA 5.1. *If the hypotheses $(h_1), (h_2), (h_4), (h_6)$ hold, then for any $\lambda \geqslant 0$ and for any function v satisfying*

$$v \in W^2(\Omega) \cap \overset{o}{W}{}^1(\Omega)\,, \qquad supp\, v \subset V\,,$$

we have for any $\epsilon \in R_+$ the bound

$$(\nu^2 - \epsilon^2)|v_{xx}|^2_{2,\Omega} \leqslant \left| -\sum_{i,j=1}^{n} e_{ij} v_{x_i x_j} + \lambda\,\beta\, v \right|^2_{2,\Omega} + c(\epsilon)|\tilde{f}\, v_x|^2_{2,\Omega}\,. \tag{5.1}$$

Moreover if also (h_5) is verified

$$|v_{xx}|_{2,\Omega} \leqslant c\left(|L_o v + \lambda\, g^{-1}\beta\, v|_{2,\Omega} + |\tilde{f}\, v_x|_{2,\Omega}\right)\,, \tag{5.2}$$

where $c = c\,(\Omega, \nu, \|a_{ij}\|_\infty, \|e_{ij}\|_\infty)$.

Proof. Inequality (5.1) can be proved as in [10, Section 7].

We proceed using hypothesis of Chicco type to get the result. Indeed il we set

$$h = \operatorname*{ess\,sup}_{\Omega} \left(\sum_{i,j=1}^{n} |e_{ij} - g a_{ij}|^2 \right)^{1/2},$$

from inequality (5.1) we get

$$\begin{aligned}(\nu - \epsilon)|v_{xx}|_{2,\Omega} &\leqslant \left| -\sum_{i,j=1}^{n} (e_{ij} - g\, a_{ij}) v_{x_i x_j} + g\, L_o\, v + \lambda\, \beta\, v \right|_{2,\Omega} + c(\epsilon)^{\frac{1}{2}} |\tilde{f}\, v_x|_{2,\Omega} \\ &\leqslant h\, |v_{xx}|_{2,\Omega} + \|g\|_\infty \left| L_o v + \lambda\, g^{-1} \beta\, v \right|_{2,\Omega} + c(\epsilon)^{\frac{1}{2}} |\tilde{f}\, v_x|_{2,\Omega},\end{aligned}$$

from which, by condition (h_5), we deduce (5.2) for $\epsilon < \nu - h$. □

REMARK 5.1. If e_{ij}, for $i,j = 1\dots n$, are constant functions satisfying (h_4), inequality (5.1) takes the form

$$\nu^2 |v_{xx}|^2_{2,\Omega} \leqslant \left| -\sum_{i,j=1}^{n} e_{ij} v_{x_i x_j} + \lambda\, \beta\, v \right|^2_{2,\Omega} + |\tilde{f}\, v_x|^2_{2,\Omega},$$

where $\tilde{f}$ is defined at beginning of the Section. As a consequence, by Chicco type condition (h_8) we deduce (5.2).

We are able to prove the following Lemma using as tools Lemma 5.1 and Lemma 3.2. A similar lemma was proved in [7] under different hypotheses on coefficients of the operator L.

LEMMA 5.2. *If the conditions $(h_1) - (h_7)$ hold and λ_1 is a real number, then there exists a constant $c \in R_+$ such that for any $\lambda \in [\lambda_1, +\infty[$ and for any function v satisfying*

$$v \in W^2(\Omega) \cap W^1_0(\Omega), \qquad supp\, v \subset V,$$

we get

$$|v_{xx}|_{2,\Omega} \leqslant c \left(|Lv + \lambda\, g^{-1}\, \beta\, v|_{2,\Omega} + |v_x|_{2,\Omega} + |v|_{2,\Omega} \right), \tag{5.3}$$

where c is a positive constant depending on Ω, ν, n, s, t, $\|a_{ij}\|_\infty$, $\|e_{ij}\|_\infty$, $\|g\|_\infty$, $\|\alpha_{ij}\|_\infty$, $\tau[(\alpha_{ij})_x]$, $\tau[\zeta_k\,(e_{ij})_x]$, $\tau[\beta]$, $\tau[\delta]$, $\tau[a_i]$, $\tau[a]$, $r[(\alpha_{ij})_x]$, $r[\zeta_k\,(e_{ij})_x]$, $r[\beta]$, $r[\delta]$, $r[a_i]$, $r[a]$.

Proof. Step 1 (*Estimates at infinity*).

Let us suppose $\lambda \geqslant 0$ and consider the functions ζ_k, $k \in R_+$, introduced in Section 2. Applying (5.1) in Lemma 5.1 to the function $(1-\zeta_k)\, v$ with $e_{ij} = \alpha_{ij}$, we get

$$\begin{aligned}\left| \big((1-\zeta_k) v\big)_{xx} \right|_{2,\Omega} \leqslant c_1 \Bigg(\Bigg| -\sum_{i,j=1}^{n} \alpha_{ij} \big((1-\zeta_k)\, v\big)_{x_i x_j} \\ + \lambda\, \beta\, (1-\zeta_k)\, v \Bigg|_{2,\Omega} + |\tilde{g}((1-\zeta_k) v)_x|_{2,\Omega} \Bigg)\end{aligned} \tag{5.4}$$

where $\tilde{g} = \left(1 + \sum_{i,j=1}^{n} |\alpha_{ij}|\,\delta + \sum_{i,j=1}^{n} (\alpha_{ij})_x\right)$.

Moreover we have by hypothesis (h_7) that the first term on the right hand in (5.4) is bounded as follows

$$\begin{aligned}
&\left|-\sum_{i,j=1}^{n} \alpha_{ij} \left((1-\zeta_k)\,v\right)_{x_i x_j} + \lambda\,\beta\,(1-\zeta_k)\,v\right|_{2,\Omega} \\
&\quad \leqslant \left|g\,L_o\big((1-\zeta_k)\,v\big) + \lambda\,\beta\,(1-\zeta_k)\,v\right|_{2,\Omega} + \left|-\sum_{i,j=1}^{n} (\alpha_{ij} - g a_{ij}) \left((1-\zeta_k)\,v\right)_{x_i x_j}\right|_{2,\Omega} \\
&\quad \leqslant c_2 \left(\left|(1-\zeta_k)\left(L_o v + \lambda\, g^{-1}\,\beta\, v\right)\right|_{2,\Omega} + \left|(1-\zeta_k)_x\, v_x\right|_{2,\Omega} + \left|(1-\zeta_k)_{xx}\, v\right|_{2,\Omega} \right) \\
&\qquad + \gamma(k) \left| \left((1-\zeta_k)\,v\right)_{xx}\right|_{2,\Omega}. \qquad (5.5)
\end{aligned}$$

Since $\tilde{g} \in \tilde{M}^{s,n-s}(\Omega)$, we can use Lemma 3.2 to estimate the last term in (5.4). So we obtain by (5.4) and (5.5)

$$\begin{aligned}
\left| \left((1-\zeta_k)\,v\right)_{xx}\right|_{2,\Omega} &\leqslant c_3 \left(\left|L_o\, v + \lambda\, g^{-1}\,\beta\, v\right|_{2,\Omega} + |v_x|_{2,\Omega} + |v|_{2,\Omega} \right) \\
&\quad + \left(c_1 H\, \tau[\tilde{g}] \left(\frac{1}{k+1}\right) + \gamma(k) \right) \left| \left((1-\zeta_k)\,v\right)_{xx}\right|_{2,\Omega}. \qquad (5.6)
\end{aligned}$$

Step 2 (*Estimates on bounded sets*).

Now applying Lemma 5.1 to the function $\zeta_k v$ we get

$$|(\zeta_k v)_{xx}|_{2,\Omega} \leqslant c_4 \left(\left|L_o(\zeta_k v) + \lambda\,\beta\, g^{-1}\,\zeta_k v\right|_{2,\Omega} + |\tilde{f}\,(\zeta_k v)_x|_{2,\Omega} \right) \qquad (5.7)$$

(we recall that $\tilde{f} = 1 + \sum_{i,j=1}^{n} |e_{ij}|\,\delta + \sum_{i,j=1}^{n} (e_{ij})_x$).

For any $k \in R_+$ let $r \geqslant 2k$ so that $\zeta_{r|_{supp\,\zeta_k}} = 1$. The function $\zeta_r \tilde{f}$ belongs to the space $\tilde{M}^{s,n-s}(\Omega)$ (see Lemma 2.1) and then we can use Lemma 3.2 to estimate the last term in (5.7).

Proceeding as in Step 1 we obtain

$$\begin{aligned}
|(\zeta_k v)_{xx}|_{2,\Omega} \leqslant c_5 \Big(&|L_o v + \lambda g^{-1} \beta v|_{2,\Omega} + |v_x|_{2,\Omega} \\
&+ |v|_{2,\Omega} + H\,\tau[\zeta_r \tilde{f}] \left(\frac{1}{k+1}\right) |(\zeta_k v)_{xx}|_{2,\Omega} \Big). \qquad (5.8)
\end{aligned}$$

By definition of modulus of continuity given in Section 3 and by hypothesis (h_7) it follows that there exists $k_0 \in R_+$ such that from (5.6) and (5.8) we can deduce that

$$\begin{aligned}
|v_{xx}|_{2,\Omega} &\leqslant \left| \left((1-\zeta_{k_o})v\right)_{xx}\right|_{2,\Omega} + |(\zeta_{k_o} v)_{xx}|_{2,\Omega} \\
&\leqslant c_6 \left(|L_o\, v + \lambda\, g^{-1} \beta\, v|_{2,\Omega} + |v_x|_{2,\Omega} + |v|_{2,\Omega} \right). \qquad (5.9)
\end{aligned}$$

If $\lambda_1 < 0$, we fix $\lambda \in [\lambda_1, 0[$.

Using (h_6) and applying to β Lemma 3.2 we get the bound

$$|\lambda\, g^{-1}\, \beta\, v|_{2,\Omega} \leqslant c_7|\lambda_1|\,(\operatorname{ess\,inf} g)^{-1}\left(|v_x|_{2,\Omega} + |v|_{2,\Omega}\right). \tag{5.10}$$

Now if we consider the inequality (5.9) with $\lambda = 0$, from (5.10) we easly deduce (5.3) with L_o instead of L.

Finally, applying Lemma 3.2 to the functions a_i and a verifying hypothesis (h_3) we obtain the result. □

REMARK 5.2. We can prove Lemma 5.2 without any assumptions of convergence at infinity as in (h_7). If we substitute hypothesis (h_7) with (h_8) or (h_9), we can modify the proof of Lemma 5.2. In fact, in Step 1, if we consider (5.2), we can apply to $\tilde{f}$ embedding theorem to estimate the last term. If e_{ij} are constant functions we do not need any further assumptions while in the general case we can suppose (h_9) to use Lemma 3.2.

In Step 2 we do not need hypotheses (h_7), (h_8) or (h_9), but anyway we are able to apply embedding results.

Then using definition of modulus of continuity and assumptions on functions β, a_i and a we proceed as in the the proof of Lemma 5.2.

We remark that hypothesis (h_5) of Chicco type is not sufficient to get local estimates for $|x|$ large enough without further assumptions (see (h_7) or (h_9)) or without limit oneself to the case e_{ij} =const.

6. A priori bounds

We assume the following further hypotheses:

(h_{10}) $$a_i \in M_0^{s,n-s}(\Omega)\,, \quad i = 1,\dots,n\,, \quad \operatorname{ess\,inf}_\Omega a > 0\,;$$

(h_{11}) $$(\alpha_{ij})_{x_h} \in M_0^{s,n-s}(\Omega)\,, \quad i,j,h = 1,\dots,n\,.$$

Local a priori bound stated in Lemma 5.2 allows us to prove the following result.

THEOREM 6.1. *If $(h_1) - h_7)$ and $(h_{10}) - (h_{11})$ hold, then there exist a constant $c \in R_+$ and a bounded open set $\Omega_o \subset\subset \overline{\Omega}$ such that*

$$\|u\|_{W^2(\Omega)} \leqslant c\left(|Lu + \lambda\, g^{-1}\, \beta\, u|_{2,\Omega} + |u|_{2,\Omega_o}\right) \tag{6.1}$$

$$\forall\, u \in W^2(\Omega) \cap W_0^1(\Omega)\,, \qquad \forall\, \lambda \geqslant 0\,,$$

where c is a positive constant depending on Ω, ν, n, s, t, a_i, a, $\|a_{ij}\|_\infty$, α_{ij}, $\|e_{ij}\|_\infty$, $\|g\|_\infty$, $\|\alpha_{ij}\|_\infty$, $\tau[(\alpha_{ij})_x]$, $\tau[\zeta_k\,(e_{ij})_x]$, $\tau[\beta]$, $\tau[\delta]$, $\tau[a]$, $r[(\alpha_{ij})_x]$, $r[\zeta_k\,(e_{ij})_x]$, $r[\beta]$, $r[\delta]$, $r[a]$.

Proof.

Step 1 (*Estimates at infinity*).

If the principal coefficients of L are regular 'enough', we can use Corollary 5.2 in [10] to get the bound (6.2). Therefore if

$$\tilde{L}_o = -\sum_{i,j=1}^{n} \alpha_{ij} \frac{\partial^2}{\partial x_i \partial x_j},$$

we have that

$$\|(1-\zeta_k)\,u\|_{W^2(\Omega)} \leqslant c_1 \left| \tilde{L}_o((1-\zeta_k)\,u) + (ga+\lambda\beta)(1-\zeta_k)\,u \right|_{2,\Omega}, \tag{6.2}$$

from which

$$\begin{aligned} \|(1-\zeta_k)\,u\|_{W^2(\Omega)} \leqslant c_1 \Bigg(\Bigg| &-\sum_{i,j=1}^{n} \big(\alpha_{ij} - g a_{ij}\big)\big((1-\zeta_k)\,u\big)_{x_i x_j} \\ &- g \sum_{i,j=1}^{n} a_{ij}\big((1-\zeta_k)\,u\big)_{x_i x_j} + (ga+\lambda\,\beta)(1-\zeta_k)\,u \Bigg|_{2,\Omega} \Bigg) \\ &\leqslant c_1 \Big(\|g\|_\infty |L_o\big((1-\zeta_k)\,u\big) + (a+\lambda\,g^{-1}\,\beta)(1-\zeta_k)\,u|_{2,\Omega} \\ &\quad + \gamma(k)|((1-\zeta_k)\,u)_{xx}|_{2,\Omega} \Big). \end{aligned} \tag{6.3}$$

Taking in mind (h_7), by a suitable choice $k = k_0 \in R_+$ we get from (6.3)

$$\|(1-\zeta_{k_0})\,u\|_{W^2(\Omega)} \leqslant c_2 \left| L_o\big((1-\zeta_{k_0})\,u\big) + (a+\lambda\,g^{-1}\,\beta)(1-\zeta_{k_0})\,u \right|_{2,\Omega}. \tag{6.4}$$

Step 2 (*Estimates on bounded sets*).

Locally we can apply Lemma 5.2 with $L = L_0 + a$. Then, reasoning as in [7], we get the estimate

$$\|\zeta_{k_0} u\|_{W^2(\Omega)} \leqslant c_3 \left(\left|(L_o(\zeta_{k_0} u) + (a+\lambda g^{-1}\beta)\zeta_{k_0} u\right|_{2,\Omega} + |\zeta_{k_0} u|_{2,\Omega} \right). \tag{6.5}$$

Using the well known inequality (see [1])

$$|u_x|_{2,supp\zeta_{k_0}} \leqslant K(\epsilon |u_{xx}|_{2,supp\zeta_{k_0}} + \epsilon^{-1} |u|_{2,supp\zeta_{k_0}}),$$

where $K = K(n,\Omega)$ and $0 < \epsilon < \epsilon_0$, $\epsilon_0 > 0$, inequalities (6.4) and (6.5) imply

$$\|u\|_{W^2(\Omega)} \leqslant c_4\big(|L_o\,u + (a+\lambda\,g^{-1}\,\beta)\,u|_{2,\Omega} + |u|_{2,\Omega'_o}\big), \tag{6.6}$$

with $\Omega'_o =$ supp ζ_{k_0}. Moreover from Lemma 3.4 in [10] we have that for any $\epsilon \in R_+$ there exist $c(\epsilon) \in R_+$ and an open set $\Omega_\epsilon \subset\subset \Omega$ such that

$$\sum_{i=1}^{n} \|a_i u_{x_i}\|_{L^2(\Omega)} \leqslant \epsilon\,\|u\|_{W^2(\Omega)} + c(\epsilon)\,|u|_{2,\Omega_\epsilon}. \tag{6.7}$$

From (6.6) and (6.7) we deduce the assertion with $\Omega_o = \Omega'_o \cup \Omega_\epsilon$. □

REMARK 6.1. We observe that in Theorem 6.1 we can suppose in place of the condition $\operatorname{ess\,inf}_\Omega a > 0$ in (h_{10})

$$a = a' + a'' , \qquad a' \in M_0^t(\Omega) , \qquad \operatorname*{ess\,inf}_\Omega a'' > 0 .$$

REMARK 6.2. A particular case of convergence at infinity can be obtained assuming $\alpha_{ij} = const$. So we can avoid introduction of further hypotheses on derivatives of α_{ij} (see (h_{11})).

From Theorem 6.1 we get the following result as in [7].

COROLLARY 6.2. *In the same hypotheses of Theorem 6.1 and if*

$$\beta^{-1} \in L^\infty_{loc}(\Omega)$$

then for any $s \in R$ *there exist* c, $\lambda_0 \in R_+$ *such that*

$$\|u\|_{W^2(\Omega)} \leqslant c \left|Lu + \lambda\, g^{-1}\, \beta\, u\right|_{2,\Omega} \tag{6.8}$$

$$\forall\, u \in W^2(\Omega) \cap W^1_0(\Omega) , \quad \forall\, \lambda \geqslant \lambda_0 ,$$

where c *has the same dependence of the constant in Theorem 6.1.*

REMARK 6.3. Inequality (6.8) can be obtained under different assumptions if we suppose coefficients of the operator L more regular. We refer to the paper [10] where we can find some results. We remark that Theorem 6.1 allows us to obtain the result stated in Corollary 6.2 under hypotheses considerably weakened with respect to previous papers.

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A. Canale
Università degli Studi di Salerno
Facoltà di Scienze MM.FF.NN.
Dipartimento di Matematica e Informatica
Via Ponte don Melillo
84084 Fisciano (Salerno)
Italia
e-mail: acanale@unisa.it