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ON INEQUALITIES FOR POLYNOMIALS IN TWO VARIABLES

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Abstract. Necessary and sufficient conditions are established for the comparison of the powers of polynomials in two variables with real coefficients.

0. Introduction, auxiliary results

In this paper for a given polynomial $P(\xi) = P(\xi_1, \xi_2)$ in two variables $(\xi_1, \xi_2) \in R^2$, with real coefficients we describe all homogeneous polynomials $Q(\xi) = Q(\xi_1, \xi_2)$ which are of less power than P (briefly $Q < P$), i.e. for some $C > 0$

$$|Q(\xi)| \leqslant C[|P(\xi)| + 1] \quad \forall \xi \in R^2. \tag{0.1}$$

Such inequalities often are used in the general theory of linear partial differential operators (see [1]–[3]). Many problems in this theory are reduced to the comparison of the characteristic polynomials (symbols) of differential operators. While one can add any lower order terms to an elliptic (or semi-elliptic) operator without violating its ellipticity (or semi-ellipticity), addition of lower order terms may violate the hypoellipticity (by L. Hörmander) or hyperbolicity (by I. Petrovski or by L. Görding) of operators.

Therefore naturally arises the problem of the description of lower order terms such that their addition to the given operator (polynomial) does not change its strength (by L. Hörmander) or power and consequently does not violate its type.

In [4] S. M. Nikolskii proved the uniqueness of the solution of the first boundary value problem for linear equations, for which all the monomials entering the characteristic polynomial are estimated via it.

In [5]–[6] V. I. Burenkov established connection between the behaviour of solutions of linear partial differential equations at infinity and their differential properties.

In [7] V. P. Mikhailov introduced the class of non-degenerate complete polynomials to which one can add any lower order terms without changing their powers. This class is a proper subset of hypoelliptic polynomials and all semi-elliptic polynomials.

Similar results, but in different terms, have been obtained by many authors (see for example [8]–[12]).

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In [13] the following result has been obtained. Let P be a hypoelliptic polynomial and let Q be a homogeneous polynomial such that $Q < P$. Then there exists a number $\Delta > 0$ such that the polynomial $P + aQ$ is hypoelliptic for all a satisfying $|a| \leqslant \Delta$ and, in general, $P + aQ$ is not hypoelliptic for a satisfying $|a| > \Delta$.

In the present paper for a given polynomial $P(\xi) = P(\xi_1, \xi_2)$ in two variables $(\xi_1, \xi_2) \in R^2$ with real coefficients we describe all homogeneous polynomials $Q(\xi) = Q(\xi_1, \xi_2)$ for which $Q < P$. Our method does not work in the case $n > 2$.

In [14] we have obtained necessary and sufficient conditions for $|P(\xi_1, \xi_2)| \to \infty$ as $|\xi| = \sqrt{\xi_1^2 + \xi_2^2} \to \infty$. We denote the set of all such polynomials by I_2.

Methods and results of the works [13]–[16] are used here.

Let P be represented as the sum of homogeneous polynomials:

$$P(\xi) = \sum_{i=0}^{M} P_i(\xi) = \sum_{i=0}^{M} \sum_{|\alpha|=d_i} \gamma_\alpha \xi^\alpha, \tag{0.2}$$

where $\alpha = (\alpha_1, \alpha_2)$ are multi-indices, $|\alpha| = \alpha_1 + \alpha_2$, $\xi^\alpha = \xi^{\alpha_1} \cdot \xi^{\alpha_2}$, γ_α are real numbers and d_i are the orders of the homogeneous polynomials P_i $(i = 0, 1, ...M)$, $d_0 > d_1 > \cdots > d_M \geqslant 0$.

Let $Q(\xi) = Q(\xi_1, \xi_2)$ be a homogeneous polynomial of order $d > 0$ and $\Sigma(Q) = \{\xi \in R^2, |\xi| = 1, Q(\xi) = 0\}$. For $\eta \in R^2$ with $|\eta| = 1$ we denote by $l(\eta) = l(\eta, Q)$ the order of zero η if $\eta \in \Sigma(Q)$ and assume that $l(\eta) = 0$ if $\eta \notin \Sigma(Q)$. Since $n = 2$ $\Sigma(Q)$ consists of finite number of points and $l(\eta) \leqslant d$.

For polynomials P represented in form (0.2) we define $\Sigma_0 \equiv \Sigma_0\ (P) = \Sigma(P_0), \Sigma_j \equiv \Sigma_j(P) = \{\xi \in \Sigma_{j-1}, P_j(\xi) = 0\}$ $(j = 1, 2, \cdots, M)$. For $\eta \in \Sigma_j$ by $l_j(\eta)$ we denote the order of zero η of the polynomial P_j $(0 \leqslant j \leqslant M)$ and set

$$\chi(\eta, \delta) = \chi(\eta, \delta, P) = \max_{0 \leqslant i \leqslant M} \{d_i - l_i(\eta)\delta\}, \quad \delta \in (0, \infty). \tag{0.3}$$

Let $\eta \in \Sigma_0$. Denote by $A(\eta) = A(\eta, P)$ the set of all numbers $\delta \geqslant 0$ for which there exist integer numbers $0 \leqslant i, j \leqslant M$, such that $i \neq j$ and

$$d_i - l_i(\eta) \cdot \delta = d_j - l_j(\eta) \cdot \delta = \chi(\eta, \delta).$$

It is easily verified that 1) for $\eta \in \Sigma_0$ the set $A(\eta)$ is finite, 2) if $0 \neq \eta \in R^2, Q(\eta) \neq 0$ and $\eta \in \Sigma_M$, or $\eta \in \Sigma_{M-1}$ and $d_M = 0$, then $Q \not< P$.

Finally for $\eta \in \Sigma_0$, $\delta \in A(\eta)$ we set

$$J(\eta, \delta) = J(\eta, \delta, P) = \{i; 0 \leqslant i \leqslant M, d_i - l_i(\eta) \cdot \delta = \chi(\eta, \delta), c(\eta, \delta) = \operatorname{card} J(\eta, \delta)\}.$$

Let $J(\eta, \delta) = \{i_0, i_1, ..., i_c\}, 0 \leqslant i_0 < i_1 < ... < i_c \leqslant M$, where $c = c(\eta, \delta)$, then it follows immediately that $l_{i_0}(\eta) > l_{i_1}(\eta) > ... > l_{i_c}(\eta)$.

LEMMA 0.1. ([14], Lemma 1.2) *Let Q be a homogeneous polynomial of order $d > 0, \Sigma(Q) \neq \emptyset$. Then, for each $\eta \in \Sigma(Q)$ there exists $\varepsilon = \varepsilon(\eta, Q) > 0$ such that*

$$\frac{1}{2} \cdot |D_\tau^l(\eta)| \cdot |(\xi, \eta)|^{d-l} \cdot |(\xi, \tau)|^l \leqslant |Q(\xi)| \leqslant \frac{3}{2} |D_\tau^l Q(\eta)| \cdot |(\xi, \eta)|^{d-l} \cdot |(\xi, \tau)|^l$$

for all $\xi \in G_\varepsilon(\eta) \equiv \{z \in R^2, |(z,\tau)| \leqslant \varepsilon|(z,\eta)|\}$. *Here* $l = l(\eta, Q)$ *is the order of zero* $\eta \in \Sigma(Q)$, $\tau = \tau(\eta) \in R^2$, $|\tau| = 1$, $(\tau, \eta) = 0$,

$$D^l_\tau Q(\eta) = \sum_{|\alpha|=l} \frac{D^\alpha Q(\eta)}{\alpha!} \cdot \tau^\alpha \neq 0. \tag{0.4}$$

In the sequel all unimportant positive constants will be denoted by C.

DEFINITION 0.1. We say that a polynomial P is more powerful than a polynomial Q relative to a point $\eta \in R^2$ and write $Q <^\eta P$ if there exist $C, \varepsilon > 0$ such that

$$|Q(\xi)| \leqslant C \cdot [|P(\xi)| + 1] \; \forall \xi \in G_\varepsilon(\eta). \tag{0.5}$$

Since $n = 2$, it is easy to verify that $Q < P$ if and only if $d \leqslant d_0$ and $Q <^\eta P$ for all $\eta \in \Sigma_0$.

LEMMA 0.2. *Let* Q *be a homogeneous polynomial of order* $d > 0$, P *be a polynomial represented in form (0.2),* $\eta \in \Sigma_0$ *and* $Q <^\eta P$. *Then*

1) $d \leqslant d_0$

2) $d - l(\eta)\delta \leqslant \chi(P, \eta, \delta)$ *for all* $\delta \in [0, \sigma(\eta))$

3) if $A(\eta, P, Q) \equiv A(\eta, P) \cap [0, \sigma(\eta)) = \emptyset$ *then* $\eta \in \Sigma(Q)$ *and* $\sigma(\eta) \leqslant \sigma_0(\eta)$

4) if $\eta \notin \Sigma(Q)$ *then* $A(\eta, P) \neq \emptyset$.

Here $\sigma(\eta) = d/l(\eta)$ *if* $\eta \in \Sigma(Q)$ *and* $\sigma(\eta) = \infty$ *if* $\eta \notin \Sigma(Q)$, $\sigma_0(\eta) = d_0/l_0(\eta)$ *if* $\eta \in \Sigma_0$ *and* $\sigma_0(\eta) = \infty$ *if* $\eta \notin \Sigma_0$.

Proof. Statement 1) is obvious. To prove statement 2) suppose,to the contrary, that there exists $\delta \in (0, \sigma(\eta))$ such that

$$d - l(\eta)\delta > \chi(\eta, \delta, P). \tag{0.6}$$

Let $s \in N, \xi^s = s\eta^s \equiv s(\eta + s^{-\delta}\tau)$. Then $(\xi^s, \eta) = s$, $(\xi^s, \tau) = s^{1-\delta}$, $\xi^s \in G_\varepsilon(\eta)$ for any $\varepsilon > 0$ and for sufficiently large s. By Taylor's formula (see (0.4)) and Lemma 0.1 we obtain that for some $\varepsilon = \varepsilon(\eta, Q, P_0, \cdots, P_M)$ and for sufficiently large s

$$|Q(\xi^s)| \geqslant \frac{1}{2}|D^l_\tau Q(\eta)| \cdot |(\xi^s, \eta)|^{d-l} \cdot |(\xi^s, \tau)|^l = \frac{1}{2}|D^l_\tau Q(\eta)| s^{d-l} s^{(1-\delta)\cdot l} \geqslant C s^{d-l\cdot\delta}.$$

Similarly for the polynomial P

$$|P(\xi^s)| \leqslant \sum_{i=0}^{M} |P_i(\xi^s)| \leqslant \sum_{i=0}^{M} \frac{3}{2}|D^{l_i}_\tau P_i(\eta)| s^{d_i - l_i\cdot\delta} \leqslant C s^{\chi(\eta,\delta,P)}.$$

These inequalities together with (0.6) contradict the condition $Q <^\eta P$ and hence statement 2) follows.

To prove the first part of statement 3) we assume, to the contrary, that $A(\eta, P, Q) = \emptyset$ and $Q(\eta) \neq 0$. Then by simple geometric considerations it is easy to see that $P_i(\eta) = 0$ $(i = 0, 1, ..., M)$ and $Q(s\eta)| = s^d |Q(\eta)| \to \infty$ as $s \to \infty$, $P(s\eta) = 0$ $(s \in N)$, which contradicts (0.5).

To prove the second part of statement 3), we assume that $\eta \in \Sigma(Q)$, $A(\eta,P,Q) = \emptyset$ and $\sigma(\eta) > \sigma_0(\eta)$. Then $A(\eta,P) \cap [0,d/l(\eta)) = \emptyset$ and it follows that in this case $P_i(\eta) = 0$, $\chi(\eta,0,P) = d_0$ and $\chi(\eta,\delta,P) = d_0 - l_0(\eta)\delta > d_i - l_i(\eta)\delta$ for all $\delta \in [0,\sigma_0(\eta)]$, $i = 1,\cdots,M$. As in the proof of statement 2) we have $|P(\xi^s)| \leqslant C$ $(s \in N)$, $|Q(\xi^s)| \geqslant Cs^{d-l(\eta)\sigma_0} \to \infty$ as $s \to \infty$, which contradicts the condition $Q <^{\eta} P$.

To prove statement 4) it is sufficient to note that $d - l(\eta)\delta = d > \chi(P,\eta,\delta) = d_0 - l_0(\eta)\delta$ for $\delta > (d_0 - d)/l_0(\eta)$ when $\eta \in \Sigma_0 \backslash \Sigma(Q)$ and $A(\eta,P) \neq \emptyset$. Lemma 0.2 is proved. $\square$

LEMMA 0.3. *Let P and Q be as in Lemma 0.2 and $A(\eta,P,Q) = \emptyset$ for all $\eta \in \Sigma_0$. Then $Q < P$ if and only if 1) $d \leqslant d_0$, 2) $\eta \in \Sigma(Q)$ and $\sigma(\eta) \leqslant \sigma_0(\eta)$ for all $\eta \in \Sigma_0$.*

Proof. The necessity follows from Lemma 0.2. To prove the sufficiency first we note that conditions 1)–2) and condition $A(\eta,P,Q) = \emptyset$ for all $\eta \in \Sigma_0$ imply that for all $\eta \in \Sigma_0$ and $i = 1,2,\cdots,M$

$$d_0 - l_0(\eta)\delta \geqslant d - l(\eta)\delta, \quad d_0 - l_0(\eta)\delta > d_i - l_i(\eta)\delta, \quad 0 \leqslant \delta < \sigma(\eta). \tag{0.7}$$

Let, to the contrary, there exist a sequence $\{\xi^s\}$ such that

$$s \to \infty: \quad |\xi^s| \to \infty, \quad |Q(\xi^s)|/[|P(\xi^s)| + 1] \to \infty. \tag{0.8}$$

Denote $\eta^s = \xi^s/|\xi^s|$ $(s \in N)$. By choosing a subsequence (we denote this subsequence and all the subsequences coming henceforth again by $\{\eta^s\}$) one may assume that $\eta^s \to \eta$ as s$\to \infty$, for some $\eta \in R^2, |\eta| = 1$. It is easy to verify that $\eta \in \Sigma_0$. Then by condition 2) $\eta \in \Sigma(Q)$. Let us expand the vectors $\{\xi^s\}$ via the orthonormal basis $\{\eta,\tau\}$:

$$\xi^s = \varphi_s \cdot \eta + \psi_s \cdot \tau = (\xi^s,\eta) \cdot \eta + (\xi^s,\tau) . \tau \quad (s \in N).$$

Without loss of generality one can assume that $\varphi_s \geqslant 1$, $\psi_s > 0$ $(s \in N)$ (see [14]). Since $\eta^s \to \eta$ we have $\psi_s/\varphi_s \to 0$ as $s \to \infty$ and $\xi^s \in G_\varepsilon(\eta)$ for any $\varepsilon > 0$ and sufficiently large s. Let $\varepsilon = \min\{\varepsilon(\eta,Q),\varepsilon(\eta,P_0),...,\varepsilon(\eta,P_M)\}$ where $\varepsilon(\eta,Q),\varepsilon(\eta,P_0),...,\varepsilon(\eta,P_M)$ are defined in lemma 0.1.

Denote $\rho_s = 1 - \ln\psi_s/\ln\varphi_s \Longleftrightarrow \psi_s = \varphi_s^{1-\rho_s}$ $(s \in N)$.

If $\rho_s \geqslant \sigma(\eta)$ for all $s \in N$, then by Lemma 0.1

$$|Q(\xi^s)| \leqslant \frac{3}{2}|D_\tau^l Q(\eta)| \cdot \varphi_s^{d-l}\psi_s^l = \frac{3}{2}|D_\tau^l Q(\eta)| \cdot \varphi_s^{d-\rho_s l} \leqslant \frac{3}{2}|D_\tau^l Q(\eta)|,$$

which contradicts (0.8). Similarly we arrive at a contradiction if the inequality $\rho_s \geqslant \sigma(\eta)$ holds for infinitely many $s \in N$.

Let now $\rho_s < \sigma = \sigma(\eta)$ $(s \in N)$. Without loss of generality we assume that $\rho_s \to \delta$ as $s \to \infty$ where $\delta \leqslant \sigma$. Consider the following cases a) $\delta < \sigma$ and b) $\delta = \sigma$. In the case a) we get by Lemma 0.1 that for all $s \in N$ and $i = 1,\cdots,M$

$$|P_0(\xi^s)| \geqslant \frac{1}{2}|D_\tau^{l_0} P_0(\eta)| \cdot \varphi_s^{d_0 - l_0\rho_s}, \quad |P_i(\xi^s)| \leqslant \frac{3}{2}|D_\tau^{l_i} P_i(\eta)| \cdot \varphi_s^{d_i - l_i\rho_s}.$$

From these inequalities we get by (0.7)

$$|P(\xi^s)| \geqslant |P_0(\xi^s)| - \sum_{i=1}^{M} |P_i(\xi^s)| \geqslant \frac{1}{4}|D_\tau^{l_0} P_0(\eta)| \varphi_s^{d_0 - l_0 \cdot \rho_s} \quad (s \in N).$$

For the polynomial Q we have by Lemma 0.1 $|Q(\xi^s)| \leqslant (3/2)|D_\tau^l Q(\eta)| \varphi_s^{d-l\rho_s}$ $(s \in N)$. The last two inequalities together with (0.7) contradict (0.8).

In case b) we write $x_s = \psi_s \cdot \varphi_s^{-(1-\sigma)} = \varphi_s^{\sigma-\rho_s}$ $(s \in N)$ and consider the following possible subcases: b.1) $\{x_s\}$ is bounded: $0 < x_s \leqslant C$ $(s \in N)$, b.2) $\{x_s\}$ is unbounded: without loss of generality $x_s \to \infty$ as $s \to \infty$. In the case b.1) we obtain by Lemma 0.1 for all $s \in N$

$$|Q(\xi^s)| \leqslant \frac{3}{2}|D_\tau^l Q(\eta)| \cdot \varphi_s^{d-l} \psi_s^l = \frac{3}{2}|D_\tau^l Q(\eta)| \cdot \varphi_s^{d-l\sigma} x_s^l = \frac{3}{2}|D_\tau^l Q(\eta)| x_s^l \leqslant C,$$

which contradicts (0.8).

Case b.2) in its turn we split into the following subcases : b.2.1) $\sigma \notin A(\eta, P)$, b.2.2) $\sigma \in A(\eta, P)$. In case b.2.1) we get by Lemma 0.1

$$|Q(\xi^s)| \leqslant \frac{3}{2}|D_\tau^l Q(\eta)| \cdot x_s^l \quad (s \in N), \tag{0.9}$$

$$|P(\xi^s)| \geqslant \frac{1}{2}|D_\tau^{l_0} P_0(\eta)| \varphi_s^{d_0 - l_0\sigma} x_s^{l_0}; \quad |P_i(\xi^s)| \leqslant \frac{3}{2}|D_\tau^{l_i} P_i(\eta)| \varphi^{d_i - l_i\sigma} x_s^{l_i}, \ (1 \leqslant i \leqslant M).$$

Since $x_s^m . \varphi_s^{-\varepsilon} = \varphi_s^{m(\sigma-\rho_s)} \to 0$ as $s \to \infty$ for any $m \geqslant 0$, the last two inequalities together with (0.7) imply that for sufficiently large s

$$|P(\xi^s)| \geqslant |P_0(\xi^s)| - \sum_{j=1}^{M} |P_j(\xi^s)| \geqslant \frac{1}{4}|D_\tau^{l_0} P_0(\eta)| \varphi^{d_0 - l_0\sigma} x_s^{l_0}. \tag{0.10}$$

If $\sigma(\eta) < \sigma_0(\eta)$ then $d_0 - l_0\sigma(\eta) > d_0 - l_0\sigma_0(\eta) = 0$ and arguing as above we see that inequalities (0.9)–(0.10) contradict (0.8). If $\sigma(\eta) = \sigma_0(\eta)$, i.e. $d_0 - l_0\sigma(\eta) = 0$, then it is obvious that $l_0(\eta) \geqslant l(\eta)$ and again we arrive at a contradiction.

In case b.2.2) $\chi(\eta, \sigma, P) = d_0 - l_0\sigma = d_i - l_i\sigma$, $l_0 > l_i$ for all $0 \neq i \in J(P, \eta, \sigma)$ and $\chi(\eta, \sigma, P) > d_i - l_i\sigma$ for $i \notin J(\eta, \sigma, P)$. Then by Lemma 0.1 we have that for sufficiently large s $(J = J(\eta, \sigma, P))$

$$\begin{aligned} |P(\xi^s)| &\geqslant |P_0(\xi^s)| - \sum_{0 \neq i \in J} |P_i(\xi^s)| - \sum_{i \in J} |P_i(\xi^s)| \geqslant \frac{1}{2}|D_\tau^{l_0} P_0(\eta)| \varphi_s^{d_0 - l_0\sigma} x_s^{l_0} \\ &- \frac{3}{2}\sum_{i=0}^{M} |D_\tau^{l_i} P_i(\eta)| \varphi_s^{d_i - l_i\sigma} x_s^{l_i} = \frac{1}{2}\varphi_s^{d_0 - l_0\sigma} x_s^{l_0} [|D_\tau^{l_0} P_0(\eta)| - 3\sum_{0 \neq i \in J} |D_\tau^{l_i} P_i(\eta)| x_s^{l_i - l_0}] \\ &+ o(\varphi_s^{d_0 - l_0\sigma}) \\ &= \frac{1}{2}|D_\tau^{l_0} P_0(\eta)| \varphi_s^{d_0 - l_0\sigma} x_s^{l_0} [1 + o(1)]. \end{aligned} \tag{0.11}$$

If $\sigma < \sigma_0$ then $d_0 - l_0\sigma > d - l\sigma$ and (0.9), (0.11) contradict (0.8). If $\sigma = \sigma_0$ then it is obvious that $l_0 \geqslant l$ and again we arrive at a contradiction. Lemma 0.3 is proved. □

EXAMPLE 0.1. Let $P(\xi) = P_0(\xi) + P_1(\xi) = (\xi_1 - \xi_2)^3 \cdot (\xi_1^2 + \xi_2^2)^2 + \xi_1$, $Q(\xi) = (\xi_1 - \xi_2)^2 \cdot (\xi_1^2 + \xi_2^2)$. Here $d_0 = 7, d_1 = 1$, $d = 4$, $\Sigma_0 = \{\eta^{\pm} = \{(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}})\}$, $l_0(\eta^{\pm}) = 3$, $l(\eta^{\pm}) = 2$, $\sigma(\eta^{\pm}) = 2$. Since $d_0 > d$, $d_0/l_0(\eta^{\pm}) > d/l(\eta^{\pm})$ and $A(\eta^{\pm}, P) \cap [0, \sigma(\eta^{\pm})) = \emptyset$, by Lemma 0.3 $Q < P$. It is interesting to note that $P_1 \not< P$ and $P \notin I_2$ (see [14]).

Our further efforts will be devoted to obtaining conditions ensuring that $Q < P$ when $A(\eta, P, Q) \neq \emptyset$ for some $\eta \in \Sigma_0$. We denote by $\Sigma^0 = \Sigma^0(P_0)$ the set of points $\eta \in \Sigma_0$ for which $A(\eta, P, Q) \neq \emptyset$. Let $\Sigma^1 = \Sigma_0 \backslash \Sigma^0$ and let $B(P, Q)$ be the set of all pairs (η, δ) such that $\eta \in \Sigma^0$, $\delta \in A(\eta, P, Q)$.

Let $r(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_k x^k$ be a polynomial with real coefficients, $0 \leqslant k < n$, $a_n \cdot a_k \neq 0$, $X(r) = \{x : 0 \neq x \in R^1, r(x) = 0\}$, $\kappa_0(r) = \frac{1}{2}\min\{|x| : x \in X(r)\}$, $\kappa_1(r) = 2\max\{x : x \in X(r)\}$. Let $c(r)$ be the number of nonzero coefficients of r.

We shall need the following elementary statement.

LEMMA 0.4. *1) There exists $C > 0$ such that $|\ r(x)| \geqslant C|x|^k$ for $|x| \leqslant \kappa_0(r)$ and $|r(x)| \geqslant C|x|^n$ for $|x| \geqslant \kappa_1(r)$; 2) $c(r) \geqslant 3$ when $r(x) \geqslant 0$ for all $x \in R^1$ and $X(r) \neq \emptyset$.*

Let P be a polynomial represented in form (0.2), $(\eta, \delta) \in B(P, Q)$ and

$$r_0(x) = r_0(x, \eta, \delta, P) = \sum_{i \in J(\eta, \delta, P)} D_\tau^{l_i(\eta)} P_i(\eta) \cdot x^{l_i(\eta)}, \quad x \in R^1. \tag{0.12}$$

We define the numbers $\kappa_j(r_0, \eta, \delta)$ $(j = 0, 1)$ as above and set

$$\kappa_0 = \min\{\kappa_0(r_0, \eta, \delta) : (\eta, \delta) \in B(P, Q)\} \tag{0.13}$$

$$\kappa_1 = \max\{\kappa_1(r_0, \eta, \delta) : (\eta, \delta) \in B(P, Q)\}, \tag{0.14}$$

$$\xi(t, x) = \xi(t, x, \eta, \delta) = t(\eta + t^{-\delta} x \cdot \tau), \quad x \in R^1, \quad t \in (0, \infty).$$

The following result was proved in [14] by applying Lemmas 0.1–0.4 (We gave here the proofs of Lemmas 0.2–0.3 because they were stated in [14] without proofs.)

THEOREM 0.1. *$Q < P$ if and only if conditions 1)–2) of Lemma 0.3 for all $\eta \in \Sigma^0$ and the condition*

3) for some $C > 0$ and for all $(\eta, \delta) \in B(P, Q)$

$$t^{d - l(\eta).\delta} \leqslant C. \min_{\kappa_0 \leqslant |x| \leqslant \kappa_1} [|P[\xi(t, x)]| + 1 \quad \forall t \in (0, \infty) \tag{0.15}$$

are satisfied.

Thus the problem of the comparison of the powers of a homogenous polynomial Q and a polynomial P when $B(P, Q) \neq \emptyset$, is reduced to establishing estimate (0.15) for the finite set of pairs $(\eta, \delta) \in B(P, Q)$.

Denote $f(t,x) = f(t,x,\eta,\delta) = |P[\xi(t,x,\eta,\delta)]|$. In [14] it is proved that f can be represented in the form

$$f(t,x) = f(t,x,\eta,\delta) = \sum_{i=0}^{M_0} t^{\chi(\eta,\delta,P)-\frac{i}{q}} \cdot r_i(x), \tag{0.16}$$

where $M_0 = M_0(\eta,\delta) \in N$, $\chi(\eta,\delta,P)$ is defined by (0.3), $q = q(\eta,\delta)$ is the smallest natural number for which $q\delta \in N$ (it is easy to see that the number $\delta \in A(\eta,P)$ is rational), $\{r_i\}$ are polynomials in one variable $x \in R^1$, which are uniquely defined by (P,η,δ), namely

$$r_j(x) = r_j(x,\eta,\delta) = \sum_{i \in J(\eta,\delta,P)} D_\tau^{l_i+j} P_i(\eta) \cdot x^{l_i+j} \quad (j = 0,1,\cdots,M_0). \tag{0.17}$$

Note that the polynomial r_0 was already defined by formula (0.12).

1. Investigation of functions generated by a given polynomial

Let

$$f(t,x) = \sum_{i=0}^{M_0} t^{m-\frac{i}{q}} \cdot r_i(x) \quad t \in (0,\infty) \quad x \in R^1 \tag{1.1}$$

be a function of type (0.16), $X_0 = X_0(f) = \{0 \neq x \in R^1, r_0(x) = 0\}$. For $0 \neq x \in R^1$ let $l_j(x_0)$ be the order of the zero x_0 of the polynomial r_j if $x_0 \in X_0$ and $l_j(x_0) = 0$ if $x_0 \notin X_0$, $(0 \leqslant j \leqslant M_0)$. We set

$$\chi(f,x_0,\Delta) = \max_{0 \leqslant i \leqslant M_0} \left\{ m - \frac{i}{q} - l_i(x_0)\Delta \right\}, \quad \Delta \geqslant 0.$$

We denote by $A(x_0,f)$ the set of all numbers $\Delta \geqslant 0$ for which there are indices $i \neq j \,:\, 0 \leqslant i,j \leqslant M_0$ such that $m - \frac{i}{q} - l_i(x_0)\Delta = m - \frac{j}{q} - l_j(x_0)\Delta = \chi(f,x_0,\Delta)$.

For a pair $(x_0,\Delta_0) : x_0 \in X_0, \Delta_0 \in A(x_0,f)$ we denote by $J(x_0,\Delta_0,f)$ the set of all integers $0 \leqslant k \leqslant M_0$ for which

$$m - \frac{k}{q} - l_k(x_0)\Delta_0 = \chi(f,x_0,\Delta_0) \tag{1.2}$$

and set $c(f,x_0,\Delta_0) = \operatorname{card} J(x_0,\Delta_0,f)$.

We have introduced the notations $\{l_i,\chi,A,J,c\}$ both for polynomials in two variables and for functions of one variable. We hope it will not cause any misunderstanding.

Thus every pair $(\eta,\delta) \in B(P,Q)$ generates the unique function $f(t,x,\eta,\delta)$ of type (1.1). Let (η,δ) be such a pair and

$$f_0(t,x,\eta,\delta) = |P[t(\eta + t^{-\delta}.x.\tau)]| = \sum_{i=0}^{M_0} t^{m_0-\frac{i}{q_0}} \cdot r_i^0(x,\eta,\delta), \tag{1.3}$$

where $M_0 = M_0(\eta,\delta)$, $m_0 = \chi(\eta,\delta,P)$, $q_0 = q_0(\delta)$ is the smallest natural number for which $q_0\delta \in N$, the polynomials $\{r_i^0\}$ are defined by (0.17), $t \in (0,\infty)$, $\kappa_0^0 \leqslant |x| \leqslant \kappa_1^0$, the numbers κ_j^0 $(j=0,1)$ are defined by (0.13)–(0.14).

If $X_0(f_0,\eta,\delta) = \emptyset$ for all pairs $(\eta,\delta) \in B(P,Q)$ then $f_0(t,x,\eta,\delta) \geqslant Ct^{m_0}$: $t \geqslant 0$ and by Theorem 0.1 $Q < P$ when (necessary) conditions 1)–2) of Lemma 0.2 and condition 2) of Lemma 0.3 are satisfied. If $X_0(f_0,\eta,\delta) \neq \emptyset$ for a pair $(\eta,\delta) \in B(P,Q)$ then first we will assume that there is a unique such pair (η,δ) and for simplification of notations we sometimes omit symbols η or δ. If for some $x_0 \in X_0 = X_0(f_0,\eta,\delta)$ $r_j^0(x_0) = 0$ for all $j = 0,1,...,M_0$ and $\delta \in [0,\sigma(Q))$ i.e. $d - l(\eta)\delta > 0$ then $Q \not< P$ by Theorem 0.1. Assume therefore that for each $x_0 \in X_0$ there exists a number $j_0 = j_0(x_0)$ such that $0 < j_0 \leqslant M_0$ and $r_j^0(x_0) = 0$ $j = 0,1,\cdots,j_0-1, r_{j_0}^0(x_0) \neq 0$. By Theorem 0.1 $Q < P$ implies that

$$d - l(\eta)\delta \leqslant m_0 - \frac{j_0(x_0)}{q_0} \quad \forall x_0 \in X_0. \tag{1.4}$$

Denote by X_0^0 the set of all points $x_0 \in X_0$ for which $A(x_0,f_0,\eta,\delta) = \emptyset$, $X_0^1 = X_0 \backslash X_0^0$.

Applying Theorem 0.1 one can easily see that inequality (1.4) together with (necessary) conditions 1)–2) of Lemma 0.3 are sufficient for $Q < P$ if $X_0^1 = \emptyset$.

Thus the problem of the comparison of polynomials P and Q is solved when $X_0^1 = \emptyset$. We have therefore to consider only the case when $X_0^1 \neq \emptyset$, i.e. when $B_0 = B_0(f_0,\eta,\delta) = \{(x,\Delta) : x \in X_0(f_0,\eta,\delta), \Delta \in A(x,f_0,\eta,\delta)\} \neq \emptyset$.

For simplification of notations in the remainder of this section we will assume that the set B_0 consists of a unique pair (x_0,Δ_0) and sometimes omit symbols x_0 or Δ_0.

In [14] it is proved that

$$c(P,\eta,\delta) - 1 = \operatorname{card} J(P,\eta,\delta) - 1 \geqslant l_0^0(x_0) \geqslant c(f_0,x_0,\Delta_0) - 1. \tag{1.5}$$

Next we write

$$\begin{aligned} D^k r_i^0(x_0) &= \frac{1}{k!}\cdot\frac{d^k}{dx^k} r_i^0(x_0) \quad (1 \leqslant i \leqslant M_0, k = 0,1,\cdots), \\ r_j^1(y) &= \sum_{i \in J(f_0,x_0,\Delta_0)} D^{l_i(x_0)+j} r_i^0(x_0)\cdot y^{l_i(x_0)+j} \quad (j = 0,1,\cdots). \end{aligned} \tag{1.6}$$

In [14] it is proved (see Theorem 2.1) that 1) the problem of the behaviour of the function f_0 at infinity reduces to the study of its behaviour on the set $\{(t, x_0 + t^{-\Delta_0} y), y \in R^1\}$ as $t \to \infty$, 2) $f_1(t,y) = f_1(t,y,x_0,\Delta_0) = f_0(t, x_0 + t^{-\Delta_0}.y)$ can be represented in the form

$$f_1(t,y) = \sum_{i=0}^{M_1} t^{m_1 - \frac{i}{q_0 \cdot q_1}} \cdot r_i^1(y), \tag{1.7}$$

where $M_1 = M_1(f_0,x_0,\Delta_0), m_1 = \chi(f_0,x_0,\Delta_0) < m_0, q_1 = q_1(f_0,x_0,\Delta_0)$ is the smallest natural number for which $(q_0q_1)\Delta_0 \in N$.

We introduce also the following notations: $X_1 = X_1(f_1, x_0, \Delta_0) = \{0 \neq y \in R^1,\ r_0^1(y) = 0\}$,

$$\kappa_0^1 = \kappa_0^1(x_0, \Delta_0) = \frac{1}{2} \cdot \min_{y \in X_1} |y|, \quad \kappa_1^1 = \kappa_1^1(x_0, \Delta_0) = 2 \cdot \max_{y \in X_1} |y|. \tag{1.8}$$

If $X_1 = \emptyset$ ($X_1(f_1, x_0, \Delta_0) = \emptyset$ for all $(x_0, \Delta_0) \in B_0$, when the set B_0 consists of more than one pair) then $f_1(t,y) \geqslant C \cdot t^{m_1}$ for all $t \geqslant 0$, $y \in R^1$. Hence by Theorem 0.1 $Q < P$ if and only if conditions 1)–2) of Lemma 0.3, condition (1.4) and condition $d - l(\eta)\delta \leqslant m_1$ (conditions $d - l(\eta) \cdot \delta \leqslant m_1(x_0, \Delta_0)$ for all $(x_0, \Delta_0) \in B_0$ when B_0 consists of more than one pair) hold. If $X_1 \neq \emptyset$, $r_1^1(y_1) = 0$ for all $j = 0, 1, \cdots, M_1$ and for some $y_1 \in X_1$ then by Theorem 0.1 $Q \not< P$ when $d - l(\eta)\delta > 0$. Assume therefore that for each $y_1 \in X_1$ there exists a number $j_1 = j_1(y_1)$ such that $0 < j_1 \leqslant M_1$ and $r_j^1(y_1) = 0$, $j = 0, 1, \cdots, j_1 - 1$, $r_{j_1}^1(y_1) \neq 0$. By Theorem 0.1 $Q < P$ implies that

$$d - l(\eta)\delta \leqslant m_1 - \frac{j_1}{q_0 \cdot q_1} \quad \forall y_1 \in X_1. \tag{1.9}$$

Denote by $X_1^0 = X_1^0(f_1, x_0, \Delta_0)$ the set of all points $y_1 \in X_1$ for which $A(y_1, f_1, x_0, \Delta_0) = \emptyset$ and $X_1^1 = X_1 \backslash X_1^0$.

Applying Theorem 0.1 one can easily see that inequality (1.9) together with (necessary) conditions 1)–2) of Lemma 0.3 and condition (1.4) are sufficient for $Q < P$ if $X_1^1 = \emptyset$.

Thus we have to consider only the case $X_1^1 \neq \emptyset$ ($X_1^1(f_1, x_0, \Delta_0) \neq \emptyset$ for all pairs $(x_0, \Delta_0) \in B_0$ when B_0 consists of more than one pair), i.e. when $B_1 = B_1(f_1, x_0, \Delta_0) = \{(y_1, \Delta_1) : y_1 \in X_1, \Delta_1 \in A(y, f_1, x_0, \Delta_0)\} \neq \emptyset$. For simplification of notations in the remainder of this section we will assume that the set B_1 consists of a unique pair (y_1, Δ_1).

In [14] it is proved that 1) the problem of the behaiour of the function f_1 at infinity reduces to the study of its behaviour on the set $\{(t, y_1 + t^{-\Delta_1} z), z \in R^1\}$ as $t \to \infty$, 2) $f_2(t,z) = f_2(t, z, y_1, \Delta_1) = f_1(t, y_1 + t^{-\Delta_1} z)$ can be represented in the form

$$f_2(t,z) = \sum_{i=0}^{M_2} t^{m_2 - \frac{i}{q_0 q_1 q_2}} \cdot r_i^2(z), \tag{1.10}$$

where $M_2 = M_2(f_1, y_1, \Delta_1), m_2 = \chi(f_1, y_1, \Delta_1) < m_1, q_2 = q_2(f_1, y_1, \Delta_1)$ is the smallest natural number for which $(q_0 q_1 q_2)\Delta_1 \in N$. If $X_2 = X_2(f_2, y_1, \Delta_1) = \{0 \neq z \in R^1,\ r_0^2(z) = 0\} = \emptyset$ then the problem can be solved as above. If $X_2 \neq \emptyset$ then the function f_2 generates the function f_3 and so on.

Let the pair $(\eta, \delta) \in B(P, Q)$ generate the function $f_0(t, x, \eta, \delta)$ by formula (1.3) and the set $B_0(\eta, \delta) = \{(x, \Delta) : x \in X_0(\eta, \delta),\ \Delta \in A(x_0, \eta, \delta)\}$. Each pair $(x_0, \Delta_0) \in B_0(\eta, \delta)$ generates the function $f_1(t, x, x_0, \Delta_0, \eta, \delta)$ by formula (1.7) and the set $B_1(x_0, \Delta_0, \eta, \delta) = \{(x_1, \Delta_1) : x_1 \in X_1(x_0, \Delta_0, \eta, \delta),\ \Delta_1 \in A(x_1, x_0, \Delta_0, \eta, \delta)\}$ and so on.

We denote by F_0 the set of functions $\{f_0(t, x, \eta, \delta);\ (\eta, \delta) \in B(P, Q)\}$, by F_1 the set of functions $\{f_1(t, x, x_0, \Delta_0, \eta, \delta);\ (\eta, \delta) \in B(P, Q),\ (x_0, \Delta_0) \in B_0(\eta, \delta)\}$, by F_2 the set of functions $\{f_2(t,x,x_1,\Delta_1,x_0,\Delta_0,\eta,\delta);\ (\eta,\delta)\in B(P,Q),\ (x_0,\Delta_0)\in B_0(\eta,\delta),\ (x_1,\Delta_1)\in B_1(x_0,\Delta_0)\}$ and so on. Finally we write for $(\eta,\delta)\in B(P,Q)$

$$g_0(t,\eta,\delta)=1+\min_{\kappa_0^0\leqslant|x|\leqslant\kappa_1^0} f_0(t,x,\eta,\delta),\quad g_0(t)=\min_{(\eta,\delta)\in B(P,Q)} g_0(t,\eta,\delta).$$

For $(\eta,\delta)\in B(P,Q)$ and $(x_0,\Delta_0)\in B_0(\eta,\delta)$ we write

$$G_1(t,x_0,\Delta_0,\eta,\delta)=1+\min_{\kappa_0^1\leqslant|x|\leqslant\kappa_1^1} f_1(t,x,x_0,\Delta_0,\eta,\delta),$$

$$H_1(t,x_0,\eta,\delta)=\min_{\Delta\in A(x_0,\eta,\delta)} G_1(t,x_0,\Delta,\eta,\delta), H_1^1(t,\eta,\delta)=\min_{x_0\in X_0(\eta,\delta)} H_1(t,\eta,\delta),$$

$$g_1(t)=\min_{(\eta,\delta)\in B(P,Q)} H_1^1(t,\eta,\delta),$$

or, what is the same

$$g_1(t)=\min_{f_1\in F_1}\min_{\kappa_0^1\leqslant|x|\leqslant\kappa_1^1} f_1(t,x).$$

Similarly for $j\geqslant 2$

$$g_j(t)=\min_{f_j\in F_j}\min_{\kappa_0^j\leqslant|x|\leqslant\kappa_1^j} f_j(t,x). \tag{1.11}$$

Thus each pair $(\eta,\delta)\in B(P,Q)$ generates the unique chain $\{f_0,f_1,\cdots\}$ when $\operatorname{card} B(P,Q)=\operatorname{card} B_0=\operatorname{card} B_1=\cdots=1$ and the tree with branches $\{f_j,f_j\in F_j\}$ when $\operatorname{card} B_k>1$ for some $k\in N_0$.

Let $j\in N_0, F_j\neq\emptyset$, we say that $F_j\in I=I[\kappa_0^j,\kappa_1^j]$ if $g_j(t)\to\infty$ as $t\to\infty$. If $F_j\in I$ and $f_j\in F_j$ then we say that $f_j\in I$. We will prove that $f_j\in I$ implies $f_{j+1}\in I$ for every $j\in N_0$. First we prove following simple lemma.

LEMMA 1.1. *Let P and Q be as in Lemma 0.2, $Q<P$, and $\{f_j\}$ be the chain, generated by $(\eta,\delta)\in B(P,Q)$. Then $d-l(\eta)\delta\leqslant\chi(f_j,x_j,\Delta)$ for all $x_j\in X_j$ $(j=0,1,\cdots)$ and $\Delta\geqslant 0$.*

Proof. We give the proof only for the case $j=0$, the other cases being similar.

Let, to the contrary, $d-l(\eta)\delta>\chi(f_0,x_0,\Delta_0)$ for some pair $(x_0,\Delta_0):x_0\in X_0,\Delta_0\geqslant 0$ and $x(t)=x_0+t^{-\Delta_0}$, $t\geqslant 0$. By Taylor's formula we obtain for $P[\xi(t,x(t))]=P[t(\eta+t^{-\Delta_0}x(t)\tau)]$

$$|P[\xi(t,x(t))]|=|f_0(t,x(t))|=\left|\sum_{i=0}^{M_0} t^{m_0-\frac{i}{q_0}}\cdot r_i^0(x_0+t^{-\Delta_0})\right| \tag{1.12}$$

$$=\sum_{i=0}^{M_0} t^{m_0-\frac{i}{q_0}-l_i^0(x_0)\Delta_0}\sum_{j\geqslant l_i^0(x_0)}|D^j r_i^0(x_0)|\cdot t^{-(j-l_i^0(x_0))\Delta_0}\leqslant Ct^{\chi(f_0,x_0,\Delta_0)}.$$

Since $D^{l(\eta)}Q(\eta)\neq 0$, $x_0\neq 0$ and $\kappa_0^0\leqslant|x(t)|\leqslant\kappa_1^0$ for sufficiently large t, by Taylor's formula we obtain

$$|Q[\xi(t,x(t))]|=t^{d-l(\eta)\delta}\cdot|D_\tau^{l(\eta)}Q(\eta)|\cdot|x_0|^{l(\eta)}(1+o(1)) \tag{1.13}$$

for $t \to \infty$. Then (1.12)–(1.13) together with the condition $Q < P$ imply

$$t^{d-l(\eta)\delta} \leqslant C|Q[\xi(t,x(t))]| \leqslant C[|f_0(t,x(t))|+1] \leqslant Ct^{\chi(f_0,x_0,\Delta_0)}$$

for sufficiently large t, which contradicts our assumption and proves the lemma. □

THEOREM 1.1. *Let P and Q be as in Lemma 0.2 and $X_0^1(\eta,\delta) = \emptyset$ for all $(\eta,\delta) \in B(P,Q)$. Then $Q < P$ if and only if conditions 1)–2) of Lemma 0.3 for all $\eta \in \Sigma^0$ and condition 3) $d - l(\eta)\delta \leqslant 0$ for all $(\eta,\delta) \in B(P,Q)$ hold.*

Proof. We only need to prove the necessity of condition 3). Simple geometric considerations show that condition $X_0^1(\eta,\delta) = \emptyset$ implies $\chi(f_0,x,\Delta_0) = m_0 - l_0^0(x)\Delta_0 = 0$ for $x \in X_0^1(\eta,\Delta_0)$ and $\Delta_0 = m_0/l_0^0(x)$, which together with Lemma 1.1 proves the necessity of condition 3).

The sufficiency follows by Theorem 0.1. □

Our further efforts therefore will be devoted to obtaining conditions ensuring that $Q < P$ (conditions ensuring the validity of inequality (0.18)) when $d - l(\eta)\delta > 0$ and $X_0^1(\eta,\Delta_0) \neq \emptyset$ for a pair $(\eta,\delta) \in B(P,Q)$.

It is obvious (see Theorem 0.1) that $Q < P$ implies $F_j \in I$ $(j = 0,1,\cdots)$. On the other hand it is clear that if $F_j \in I$ then for each pair $(f_j,x_j) : f_j \in F_j, x_j \in X_j = X_j(f_j)$ there is a number $k_j = k_j(x_j,f_j)$ such that $0 < k_j \leqslant M_j$, $m_j - k_j/(q_0 \cdot q_1 \cdots q_j) > 0$ and

$$r_0^j(x_j) = \cdots = r_{k_j-1}^j(x_j) = 0, \quad r_{k_j}^j(x_j) \neq 0 \quad (j = 0,1,\cdots) \tag{1.14}$$

In addition we prove following simple proposition.

LEMMA 1.2. *Assume that $j \in N_0$.*

1) The inequality

$$g_j(t) \geqslant Ct^{m_j} \quad t \in [0,\infty) \tag{1.15}$$

holds if and only if $X_j(f_j) = \emptyset$ for all $f_j \in F_j$.

2) Let $X_j(f_j) \neq \emptyset$ for $f_j \in F_j$, then the inequality

$$g_j(t) \geqslant Ct^{m_{j+1}} \quad t \in [0,\infty) \tag{1.16}$$

holds if and only if $X_{j+1}(f_{j+1}) = \emptyset$ for all $f_{j+1} \in F_{j+1}$.

3) Let $F_j \in I$ and card $J(f_j) = 2$ *for all $f_j \in F_j$ then $X_{j+1}(f_{j+1}) = \emptyset$ for all $f_{j+1} \in F_{j+1}$.*

Proof. First two statements immediately follow by the definition of the set F_j. Statement 3) in turn follows by Lemma 2.3 of [14]. □

Let $j \in N_0$, $F_j \in I$, $f_j \in F_j$ and the number $k_j = k_j(x_j,f_j)$ be defined by (1.14), then $\chi(f_j,x_j,\overline{\Delta}^j) = m_j - k_j/(q_0 \cdot q_1 \cdots q_j)$ for a number $\overline{\Delta}^j > 0$. Let us write

$$\Delta_j^0 = \Delta_j^0(x_j) = \inf_{\overline{\Delta}^j}\left\{\chi(f_j,x_j,\overline{\Delta}^j) = m_j - \frac{k_j}{q_0 \cdot q_1 \cdots q_j}\right\}. \tag{1.17}$$

By continuity of χ we have $\chi(f_j,x_j,\Delta_j^0) = m_j - k_j/(q_0 \cdot q_1 \cdots q_j)$, i.e. $k_j \in J(f_j,x_j,\Delta_j^0)$.

LEMMA 1.3. *Let $j \in N_0$, $x_j \in X_j$ and $F_j \in I$. Then for each $f_j \in F_j$*
1) $\Delta_j^0 \in A(x_j,f_j)$,
2) $(\Delta_j^0,\infty) \cap A(x_j,f_j) = \emptyset$, $\Delta_j^0 = \max\{\Delta : \Delta \in A(x_j,f_j)\}$
3) $\chi(f_j,x_j,\Delta) \geqslant m_j - \frac{k_j}{q_0 \cdot q_1 \cdots q_j} = \chi(f_j,x_j,\Delta_j^0) : \Delta \in [0,\infty)$.

Proof. For simplification of notations we can assume that $j = 0$ and write $\Delta^0 = \Delta_0^0$, $l_i = l_i^0(x_0)$ $(0 \leqslant i \leqslant M_0)$, $J = J(f_0,x_0,\Delta^0)$.

Since $k_0 \in J$ then to prove statement 1) it is sufficient to prove the existence of a number $i \neq k_0$ $(0 \leqslant i \leqslant M_0)$ such that $i \in J$. Suppose, to the contrary, that

$$m_0 - \frac{i}{q_0} - l_i\Delta^0 < m_0 - \frac{k_0}{q_0} : 1 \leqslant i \leqslant M_0, \quad i \neq k_0.$$

Then there exists a number $\varepsilon \in (0,\Delta^0)$ such that

$$m_0 - \frac{i}{q_0} - l_i(\Delta^0 - \varepsilon) < m_0 - \frac{k_0}{q_0} : 1 \leqslant i \leqslant M_0, \quad i \neq k_0.$$

Since $l_{k_0} = 0$ we have

$$\chi(f_0,x_0,\Delta^0 - \varepsilon) = \max_{0 \leqslant i \leqslant M_0} \left\{ m_0 - \frac{i}{q_0} - l_i(\Delta^0 - \varepsilon) \right\} = m_0 - \frac{k_0}{q_0},$$

which contradicts the definition of the number Δ^0 (see (1.17)) and proves statement 1).

To prove statement 2), notice that for any $\Delta > \Delta^0$ and $i < k_0$

$$m_0 - \frac{i}{q_0} - l_i\Delta < m_0 - \frac{i}{q_0} - l_i\Delta^0 \leqslant \chi(f_0,x_0,\Delta^0) = m_0 - \frac{k_0}{q_0}.$$

For $i > k_0$ we have $m_0 - \frac{i}{q_0} - l_i\Delta \leqslant m_0 - \frac{i}{q_0} < m_0 - \frac{k_0}{q_0}$. This means that $J = \{k_0\}$ i.e. $\Delta \notin A(x_0,f_0)$, which proves the first part of 2). The second part of 2) is obvious. To prove statement 3), notice that by the definition of the numbers k_0 and Δ^0

$$\chi(f_0,x_0,\Delta) \geqslant m_0 - \frac{k_0}{q_0} - l_{k_0}\Delta = m_0 - \frac{k_0}{q_0} = \chi(f_0,x_0,\Delta^0)$$

for any $\Delta \geqslant 0$, which completes the proof of Lemma 1.3. $\square$

In Lemmas 1.4–1.6 below one can take f_j instead of f_0 and f_{j+1} instead of f_1 for any $j \in N_0$.

Let $U(x,\varepsilon)$ be an ε-neighbourhood of $x \in R^1$ and a number $\varepsilon > 0$ be chosen in such a way that $U(x_1,\varepsilon) \cap U(x_2,\varepsilon) = \emptyset$ for any pair $(x_1,x_2) : x_j \in X_0$ $(j = 1,2)$. We set

$$U_\varepsilon(f_0) = [\kappa_0^0, \kappa_1^0] \setminus \bigcup_{x \in X_0} U(x,\varepsilon).$$

LEMMA 1.4. *Let* $\{f_j\}$ *be the chain generated by the pair* $(\eta,\delta)\in B(P,Q)$, $F_0=\{f_0\}\in I$ *and* $\varepsilon>0$ *be chosen as above. Then*

$$\max_{\kappa_0^1\leqslant|y|\leqslant\kappa_1^1} f_1(t,y)/\min_{x\in U_\varepsilon(f_0)} f_0(t,x)\to 0,\quad t\to\infty. \tag{1.18}$$

Proof. By the definitions of set $U_\varepsilon(f_0)$ and the function f_1

$$\min_{x\in U_\varepsilon(f_0)} f_0(t,x)\geqslant Ct^{m_0},\qquad \max_{\kappa_0^1\leqslant|y|\leqslant\kappa_1^1} f_1(t,y)\leqslant C[t^{m_1}+1]: t\in(0,\infty),$$

where $m_1<m_0$, which proves Lemma 1.4. □

Let $x_0\in X_0$, $|x_s|\in U_\varepsilon(f_0)$, $t_s\geqslant 0$ $(s\in N)$ and

$$\rho_s=-\ln|x_s-x_0|/\ln t_s. \tag{1.19}$$

LEMMA 1.5. *Let the assumptions of Lemma 1.4 hold,* $x_s\to x_0$, $t_s\to\infty$ *and* $\rho_s\to\rho\in A(x_0,f_0)$ *as* $s\to\infty$. *Then*

$$H_1(t_s,x_0)\leqslant C[f_0(t_s,x_s)+1]\quad \forall s\in N. \tag{1.20}$$

Proof. Without loss of generality we can assume that $\rho_s\geqslant 0$ for all $s\in N$. Consider the two possibilities : $\rho=0$, $\rho>0$.

In the first case we have for sufficiently large s

$$f_0(t_s,x_s)\geqslant t_s^{m_0}\cdot|r_0^0(x_s)|-C_1t^{m_0-\frac{1}{q_0}},\quad C_1\geqslant 0.$$

By the definition of the number $l_0=l_0^0(x_0)$ it follows that $r_0^0(x)=(x-x_0)^{l_0}\cdot\overline{r}_0^0(x)$, where $\overline{r}_0^0(x_0)\neq 0$. Therefore (see. (1.19))

$$f_0(t_s,x_s)\geqslant C_2t_s^{m_0}|x_s-x_0|^{l_0}-C_1t_s^{m_0-\frac{1}{q_0}}=C_2t_s^{m_0-l_0.\rho_s}-C_1t_s^{m_0-\frac{1}{q_0}},\quad C_2>0.$$

Since in the first case $\rho_s l_0<1/(2q_0)$ for sufficiently large s, we have

$$f_0(t_s,x_s)\geqslant C_3t_s^{m_0-\frac{1}{q_0}},\quad C_3>0 \tag{1.21}$$

for sufficiently large s.

On the other hand by the definition of H_1, k_0 and $\Delta^0=\Delta_0^0$ we have

$$H_1(t_s,x_0)\leqslant G_1(t_s,x_0,\Delta^0)\leqslant C_4t^{\chi(f_0,x_0,\Delta^0)}\leqslant C_4t_s^{m_0-\frac{k_0}{q_0}}\leqslant C_4t_s^{m_0-\frac{1}{q_0}},\quad C_4>0.$$

Combining these inequalities with (1.21) we get (1.20).

In the second case $\rho\in A(x_0,f_0)$, which means that there is a unique number j_0 such that $0\leqslant j_0\leqslant M_0$ and

$$\chi(f_0,x_0,\rho)=m_0-\frac{j_0}{q_0}-l_{j_0}(x_0)\rho>m_0-\frac{j_0}{q_0}-l_j(x_0)\rho: j_0\neq j\in[0,M_0].$$

Then there exists $\varepsilon > 0$ such that the inequality

$$m_0 - \frac{j_0}{q_0} - l_{j_0}(x_0)\rho_s > m_0 - \frac{j_0}{q_0} - l(x_0)\rho_s : j_0 \neq j \in [0, M_0]$$

holds for $s : |\rho_s - \rho| \leqslant \varepsilon$, $|x_s| \in U_\varepsilon(f_0)$.

Arguing as in the first case we get for all $s \in N$

$$t_s^{m_0 - \frac{j_0}{q_0}} |r_{j_0}^0(x_s)| \geqslant C_5 t_s^{m_0 - \frac{j_0}{q_0}} |x_s - x_0|^{l_{j_0}(x_0)} = C_5 t_s^{m_0 - \frac{j_0}{q_0} - l_{j_0}(x_0).\rho_s}, \quad C_5 > 0,$$
$$t_s^{m_0 - \frac{j}{q_0}} |r_{j_0}^0(x_s)| \leqslant C_6 t_s^{m_0 - \frac{j}{q_0} - l_j(x_0)\rho_s}, \quad j_0 \neq j \in [0, M_0], \quad C_6 > 0.$$

Let numbers $\varepsilon > 0$ and $s_0 \in N$ be chosen in such a way that $|\rho_s - \rho| \leqslant \varepsilon$ and $|x_s| \in U_\varepsilon(f_0)$ for $s \geqslant s_0$. Then combining the last three inequalities we obtain for a constant $C_7 > 0$ that

$$f_0(t_s, x_s) \geqslant t_s^{m_0 - \frac{k_0}{q_0}} |r_{j_0}^0(x_s)| - \sum_{j \neq j_0} t_s^{m_0 - \frac{j}{q_0}} |r_{j_0}^0(x_s)| \geqslant C_7 t_s^{m_0 - \frac{j_0}{q_0} - l_{j_0}(x_0)\rho_s}$$
$$= C_7 t^{\chi(f_0, x_0, \rho_s)}, \quad s \geqslant s_0.$$

For the function H_1 similarly

$$H_1(t_s, x_0) \leqslant G_1(t_s, x_0, \Delta^0) \leqslant C_8 t^{\chi(f_0, x_0, \Delta^0)}, \quad C_8 \geqslant 0.$$

Combining the last two inequalities with statement 3) of Lemma 1.3 we get (1.20). Lemma 1.5 is proved. □

LEMMA 1.6. *Let* $(\eta, \delta) \in B(P, Q)$, $k \in N_0$, $F_k = F_k(\eta, \delta) \in I$. *Then there exists* $C > 0$ *such that*

$$C^{-1} g_k(t) \leqslant g_{k+1}(t) \leqslant C g(t), \quad t \geqslant 0. \tag{1.22}$$

Proof. We give the proof only for the case $k = 0$, the cases $k \geqslant 1$ being similar. Let a number t_0 be chosen in such a way that

$$\kappa_0^0 \leqslant |x + t^{-\Delta} y| \leqslant \kappa_1^0 \quad \forall x \in X_0, \quad \Delta \in A(x, f_0), \quad |y| \in [\kappa_0^1, \kappa_1^1], \quad t \geqslant t_0.$$

Then for any pair $(x_0, \Delta) \in B_0$

$$g_0(t) \leqslant 1 + \min_{\kappa_0^0 \leqslant |x| \leqslant \kappa_1^0} f_0(t, x, \eta, \delta) \leqslant 1 + \min_{\kappa_0^0 \leqslant |x| \leqslant \kappa_1^0} f_0(t, x_0 + t^{-\Delta} y)$$
$$= 1 + \min_{\kappa_0^0 \leqslant |x| \leqslant \kappa_1^0} f_0(t, y, x_0, \Delta) = G_1(t, x_0, \Delta),$$

i.e. $g_0(t) \leqslant g_1(t)$ for $t \geqslant t_0$. Since $g_1(t) \geqslant 1$ for all $t \geqslant 0$, this proves the left-hand-side inequality of (1.22).

To prove the right-hand-side inequality of (1.22) suppose, to the contrary, that for a sequence $\{t_s\}$:

$$t_s \to \infty, \quad g_1(t_s)/g_0(t_s) \to \infty, \quad s \to \infty. \tag{1.23}$$

By the compactness of the set $[\kappa_0^0, \kappa_1^0]$ and by the continuity of f_0 it follows that for each $s \in N$ there is a number $x_s : |x_s| \in [\kappa_0^0, \kappa_1^0]$ such that $g_0(t_s) = f_0(t_s, x_s) + 1$. Without loss of generality one may assume that the sequence $\{x_s\}$ is convergent. Let $x_s \to \overline{x}$ as $s \to \infty$ then $|\overline{x}| \in [\kappa_0^0, \kappa_1^0]$. By assumption (1.23) and by Lemma 1.4 $\overline{x} \in X_0$. If for infinite number of $s \in N$ $x_s = \overline{x}$ then by choosing a subsequence one can assume that $x_s = \overline{x}$ for all $s \in N$. Since $F_0 = \{f_0\} \in I$, we get for the number $k_0 = k_0(\overline{x}) \geqslant 1$ defined by (1.14)

$$\begin{aligned} f_0(t_s, x_s) = f_0(t_s, \overline{x}) &\geqslant t_s^{m_0 - \frac{k_0}{q_0}} |r_{k_0}^0(\overline{x})| - \sum_{j>k_0} t_s^{m_0 - \frac{j}{q_0}} |r_j^0(\overline{x})| \\ &\geqslant \frac{1}{2} |r_{k_0}^0(\overline{x})| t_s^{m_0 - \frac{k_0}{q_0}} \end{aligned} \tag{1.24}$$

for sufficiently large s. On the other hand for $s \in N$

$$g_1(t_s) \leqslant H_1(t_s, \overline{x}) \leqslant G_1(t_s, \overline{x}, \Delta^0(\overline{x})) \leqslant C t^{\chi(t_s, \overline{x}, \Delta^0(\overline{x}))} = C t_s^{m_0 - \frac{k_0}{q_0}}, \tag{1.25}$$

where the number $\Delta^0(\overline{x}) = \Delta_0^0(\overline{x})$ is defined by (1.17). This together with (1.24) contradict (1.23).

The case when $x_s \neq \overline{x}$ for sufficiently large s is still to be considered. Let $x_s \neq \overline{x}$ for all $s \in N$ and the numbers $\{\rho_s\}$ be defined by formula (1.19) for $x_0 = \overline{x}$. If for infinite number of $s \in N$ $\rho_s > 2\Delta^0(\overline{x})$ then by choosing a subsequence one can assume that $\rho_s > 2\Delta^0(\overline{x})$ for all $s \in N$. In this case

$$f_0(t_s, x_s) \geqslant t_s^{m_0 - \frac{k_0}{q_0}} |r_{k_0}^0(x_s)| - \sum_{j<k_0} t_s^{m_0 - \frac{j}{q_0}} |r_j^0(x_s)| - \sum_{j>k_0} t_s^{m_0 - \frac{j}{q_0}} |r_j^0(x_s)|. \tag{1.26}$$

Since $r_{k_0}^0(\overline{x}) \neq 0$, we have

$$|r_{k_0}^0(x_s)| \geqslant \frac{1}{2} |r_{k_0}^0(\overline{x})| \tag{1.27}$$

for sufficiently large s.

On the other hand by the definition of the numbers $k_0 = k_0(\overline{x})$ and $\Delta^0(\overline{x})$ and by Taylor's formula we obtain for $j < k_0$ as $s \to \infty$

$$\begin{aligned} t_s^{m_0 - \frac{j}{q_0}} \cdot |r_j^0(x_s)| = t_s^{m_0 - \frac{j}{q_0}} \cdot \left| \sum_{i \geqslant l_j^0(\overline{x})} \frac{(x_s - \overline{x})^i}{i!} D^i r_j^0(\overline{x}) \right| &\leqslant C t_s^{m_0 - \frac{j}{q_0}} (x_s - \overline{x})^{l_j^0(\overline{x})} \\ = C t_s^{m_0 - \frac{j}{q_0} - 2 l_j^0(\overline{x}) \Delta^0(\overline{x})} \leqslant C t_s^{\chi(f_0, \overline{x}, \Delta^0(\overline{x}))} t_s^{-.l_j^0(\overline{x}).\Delta^0(\overline{x})} \\ = o(t_s^{\chi(f_0, \overline{x}, \Delta^0(\overline{x}))}). \end{aligned} \tag{1.28}$$

For $j > k_0$ relation (1.28) is obvious. Combining (1.26)–(1.28) we get (1.24). Similarly we get relation (1.25) which contradicts (1.23).

Thus without loss of generality we can assume that $\rho_s \in [0, 2\Delta^0(\overline{x})]$ for all $s \in N$ and that $\rho_s \to \rho \in [0, 2\Delta^0(\overline{x})]$ as $s \to \infty$. If $\rho \notin A(\overline{x}, f_0)$ then we arrive at a contradiction immediately by Lemma 1.5. Therefore we only need to consider the case $\overline{x} \in X_0$, $\rho \in A(\overline{x}, f_0)$.

Denote $y_s = (x_s - \overline{x})t_s^{\rho}$ or, which is the same, $x_s = \overline{x} + t_s^{-\rho} y_s$ $(s \in N)$. Then for any $\varepsilon > 0$ and $l > 0$

$$|y_s|^l \cdot t_s^{\varepsilon} \to \infty : |y_s|^l \cdot t_s^{-\varepsilon} \to 0, \quad s \to \infty. \tag{1.29}$$

Since for each $i \notin J = J(f_0, \overline{x}, \rho)$ $m_0 - \frac{i}{q_0} - l_i^0(\overline{x})\rho < \chi(f_0, \overline{x}, \rho)$, by Taylor's formula and by the definition of the polynomial r_0^1 we get

$$f_0(t_s x_s) \geqslant t_s^{\chi(f_0,\overline{x},\rho)} \left| \sum_{i \in J} D^{l_i^0(\overline{x})} r_i^0(\overline{x}) y_s^{l_i^0(\overline{x})} \right| - \sum_{i \notin J} t_s^{m_0 - \frac{i}{q_0} - l_i^0(\overline{x})\rho} |D^{l_i^0(\overline{x})} r_i^0(\overline{x})| |y_s|^{l_i^0(\overline{x})}$$

$$= t_s^{\chi(f_0,\overline{x},\rho)} \cdot \left[|r_0^1(y_s)| - \sum_{i \notin J} t_s^{-\varepsilon_i} |D^{l_i^0(\overline{x})} r_i^0(\overline{x})| |y_s|^{l_i^0(\overline{x})} \right], \ (s \in N),$$

where $\varepsilon_i = \chi(f_0, \overline{x}, \rho) - \left[m_0 - \frac{i}{q_0} - l_i^0(\overline{x})\rho \right] > 0$. Hence by (1.29)

$$|f_0(t_s x_s)| \geqslant t_s^{\chi(f_0,\overline{x},\rho)} [|r_0^1(y_s)| + o(1)], s \to \infty. \tag{1.30}$$

Let us consider the following two possibilities: 1) $\rho < \Delta^0(\overline{x})$, 2) $\rho = \Delta^0(\overline{x})$. The first case in turn we divide into the following three subcases (for a subsequence of the sequence $\{y_s\}$, which we alsodenote by $\{y_s\}$):

$$1.1) \ |y_s| \geqslant \kappa_1^1, \quad 1.2) \ |y_s| \leqslant \kappa_0^1, \quad 1.3) \ \kappa_0^1 \leqslant |y_s| \leqslant \kappa_1^1 \quad (s \in N), \tag{1.31}$$

where the numbers $\kappa_j^1 = \kappa_j^1(\overline{x}, \rho)$ $(j = 0, 1)$ are defined by formula (1.8).

Applying Lemma 0.4 we get in subcases 1.1) and 1.2)

$$|r_0^1(y_s)| \geqslant C \cdot \min\{|y_s|^{l_{i_0}^0(\overline{x})}, |y_s|^{l_{i_c}^0(\overline{x})}\} \quad (s \in N), \quad C > 0,$$

where i_0 and i_c are the smallest, the largest respectively, numbers of $J(f_0, \overline{x}, \rho)$. Combining this together with (1.30) we obtain

$$|f_0(t_s x_s)| \geqslant C t_s^{\chi(f_0,\overline{x},\rho)} \cdot \min\{|y_s|^{l_{i_0}^0(\overline{x})}, |y_s|^{l_{i_c}^0}(\overline{x})\} \quad (s \in N).$$

Since in case 1) $\rho < \Delta^0(\overline{x})$ we have $\chi(f_0, \overline{x}, \rho) > \chi(f_0, \overline{x}, \Delta^0(\overline{x}))$. Let a number $\varepsilon > 0$ be chosen in such a way that $\chi(f_0, \overline{x}, \rho) - \varepsilon > \chi(f_0, \overline{x}, \Delta^0(\overline{x}))$. Then applying the last inequality together with (1.29) we obtain

$$\begin{aligned} |f_0(t_s x_s)| &\geqslant C t_s^{\chi(f_0,\overline{x},\rho)-\varepsilon} \cdot \min\{|y_s|^{l_{i_0}^0(\overline{x})}, |y_s|^{l_{i_c}^0(\overline{x})}\}.t_s^{\varepsilon} \\ &\geqslant C t_s^{\chi(f_0,\overline{x},\rho)-\varepsilon} \geqslant C t_s^{\chi(f_0,\overline{x},\Delta^0(\overline{x}))} \quad (s \in N). \end{aligned} \tag{1.32}$$

In the same manner as in the case $\rho_s > 2.\Delta^0(\overline{x})$ ($s \in N$) we get inequality (1.25) in cases 1.1) and 1.2). Then (1.25) together with (1.32) contradict (1.23). In case 1.3) we have for any $s \in N$

$$\begin{aligned} f_0(t_s x_s) &= f_0(t_s, \overline{x} + t_s^{-\rho} y_s) = f_1(t_s, y_s, \overline{x}, \rho) \\ &\geqslant \min_{\kappa_0^1 \leqslant |y_s| \leqslant \kappa_1^1} f_1(t_s, y_s \overline{x}, \rho) = G_1(t_s, \overline{x}, \rho) - 1 \geqslant g_1(t_s) - 1, \end{aligned}$$

which contradicts (1.23).

We also divide case 2) into three subcases (see (1.31)). Proceeding as in case 1), we get contradiction in subcases 2.1) and 2.2). Consider subcase 2.3) $\kappa_0^1 \leqslant |y_s| \leqslant \kappa_1^1$ $(s \in N)$. It is obvious that in this case $k_0 = k_0(\overline{x}) \in J(f_0, \overline{x}, \rho)$. Then $r_0^1(0) \neq 0$ and $|r_0^1(x)| \geqslant C > 0$ for $|x| \leqslant \kappa_0^1$. Applying this, we obtain by (1.30) for sufficiently large s

$$|f_0(t_s x_s)| \geqslant C t_s^{\chi(f_0, \overline{x}, \rho)} = C t_s^{\chi(f_0, \overline{x}, \Delta^0(\overline{x}))} = C t_s^{m_0 - \frac{k_0}{q_0}}, \quad C > 0.$$

This together with (1.25) contradict (1.23). Lemma 1.6 is proved. □

From Lemma 1.6 immediately follows

COROLLARY 1.1. *Let P and Q be as in Lemma 0.2, $(\eta, \delta) \in B(P, Q)$. Then $F_k\ (\eta, \delta) \in I$ if and only if $F_{k+1}\ (\eta, \delta) \in I$ $(k \in N_0)$.*

2. Comparison of polynomials. The main result

Let polynomials P and Q be as above, $f_0(t, x, \eta, \delta)$ be the function, generated by a pair $(\eta, \delta) \in B(P, Q)$ (see (1.3)), the polynomial r_0^0 and the numbers $\kappa_j(P)$ $(j = 0, 1)$ be defined by (0.12)–(0.14).

It is easy to verify that $F_0\ (\eta, \delta) = \{f_0\} \in I$ when $Q < P$ and $d - l(\eta)\delta > 0$. On the other hand (see Lemmas 1.1, 2.1 of [14]) $X_0(f_0, \eta, \delta) = \emptyset$ if $F_0(\eta, \delta) \in I$ and $c(r_0^0, \eta, \delta) \leqslant 2$.

We will show below that the general case of the comparison of the polynomials P and Q, when $X_0(f_0, \eta, \delta) \neq \emptyset$ is reduced to the special case $c(r_0^0, \eta, \delta) \leqslant 3$. Therefore we will first concentrate our attention on the case $c(r_0^0, \eta, \delta) = 3$.

LEMMA 2.1. *Let a pair $(\eta, \delta) \in B(P, Q)$ generate the set $\{F_i = F_i(\eta, \delta)\}$, $f_i \in F_i$ $(i = 0, 1, \cdots)$, $F_0 \in I$, $c(r_0^j) = c(r_0^j, \eta, \delta) = 3$ for some $j \in N_0$, $(x_j, \Delta_j) \in B_j$ and $X_{j+1}(f_{j+1}, x_j, \Delta_j) \neq \emptyset$. Then*

1) $c(f_j) = \operatorname{card} J_j = \operatorname{card} J(f_j, x_j, \Delta_j) = 3$, $0 \in J_j$, $k_j = k_j(x_j) \in J_j$

2) $q_{j+1} = q_{j+1}(x_j, \Delta_j) = 1$

3) $m_{j+1} \leqslant m_j - 2/(q_0 \cdot q_1 \cdots q_j)$

4) $\operatorname{card} A_j(x_j, f_j) = 1$.

Proof. By Corollary 1.1 $f_{j+1} \in I$ hence $r_0^{j+1}(y) \geqslant 0$ for all $y \in R^1$. Since $c(r_0^{j+1}) = c(f_j)$ by the definition of the polynomial r_0^{j+1} ,and $X_{j+1} \neq \emptyset$ by ourassumption hence $c(f_j) \geqslant 3$ by Lemma 0.4. On the other hand since $f_j \in I$ and $c(r_0^j) = 3$ we have $l_0^j = l_0^j(x_j) = 2$ by Lemma 4.1 of [14] (see also formula (2.6) of [17]). By the same lemma $c(f_j) = c(r_0^{j+1}) \leqslant l_0^j + 1 \leqslant c(r_0^j) = 3$, i.e. $c(f_j) = 3$.

Let $J_j = \{i_0, i_1, i_2\}$, $0 \leqslant i_0 < i_1 < i_2 \leqslant M_j$, then for $k = 0, 1, 2$

$$m_j - \frac{i_k}{q_0 \cdot q_1 \cdots q_j} - l_{i_k}^j \Delta_j = \chi(f_j, x_j, \Delta_j) \geqslant m_j - l_0^j \Delta_j = m_j - 2\Delta_j. \tag{2.1}$$

This means that $2 = l_0^j \geqslant l_{i_0}^j > l_{i_1}^j > l_{i_2}^j \geqslant 0$, i.e. $i_0 = 0$, $l_{i_0}^j = 2$, $l_{i_1}^j = 1$, $l_{i_2}^j = 0$. thus $0 = i_0 \in J_j$. On the other hand, since $i_2 \in J_j$ and $l_{i_2}^j = 0$ we have $i_2 \geqslant k_j$

and $m_i - i/(q_0 \cdot q_1 \cdots q_j) - l_i^j \Delta_j < m_k - k_j/(q_0 \cdot q_1 \cdots q_j)$ for all $i > k_j$ hence $i_2 \leqslant k_j$, i.e. $k_j = i_2 \in J_j$, which proves the first statement.

Let us prove the second statement. Since $i_2 = k_j$ and $l_{i_1}^j = 1$ then $m_j - i_1/(q_0 \cdot q_1 \cdots q_j) - \Delta_j = m_j - k_j/(q_0 \cdot q_1 \cdots q_j)$ by (2.1), which means that the number $\Delta_j(q_0 \cdot q_1 \cdots q_j)$ is natural, i.e. $q_{j+1} = 1$, which proves the second statement.

Since $i_2 = k_j \geqslant 2$ we have $m_{j+1} = \chi(f_j, x_j, \Delta_j) = m_j - k_j/(q_0 \cdot q_1 \cdots q_j) \leqslant m_j - 2/(q_0 \cdot q_1 \cdots q_j)$, which proves the third statement.

To prove statement 4) we introduce notation (1.17) and write $\Delta_j^1 = \max\{\Delta : \Delta \geqslant 0,\ m_j - 2\Delta = m_j - \ l_0^j \Delta = \chi(f_j, x_j, \Delta)\}$. Then

$$[0, \Delta_j^1] \cap A_j = \emptyset, \quad \Delta_j^1 \in A_j, \quad (\Delta_j^0, +\infty) \cap A_j = \emptyset, \quad \Delta_j^0 \in A_j. \tag{2.2}$$

It is therefore sufficient to show that $\Delta_j^0 = \Delta_j^1$. By the definition of the number Δ_j^1 $0 \in J_j$ and by the definition of the set A_j there is a number $n \in J(f_j, x_j,\ \Delta_j^1\ ),\ n > 0$, i.e.

$$m_j - 2\Delta_j^1 = m_j - \frac{n}{q_0 \cdot q_1 \cdots q_j} - l_n^j \Delta_j^1 = \chi(f_j, x_j, \Delta_j^1). \tag{2.3}$$

On the other hand since $\Delta^j \in A_j,\ k_j \in J_j$ (see statement 1)) we have

$$\chi(f_j, x_j, \Delta_j^1) = m_j - k_j/(q_0 \cdot q_1 \cdots q_j) = \chi(f_j, x_j, \Delta_j^0). \tag{2.4}$$

Since $\Delta^j \in A_j,\ \Delta_j^0 \in A_j$ hence by (2.2) $\Delta^j \leqslant \Delta_j^0$ and $\chi(f_j, x_j, \Delta_j) > m_j - k_j/(q_0 \cdot q_1 \cdots q_j)$ for $\Delta < \Delta_j^0$ we have $\Delta^j \geqslant \Delta_j^0$ by (2.4), i.e. $\Delta^j = \Delta_j^0$.

By the definition of the number Δ_j^1 $\Delta^j \leqslant \Delta_j^0$, i.e. $\Delta_j^0 = \Delta_j^1$. It remains to prove that $\Delta_j^0 \geqslant \Delta_j^1$. Let, to the contrary, $\Delta_j^0 < \Delta_j^1$ then by (2.3)–(2.4) we obtain

$$\begin{aligned} m_j - 2\Delta_j^0 &= \chi(f_j, x_j, \Delta_j^0) = m_j - k_j/(q_0 \cdot q_1 \cdots q_j) = \chi(f_0, x_j, \Delta_j^1) \\ &= m_j - n/(q_0 \cdot q_1 \cdots q_j) - l_n^j \Delta_j^1 < m_j - n/(q_0 \cdot q_1 \cdots q_j) - l_n^j \Delta_j^0. \end{aligned} \tag{2.5}$$

This means that either $l_n^j = 0$ or $l_n^j = 1$. In the first case $n \geqslant k_j$ by the definition of the number k_j. On the other hand since $n \in J(f_0, x_j, \Delta_j^1)$ and $i \notin J(f_j, x_j, \Delta_j^1)$ for $i > k_j$ then $n \leqslant k_j$, i.e. $n = k_j$. This together with statement 1) of our lemma immediately imply $\Delta_j^1 = \Delta_j = \Delta_j^0$. In the case $l_n^j = 1$ by (2.3) we have $\Delta_j^1 = n/(q_0 \cdot q_1 \cdots q_j)$ and by (2.5) $\Delta_j^0 \geqslant n/(q_0 \cdot q_1 \cdots q_j)$, i.e. $\Delta_j^0 \geqslant \Delta_j^1$, which contradicts our assumption and proves statement 4). Lemma 2.1 is proved. □

For simplification of notations, first we assume that each set $F_i(\eta, \delta)$, generated by a pair $(\eta, \delta) \in B(P, Q)$, consists of a unique function f_i $(i = 0, 1, \cdots)$, i.e. we assume that the pair (η, δ) generates the unique chain $\{f_i\}$. By Corollary 1.1 $f_j \in I$ $(j = 1, 2, \cdots)$ if $f_0 \in I$ and by the definition of the functions $\{f_j\}$

$$\operatorname{ord} f_0 = \chi(P, \eta, \delta) \geqslant \operatorname{ord} f_1 = \chi(f_0, x_0, \Delta_0) \geqslant \operatorname{ord} f_2 = \chi(f_1, x_1, \Delta_1) \geqslant \cdots .$$

If for some $j \in N_0 \operatorname{ord} f_j < d - l(\eta)\delta$ then $Q \not< P$ by Theorem 0.1. It is natural therefore to introduce the following definition.

DEFINITION 2.1. Let $\gamma \in N_0$. We call the chain $\{f_j\}$ γ-finite if either $\operatorname{ord} f_\gamma < d - l(\eta)\delta$ or $X_\gamma = X(f_\gamma) = \emptyset$ or $B_\gamma = B(f_\gamma) = \emptyset$.

LEMMA 2.2. *Let $\{f_j\}$ be the chain, generated by a pair $(\eta,\delta) \in B(P,Q)$, $f_0 \in I$ and $c(r_0^{j_0}) \leqslant 3$ for $j_0 \in N_0$. Then the chain $\{f_j\}$ is γ-finite for any γ, satisfying the inequality $\gamma \leqslant \gamma_{j_0} = [\frac{1}{2} m_0 \cdot q_0 \cdot q_1 \cdots q_{j_0}]$, where $[a]$ denotes the integer part of a.*

Proof. Without loss of generality we can assume that $j_0 = 0$. If either $\operatorname{ord} f_0 < d - l(\eta)\delta$ or $X_0 = \emptyset$ then the chain $\{f_j\}$ is 0-finite. Let $\operatorname{ord} f_0 \geqslant d - l(\eta)\delta$ and $X_0 = \emptyset$ then the chain $\{f_j\}$ is 0-finite. Let $\operatorname{ord} f_0 \geqslant d - l(\eta)\delta$ and $X_0 \neq \emptyset$. If $c(r_0^0) \leqslant 2$ then $X_1 = \emptyset$ by Lemma 1.2 and the chain $\{f_j\}$ is 1-finite.

Let now $c(r_0^0) = 3$. To prove our lemma it is sufficient to prove that $\operatorname{ord} f_\gamma < d - l(\eta)\delta$ or $X_\gamma = \emptyset$ for a number $\gamma \leqslant \gamma_0 = [\frac{1}{2} m_0 \cdot q_0]$. Let, to the contrary, $\operatorname{ord} f_0 \geqslant d - l(\eta).\delta$ and $X_\gamma \neq \emptyset$ for all $\gamma = 0, 1, \cdots, \gamma_0$.

Since $f_\gamma \in I$ for all $\gamma = 0, 1, \cdots$ (see. Corollary 1.1) then $r_0^\gamma(x) \geqslant 0$ for all $\gamma = 0, 1, \cdots$ and $c(r_0^\gamma) \geqslant 3$ for all $\gamma = 0, 1, \cdots, \gamma_0$ by Lemma 1.2

On the other hand by Lemma 4.1 of [14] $3 \leqslant c(r_0^{\gamma_0}) \leqslant c(r_0^{\gamma_0 - 1}) \leqslant \cdots \leqslant c(r_0^0) = 3$, i.e. $c(r_0^\gamma) = 3$ $(\gamma = 0, 1, \cdots, \gamma_0)$.

Now let us note that by Lemma 2.1 and by the definition of the number $\gamma_0 \operatorname{ord} f_{\gamma_0+1} = \chi(f_{\gamma_0}, x_{\gamma_0}, \Delta_{\gamma_0}) \leqslant m_{\gamma_0} - \frac{2}{q_0} \leqslant m_{\gamma_0 - 1} - \frac{2}{q_0} - \frac{2}{q_0} \leqslant \cdots \leqslant m_{\gamma_0} - \frac{2(\gamma_0+1)}{q_0} \leqslant 0$, i.e. $f_{\gamma_0+1} \notin I$, which contradicts the assumption $f_0 \in I$ and proves Lemma 2.2. □

In [14] it is proved that for the chain $\{f_j\}$, $f_0 \in I$ there exists a number $\gamma_1 \in N_0$ for which either $B_{\gamma_1} = \emptyset$ or $c(r_0^{\gamma_1}) \leqslant 3$. This, together with Lemma 2.2, imply

COROLLARY 2.1. *Let $\{f_j\}$ be the chain generated by a pair $(\eta,\delta) \in B(P,Q)$ and $f_0 \in I$, then the chain $\{f_j\}$ is finite.*

Let now $\operatorname{card} B(P,Q) \geqslant 1$ and $(\eta,\delta) \in B(P,Q)$. Then the pair (η,δ) generates the function $f_0 = f_{0,(\eta,\delta)}$ of type (1.3). If $\operatorname{ord} f_0 = \chi(P,\eta,\delta) < d - l(\eta)\delta$ for such a pair then $Q \not\prec P$. Let us assume therefore that $\operatorname{ord} f_0 \geqslant d - l(\eta)\delta$ for all $(\eta,\delta) \in B(P,Q)$. If either $X_0(f_0,\eta,\delta) = \emptyset$ or $B_0(f_0,\eta,\delta) = \emptyset$ then we write $m_0(\eta,\delta) = \operatorname{ord} f_{0,(\eta,\delta)}$.

If $X_0(f_0,\eta,\delta) \neq \emptyset$ and $B_0(f_0,\eta,\delta) \neq \emptyset$ then the function $f_{0,(\eta,\delta)}$ generates the finite set of pairs $(x,\Delta) \in B_0(f_0,\eta,\delta)$ and the set $F_1 = F_1(\eta,\delta)$ of functions of types(1.6)–(1.7).

If $\operatorname{ord} f_1 = \chi(f_0,x,\Delta) < d - l(\eta)\delta$ for some $f_1 \in F_1$ then $Q \not\prec P$ by Lemma 1.1. Let us assume therefore that $\operatorname{ord} f_1 \geqslant d - l(\eta)\delta$ for all $(\eta,\delta) \in B(P,Q)$ and for all $f_1 \in F_1(\eta,\delta)$ and denote

$$m_1(\eta,\delta) = \min_{f_1 \in F_1^0(\eta,\delta)} \operatorname{ord} f_1, \quad (\eta,\delta) \in B(P,Q),$$

where $F_1^0(\eta,\delta)$ is the set of functions $f_2 \in F_2(\eta,\delta)$ for which either $X_1(f_1) = \emptyset$ or $B_1(f_1) = \emptyset$.

If $F_1^1(\eta,\delta) = F_1(\eta,\delta) \backslash F_1^0(\eta,\delta) = \emptyset$ for all $(\eta,\delta) \in B(P,Q)$ then the process terminates. If $F_1^1(\eta,\delta) \neq \emptyset$ for apair $(\eta,\delta) \in B(P,Q)$ then each function $f_1 \in$

$F_1^1(\eta,\delta)$ generates the finite set $F_2(f_1) = \{f_2\}$ of functions of type (1.10). Let us denote

$$F_2(\eta,\delta) = \bigcup_{f_1 \in F_1^1(\eta,\delta)} F_2(f_1), \quad m_2 = \min_{f_2 \in F_2^0(\eta,\delta)} \mathrm{ord} f_2,$$

where $F_2^0(\eta,\delta)$ is the set of functions $f_2 \in F_2(\eta,\delta)$ for which either $X_2(f_2) = \emptyset$ or $B_2(f_2) = \emptyset$.

If $F_2^1(\eta,\delta) = F_2(\eta,\delta) \backslash F_2^0(\eta,\delta) = \emptyset$ for all $(\eta,\delta) \in B(P,Q)$ then the process terminates. If $F_2^1(\eta,\delta) \neq \emptyset$ for a pair $(\eta,\delta) \in B(P,Q)$ then each function $f_2 \in F_2^1(\eta,\delta)$ generates the finite set $F_2(f_1) = \{f_2\}$ etc.

To sum up, the pair (P,Q) generates a tree. The branches of this tree are a finite number of the chains $\{f_0, f_1, \cdots\}$. We have proved above that each such chain is γ-finite. Finally for $(\eta,\delta) \in B(P,Q)$ we denote

$$m(\eta,\delta) = \min_{0 \leqslant k \leqslant \gamma} m_k(\eta,\delta) = \min_{0 \leqslant k \leqslant \gamma} \chi(f_{k-1}, x_{k-1}, \Delta_{k-1}). \tag{2.6}$$

By the definition of the number $\gamma = \gamma\,(\eta,\delta)$ for any $a \in (0,1)$ there exists a constant $c = c(a) > 0$ such that for all $f_k \in F_k^0(\eta,\delta)$ $(k = 0,1,\cdots,\gamma)$

$$c^{-1}[t^{\mathrm{ord} f_k} + 1] \leqslant f_k(t,x) + 1 \leqslant c[t^{\mathrm{ord} f_k} + 1], \quad t > 0, \quad a \leqslant |x| \leqslant a^{-1}. \tag{2.7}$$

THEOREM 2.1. *Let P and Q be as in Lemma 0.2, $(\eta,\delta) \in B(P,Q)$, $\xi(t,x) = \xi(t,x,\eta,\delta) = t(\eta + t^{-\delta} x \cdot \tau)$. Then there exists $C > 0$ such that the inequality*

$$|Q[\xi(t,x)]| \leqslant C[|P[\xi(t,x)]| + 1], \quad t > 0, \quad \kappa_0^0 \leqslant |x| \leqslant \kappa_1^0, \tag{2.8}$$

holds if and only if 1) $f_0 = f_{0,(\eta,\delta)}$ $(t,x) = |P[\xi(t,x)]| \in I$ and 2) $d - l(\eta)\delta \leqslant m(\eta,\delta)$.

Proof. Necessity. Let (2.8) hold. Since $\xi(t,x) \in D_\varepsilon(\eta)$ for any $\varepsilon > 0$ and for sufficiently large t by Lemma 0.1 we have

$$\frac{1}{2}|D_\tau^{l(\eta)} Q(\eta)| \cdot t^{d-l(\eta)\delta} \leqslant |Q[\xi(t,x)]| \leqslant C[|P[\xi(t,x)]| + 1]$$

for all $|x| \in [\,\kappa_0^0, \kappa_1^0]$ and for sufficiently large t, where $D_\tau^{l(\eta)} Q(\eta) \neq 0$, i.e.

$$t^{d-l(\eta)\delta} \leqslant C \cdot g_0(t). \tag{2.9}$$

Since $d - l(\eta)\delta > 0$, this means that $f_0 \in I$, which proves the necessity of statement 1).

Let $\{f_0, f_1, \cdots, f_\gamma\}$ be a chain for which $\mathrm{ord} f_\gamma = m(\eta,\delta)$. Applying Lemma 1.6 and inequalities (2.7) and (2.9) we have

$$t^{d-l(\eta)\delta} \leqslant C \cdot g_0(t) \leqslant C_1 \cdot g_1(t) \leqslant \cdots \leqslant C_\gamma \cdot g_\gamma(t) \leqslant C_{\gamma+1} \cdot [t^{m(\eta,\delta)} + 1]$$

for sufficiently large t, which proves the necessity of statement 2).

Sufficiency. Let $\{f_j\}$ be the chain generated by (η,δ). This chain is γ-finite. Then by Lemma 1.6 and inequality (2.7) we have for some $C'_j > 0$ $(j = 1,2,\cdots,\gamma+1)$

$$g_0(t) \geqslant C'_1 \cdot g_1(t) \geqslant \cdots \geqslant C'_\gamma \cdot g_\gamma(t) \geqslant C'_{\gamma+1} \cdot t^{m(\eta,\delta)}, \quad \forall t > 0. \tag{2.10}$$

On the other hand by Lemma 0.1

$$\max_{\kappa_0^0 \leqslant |x| \leqslant \kappa_1^0} |Q[\xi(t,x)]| \leqslant \frac{3}{2} |D_\tau^{l(\eta)} Q(\eta)| t^{d-l(\eta)\delta}$$

for sufficiently large t. This together with (2.10) and condition 2) prove inequality (2.8). Theorem 2.1 is proved. □

The complete solution of the problem we have set at the beginning of the paper is given by the following main theorem.

THEOREM 2.2. *Let P and Q be as in Lemma 0.2, $d \leqslant d_0, \Sigma_0 \neq \emptyset$. Then $Q < P$ if and only if $\eta \in \Sigma(Q)$ and $\sigma(\eta) \leqslant \sigma_0(\eta)$ for such $\eta \in \Sigma_0$ that $A(\eta,P,Q) = \emptyset$ and conditions 1)–2) of Theorem 2.1 hold for such $(\eta,\delta) \in B(P,Q)$ that $A(\eta,P,Q) \neq \emptyset$.*

Proof. The necessity is proved above (see Theorem 2.1 and Lemma 0.3).

Sufficiency. Let $\varepsilon \in (0,1)$ be arbitrary when $\operatorname{card}\Sigma_0 = 1$ and $\varepsilon \in (0,1)$ be chosen in such a way that $U_\varepsilon(\eta^1) \cap U_\varepsilon(\eta^2) = \emptyset$ for any pair $(\eta^1,\eta^2) : \eta^j \in \Sigma_0$ $(j = 1,2)$ when $\operatorname{card}\Sigma(P_0) > 1$. Here $U_\varepsilon(\eta)$ is an ε-neighbourhood of $\eta \in R^2$. Since the points $\eta \in \Sigma_0$ of the homogeneous polynomial P_0 are isolated, the existence of such $\varepsilon \in (0,1)$ is obvious.

Let for $\eta \in \Sigma_0$ and $\varepsilon \in (0,1)$ the set $G_\varepsilon(\eta)$ is defined as in Introduction then $G_\varepsilon(\eta^1) \cap G_\varepsilon(\eta^2) = \emptyset$ for any pair (η^1,η^2) : $\eta^j \in \Sigma_0$, $(j = 1,2), \eta^1 \neq \eta^2$. Let us denote

$$G_\varepsilon(\Sigma_0) = \bigcup_{\eta\in\Sigma_0} G_\varepsilon(\eta), \quad G_\varepsilon^0(\Sigma_0) = R^2 \backslash G_\varepsilon(\Sigma_0).$$

It is easy to verify that $Q < P$ if and only if $Q <^\eta P$ for all $\eta \in \Sigma(P_0)$ (see Definition 0.1).

Let $\Sigma^0 = \{\eta \in \Sigma_0,\ A(\eta,P,Q) \neq \emptyset\}$, $\Sigma^1 = \Sigma_0 \backslash \Sigma^0$ then $G_\varepsilon(\Sigma_0) = G_\varepsilon(\Sigma^0) \cup G_\varepsilon(\Sigma^1)$.

By Theorem 0.1 $Q <^\eta P$ for all $\eta \in \Sigma^0$ if inequality (2.8) holds for all $\delta \in A(\eta,P,Q)$. By Theorem 2.1 inequality (2.8) holds if the conditions 1)–2) of Theorem 2.1 are satisfied. By Lemma 0.2 and Lemma 0.3 $Q <^\eta P$ for all $\eta \in \Sigma^1$ if condition 3) of our theorem is satisfied. Theorem 2.2 is proved. □

For illustration of results consider following examples

EXAMPLE 2.1. Let $P(\xi) = P(\xi_1,\xi_2) = P_0(\xi) + P_1(\xi) + P_2(\xi) = \xi_1^6(\xi_1^4 + \xi_2^4) - \xi_1^3\xi_2^5 + \xi_2^6$. Then $\operatorname{ord} P = 10$, $\Sigma_0 \equiv \Sigma(P_0) = \{\pm\eta\} = \{(0,\pm 1)\}$, $\tau = (0,1)$, $A(\eta,P) = \{2/3\}$, $\chi(\pm\eta, 2/3, P) = 6$, $q_0 = 3$, $\xi(t,x,\pm\eta,2/3) \equiv t(\pm\eta + t^{-2/3}x\cdot\tau) = (t^{1/3}x, \pm t)$ and

$$f_0(t,x,\pm\eta) \equiv P(t^{1/3}x, \pm t) = t^6 r_0^0(x) + t^{6-8/3} r_8^0(x) \equiv t^6(x^6 - x^3 + 1) + t^{10/3}x^{10},$$

$r_1^0(x) \equiv \cdots \equiv r_7^0(x) \equiv 0$. Since $r_0^0(x) = x^6 - x^3 + 1 \neq 0$ for all $x \in R^1$, $m(\pm\eta, 2/3, P) = 6$.

Let $Q(\xi_1, \xi_2) = \xi_1^3\xi_2^5 + \xi_1^4\xi_2^4 = \xi_1^3\xi_2^4(\xi_1 + \xi_2)$ be a homogeneous polynomial of order 8. Since $\pm\eta \in \Sigma(Q)$, $l = l(\pm\eta, Q) = 3$, and $\operatorname{ord} Q - (2/3)l = 8 - (2/3)3 = 6 \leqslant 6 = m(\pm\eta, 2/3, P)$, by Theorem 2.1. $Q < P$.

On the other hand for homogeneous polynomial $R(\xi_1, \xi_2) = \xi_1\xi_2^6$ also $\pm\eta \in \Sigma(R)$. Since $l = l(\pm\eta, R) = 1$ and $\operatorname{ord} R - (2/3)l = 7 - 2/3 = 19/3 > 6 = m(\pm\eta, 2/3, P)$, by Theorem 2.1. $R \not< P$.

EXAMPLE 2.2. Let $P(\xi) = P(\xi_1, \xi_2) = P_0(\xi) + \cdots + P_5(\xi) = \xi_1^8(\xi_1^8 + \xi_2^8) + \xi_1^8\xi_2^6 - 2\xi_1^4\xi_2^9 - \xi_1^4\xi_2^7 + \xi_2^{10} + \xi_2^6$. Then $\operatorname{ord} P = 16$, $\Sigma_0 = \{\pm\eta\} = \{(0, \pm 1)\}$, $\tau = (1, 0)$, $A(\pm\eta, P) = \{3/4\}$, $q_0 = 4$, $\xi(t, x, \pm\eta, 3/4) = t(\pm\eta + t^{-3/4}x \cdot \tau) = (t^{1/4}x, \pm t)$ and $f_0(t, x, \pm\eta) \equiv P(t^{1/4}x, \pm t) = t^{10}r_0^0(x) + t^{10-8/4}r_8^0(x) + t^{10-16/4}r_{16}^0(x) + t^{10-24/4}r_{24}^0(x) = t^{10}(x^8 \mp 2x^4 + 1) + t^8(x^8 \mp x^4) + t^6 + t^4x^{10}$, $r_1^0(x) \equiv \cdots \equiv r_7^0(x) \equiv r_9^0(x) \equiv \cdots \equiv r_{15}^0(x) \equiv r_{17}^0(x) \equiv \cdots \equiv r_{23}^0(x) \equiv 0$.

Since $X_0^0(f_0) = \{\pm 1\}, A(\pm 1, f_0) = \{2\}, x \equiv \pm 1 + t^{-2}y$, one can easily see that $f_1(t, y, \pm\eta, \pm 1) \equiv f_0(t, \pm 1 + t^{-2}y, \pm\eta) = t^6(16y^2 + 4y + 1) + o(t^6) \equiv t^6 r_0^1(y) + o(t^6)$ as $t \to \infty$. Since $r_0^1(y) \neq 0$ for all $y \in R^1$, $m(\pm\eta, 3/4) = 6 = (\operatorname{ord} f_1)$.

Let $Q(\xi_1, \xi_2) = \xi_1^8(\xi_1^4 + \xi_2^4)$. Then $\pm\eta \in \Sigma(Q)$, $l(\pm\eta) = 8$. Since $\operatorname{ord} Q - (3/4)l = 12 - (3/4)8 = 6 = m(\pm\eta, 3/4)$, by Theorem 2.2. $Q < P$.

On the other hand for homogeneous polynomial $R(\xi_1, \xi_2) = \xi_1^4\xi_2^{10}$ also $\pm\eta \in \Sigma(R)$. Here $l = l(\pm\eta, R) = 10$. Since $\operatorname{ord} R - (3/4)l = 14 - (3/4)10 = \, = 13/2 > 6 = m(\pm\eta, 3/4)$, by Theorem 2.2. $R \not< P$.

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