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NEIGHBORHOODS OF A CERTAIN CLASS OF p–VALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY USING A DIFFERENTIAL OPERATOR

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Abstract. In this present paper, by making use of the familiar concept of neighborhoods of p-valent functions, the author prove coefficient bounds and distortion inequalities, and associated inclusion relations for the (n,δ)-neighborhoods of a class of p-valently analytic functions with negative coefficients, which is defined by means of a certain non-homogeneous Cauchy-Euler differential equation. Relevant connections of some of the results obtained in this paper with those in earlier works are also provided.

1. Introduction and definitions

Let $T(n,p)$ denote the class of functions $f(z)$ *normalized* by

$$f(z)=z^p-\sum_{k=n+p}^{\infty}a_k z^k \quad (a_k\geqslant 0;\ n,p\in\mathbb{N}=\{1,2,3,\ldots\}), \tag{1.1}$$

which are *analytic* and p-valent in the open unit disk

$$U=\{z: z\in\mathbb{C} \text{ and } |z|<1\}.$$

Upon differentiating both sides of (1.1) q times with respect to z, we have

$$f^{(q)}(z)=\frac{p!}{(p-q)!}z^{p-q}-\sum_{k=n+p}^{\infty}\frac{k!}{(k-q)!}a_k z^{k-q} \tag{1.2}$$

$$(n,p\in\mathbb{N};\ q\in\mathbb{N}_0:=\mathbb{N}\cup\{0\};p>q).$$

Following the earlier investigations by Goodman [5], and Ruscheweyh [6], and others including Kamali [3], and Altıntaş et al. ([4] and [12]), and Srivastava and Orhan [7], and Murugusundaramoorthy and Srivastava [16], and Raina and Srivastava [17], (see

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also [10], [13], [14] and [15]), we define the (n,δ)-neighborhood of a function $f^{(q)}(z)$ when $f(z) \in T(n,p)$ by (see, for details, [9, p. 1668])

$$N_{n,p}^{\delta}(f^{(q)};g^{(q)}) := \left\{ g(z) \in T(n,p) : g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \right. \\ \left. \text{and } \sum_{k=n+p}^{\infty} \frac{k!}{(k-q)!} k\,|a_k - b_k| \leqslant \delta \right\}, \tag{1.3}$$

so that, obviously,

$$N_{n,p}^{\delta}(h^{(q)};g^{(q)}) := \left\{ g(z) \in T(n,p) : g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \right. \\ \left. \text{and } \sum_{k=n+p}^{\infty} \frac{k!}{(k-q)!} k\,|b_k| \leqslant \delta \right\}, \tag{1.4}$$

where, and in what follows,

$$h(z) = z^p \quad (p \in \mathbb{N};\ q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}). \tag{1.5}$$

We denote by $S_n^*(p,\alpha)$ and $C_n(p,\alpha)$ the classes of *p-valently starlike functions of order* α in $U(0 \leqslant \alpha < p)$ *and p-valently convex functions of order* α in $U(0 \leqslant \alpha < p)$, respectively. Thus, by the definitions, we have

$$S_n^*(p,\alpha) = \left\{ f \in T(n,p) : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \ \ (z \in U;\ 0 \leqslant \alpha < p) \right\} \tag{1.6}$$

and

$$C_n(p,\alpha) = \left\{ f \in T(n,p) : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \ \ (z \in U;\ 0 \leqslant \alpha < p) \right\}. \tag{1.7}$$

An interesting unification of the function classes $S_n^*(p,\alpha)$ and $C_n(p,\alpha)$ is provided by the class $T_n(p,\alpha,\lambda)$ of functions $f \in T(n,p)$, which also satisfy the following inequality:

$$\operatorname{Re}\left(\frac{zf'(z) + \lambda z^2 f''(z)}{\lambda z f'(z) + (1-\lambda) f(z)}\right) > \alpha, \tag{1.8}$$

$$(z \in U;\ 0 \leqslant \alpha < p;\ 0 \leqslant \lambda \leqslant 1).$$

The class $T_n(p,\alpha,\lambda)$ was investigated by Altıntaş et al. [1]. and (subsequently) by Irmak et al. [11]. In particular, the class $T_n(1,\alpha,\lambda)$ was considered earlier by Altıntaş [18].

Using the Salagean derivative operator [2]; we can write the following equalities for the function $f^{(q)}(z)$ given by (1.2)

$$D^0 f^{(q)}(z) = f^{(q)}(z)$$

$$D^1 f^{(q)}(z) = Df^{(q)}(z) = \frac{z}{p-q}\left(f^{(q)}(z)\right)' = \frac{p!}{(p-q)!}z^{p-q} - \sum_{k=n+p}^{\infty}\frac{k!(k-q)}{(k-q)!(p-q)}a_k z^{k-q}$$

$$D^2 f^{(q)}(z) = D(Df^{(q)}(z)) = \frac{p!}{(p-q)!}z^{p-q} - \sum_{k=n+p}^{\infty}\frac{k!(k-q)^2}{(k-q)!(p-q)^2}a_k z^{k-q}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$D^m f^{(q)}(z) = D(D^{m-1} f^{(q)}(z)) = \frac{p!}{(p-q)!}z^{p-q} - \sum_{k=n+p}^{\infty}\frac{k!(k-q)^m}{(k-q)!(p-q)^m}a_k z^{k-q}.$$

Now, making use of the function $f^{(q)}(z)$ given by (1.2), we introduce a new subclass $T_q^m(n,p,\alpha,\lambda)$ of the p-valently analytic function class $T(n,p)$, which consists of functions $f(z)$ satisfying the following inequality:

$$\operatorname{Re}\left(\frac{(1-\lambda)z\left(\frac{D^m f^{(q)}(z)}{(p-q)^m}\right)' + \lambda z\left(\frac{D^{m+1} f^{(q)}(z)}{(p-q)^{m+1}}\right)'}{(1-\lambda)\left(\frac{D^m f^{(q)}(z)}{(p-q)^m}\right) + \lambda\left(\frac{D^{m+1} f^{(q)}(z)}{(p-q)^{m+1}}\right)}\right) > \alpha \qquad (1.9)$$

$$(0 \leqslant \lambda \leqslant 1;\ 0 \leqslant \alpha < (p-q);\ \ p > q;\ p \in \mathbb{N};\ m,q \in \mathbb{N}_0 = \mathbb{N}\cup\{0\};\ z \in U).$$

Let $A(n)$ be the class of functions of the form:

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \ \ (a_k \geqslant 0;\ n \in \mathbb{N} = \{1,2,...\}),$$

which are analytic in the open unit disk U. Let $S_n^*(\alpha)$ denote the subclass of $A(n)$ consisting of functions which satisfy

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \ \ (z \in U) \qquad (1.10)$$

for some $\alpha(0 \leqslant \alpha < 1)$. A function $f(z)$ in $S_n^*(\alpha)$ is said to be starlike of order α in U. A function $f(z) \in A(n)$ is said to be convex of order α if it satisfies

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \ \ (z \in U) \qquad (1.11)$$

for some $\alpha(0 \leqslant \alpha < 1)$. We denote by $C_n(\alpha)$ the subclass of $A(n)$ consisting of all such functions [4].

Clearly, we have

$$T(n,1) \equiv A(n),\ T_0^0(n,1,\alpha,\lambda) \equiv T_n(1,\alpha,\lambda), T_0^0(n,p,\alpha,0) \equiv S_n^*(p,\alpha),$$
$$T_0^0(n,p,\alpha,1) \equiv C_n(p,\alpha), T_0^0(n,1,\alpha,0) \equiv S_n^*(\alpha) \text{ and } T_0^0(n,1,\alpha,1) \equiv C_n(\alpha)$$

in terms of *simpler* classes

$$S_n^*(p,\alpha), C_n(p,\alpha), T_n(p,\alpha,\lambda), T_q^m(n,p,\alpha,\lambda), S_n^*(\alpha), C_n(\alpha)$$

defined by (1.6), (1.7), (1.8), (1.9), (1.10) and (1.11) (see also [10]).

The main object of the present sequel to the aforementioned recent works is to derive several coefficient bounds and distortion inequalities, and associated inclusion relations for the (n,δ)-neighborhood of functions in the subclass $R_q^m(n,p,\alpha,\lambda;\mu)$ of the class $T(n,p)$, which consists of functions $f \in T(n,p)$ satisfying the following nonhomogeneous Cauchy-Euler differential equation:

$$z^2\frac{d^{2+q}w}{dz^{2+q}} + 2(1+\mu)z\frac{d^{1+q}w}{dz^{1+q}} + \mu(1+\mu)\frac{d^q w}{dz^q} = (p-q+\mu)(p-q+\mu+1)\frac{d^q g}{dz^q}, \quad (1.12)$$
$$\left(w = f(z) \in T(n,p),\ g(z) \in T_q^m(n,p,\alpha,\lambda) \text{ and } \mu > q-p\ (\mu \in \mathbb{R})\right).$$

2. Coefficients bounds and distortion inequalities for the class $T_{\mathbf{q}}^{\mathbf{m}}(n,p,\alpha,\lambda)$

In this section, we prove the following results which yield the coefficient inequalites for functions in the subclass $T_q^m(n,p,\alpha,\lambda)$.

THEOREM 2.1. *A function $f(z) \in T(n,p)$ is in the class $T_q^m(n,p,\alpha,\lambda)$ if and only if*

$$\sum_{k=n+p}^{\infty} \frac{(k-q)^m\,(p-q+\lambda(k-p))\,(k-q-\alpha)k!}{(k-q)!}a_k \leqslant \frac{p!(p-q)^{m+1}(p-q-\alpha)}{(p-q)!}. \quad (2.1)$$
$$\left(\begin{array}{l}\lambda(0\leqslant\lambda\leqslant 1);\ \alpha(0\leqslant\alpha<p-q);\ \ p>q;\ p,n\in\mathbb{N};\ m,q\in\mathbb{N}_0=\mathbb{N}\cup\{0\};\\ (p-q)!\leqslant p!(p-q)^m(p-q-\alpha)\end{array}\right).$$

The result is sharp with extremal function given by

$$f(z) = z^p - \frac{p!(p-q)^{m+1}(p-q-\alpha)(n+p-q)!}{(p-q)!(p-q+\lambda n)(n+p)!(n+p-q)^m(n+p-q-\alpha)}z^{n+p} \quad (2.2)$$
$$(p,n\in\mathbb{N};\ m,q\in\mathbb{N}_0=\mathbb{N}\cup\{0\}).$$

Proof. Suppose that $f(z) \in T_q^m(n,p,\alpha,\lambda)$. Then we find from (1.9) that

$$\operatorname{Re}\left(\frac{\dfrac{p!(p-q)}{(p-q)!}z^{p-q} - \displaystyle\sum_{k=n+p}^{\infty}\frac{k!(k-q)^{m+1}\,(p-q+\lambda(k-p))}{(p-q)^{m+1}(k-q)!}a_k z^{k-p}}{\dfrac{p!}{(p-q)!}z^{p-q} - \displaystyle\sum_{k=n+p}^{\infty}\frac{k!(k-q)^{m}\,(p-q+\lambda(k-p))}{(p-q)^{m+1}(k-q)!}a_k z^{k-p}}\right) > \alpha$$

$$\left(\begin{array}{l}\lambda(0 \leqslant \lambda \leqslant 1);\ \alpha(0 \leqslant \alpha < p-q);\ \ p > q;\ p,n \in \mathbb{N};\ m,q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\\ (p-q)! \leqslant p!(p-q)^m(p-q-\alpha)\end{array}\right).$$

If we choose z to be real and let $z \to 1^-$; we get the following inequality:

$$\operatorname{Re}\left(\frac{\dfrac{p!(p-q)}{(p-q)!} - \sum\limits_{k=n+p}^{\infty} \dfrac{k!(k-q)^{m+1}(p-q+\lambda(k-p))}{(p-q)^{m+1}(k-q)!} a_k}{\dfrac{p!}{(p-q)!} - \sum\limits_{k=n+p}^{\infty} \dfrac{k!(k-q)^{m}(p-q+\lambda(k-p))}{(p-q)^{m+1}(k-q)!} a_k}\right) \geqslant \alpha$$

$$\left(\begin{array}{l}\lambda(0 \leqslant \lambda \leqslant 1);\ \alpha(0 \leqslant \alpha < p-q);\ \ p > q;\ p,n \in \mathbb{N};\ m,q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\\ (p-q)! \leqslant p!(p-q)^m(p-q-\alpha)\end{array}\right)$$

or, equivalently,

$$\sum_{k=n+p}^{\infty} \frac{k!(k-q)^{m+1}(p-q+\lambda(k-p))}{(p-q)^{m+1}(k-q)!} a_k - \alpha \sum_{k=n+p}^{\infty} \frac{k!(k-q)^{m}(p-q+\lambda(k-p))}{(p-q)^{m+1}(k-q)!} a_k \leqslant \frac{p!(p-q-\alpha)}{(p-q)!}.$$

Thus we obtain

$$\sum_{k=n+p}^{\infty} \frac{(p-q+\lambda(k-p))(k-q-\alpha)(k-q)^m k!}{(k-q)!} a_k \leqslant \frac{p!(p-q)^{m+1}(p-q-\alpha)}{(p-q)!}$$

$$\left(\begin{array}{l}\lambda(0 \leqslant \lambda \leqslant 1);\ \alpha(0 \leqslant \alpha < p-q);\ \ p > q;\ p,n \in \mathbb{N};\ m,q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\\ (p-q)! \leqslant p!(p-q)^m(p-q-\alpha)\end{array}\right).$$

Conversely, we assume that the inequality (2.1) holds true and let

$$z \in \partial U = \{z : z \in \mathbb{C},\ \ |z| < 1\}.$$

Then, we find from the definition (1.9) that

$$\left|\frac{(1-\lambda)z\left(\dfrac{D^m f^{(q)}(z)}{(p-q)^m}\right)' + \lambda z\left(\dfrac{D^{m+1} f^{(q)}(z)}{(p-q)^{m+1}}\right)'}{(1-\lambda)\left(\dfrac{D^m f^{(q)}(z)}{(p-q)^m}\right) + \lambda\left(\dfrac{D^{m+1} f^{(q)}(z)}{(p-q)^{m+1}}\right)} - \frac{(p-q)^{m+1}p!(p-q-\alpha)}{(p-q)!}\right|$$

$$= \left|\frac{\dfrac{p!(p-q)}{(p-q)!}z^{p-q} - \sum\limits_{k=n+p}^{\infty} \dfrac{k!(k-q)^{m+1}(p-q+\lambda(k-p))}{(p-q)^{m+1}(k-q)!} a_k z^{k-q}}{\dfrac{p!}{(p-q)!}z^{p-q} - \sum\limits_{k=n+p}^{\infty} \dfrac{k!(k-q)^{m}(p-q+\lambda(k-p))}{(p-q)^{m+1}(k-q)!} a_k z^{k-q}} - \frac{(p-q)^{m+1}p!(p-q-\alpha)}{(p-q)!}\right|$$

$$= \left| \frac{\frac{p!(p-q)^{m+2}\left((p-q)!-p!(p-q)^{m}(p-q-\alpha)\right)}{[(p-q)!]^{2}}z^{p-q} - \sum_{k=n+p}^{\infty}\frac{k!(p-q)!(k-q)^{m+1}-k!p!(k-q)^{m}(p-q)^{m+1}(p-q-\alpha)}{(p-q)!(k-q)!}\Upsilon a_k z^{k-q}}{\frac{(p-q)^{m+1}p!}{(p-q)!}z^{p-q} - \sum_{k=n+p}^{\infty}\frac{k!(k-q)^{m}}{(k-q)!}\Upsilon a_k z^{k-q}} \right|$$

where $\Upsilon = (p-q+\lambda(k-p))$.

From here, we can write the following inequality:

$$\leqslant \frac{\left\{ \begin{array}{l} \frac{\left|p!(p-q)^{m+1}\left((p-q)(p-q)!-p!(p-q)^{m+1}(p-q-\alpha)\right)\right| \left|z^{p-q}\right|}{[(p-q)!]^{2}} \\ + \sum_{k=n+p}^{\infty}\frac{\left|k!(k-q)^{m}\left((p-q)!(k-q)-p!(p-q)^{m+1}(p-q-\alpha)\right)\right| \left|\Upsilon a_k z^{k-q}\right|}{(p-q)!(k-q)!} \end{array} \right\}}{\left|\frac{(p-q)^{m+1}p!}{(p-q)!}z^{p-q}\right| - \sum_{k=n+p}^{\infty}\left|\frac{k!(k-q)^{m}}{(k-q)!}\Upsilon a_k z^{k-q}\right|}$$

$$= \frac{\left\{ \begin{array}{l} \frac{p!(p-q)^{m+1}\left|\left((p-q)(p-q)!-p!(p-q)^{m+1}(p-q-\alpha)\right)\right|}{[(p-q)!]^{2}} \\ + \sum_{k=n+p}^{\infty}\frac{k!(k-q)^{m}\left|\left((p-q)!(k-q)-p!(p-q)^{m+1}(p-q-\alpha)\right)\right| \left|\Upsilon a_k\right|}{(p-q)!(k-q)!} \end{array} \right\}}{\frac{(p-q)^{m+1}p!}{(p-q)!} - \sum_{k=n+p}^{\infty}\frac{k!(k-q)^{m}}{(k-q)!}\Upsilon a_k}$$

$$= \frac{\left\{ \begin{array}{l} \frac{p!(p-q)^{m+1}\left\{-\left((p-q)(p-q)!-p!(p-q)^{m+1}(p-q-\alpha)\right)\right\}}{[(p-q)!]^{2}} \\ + \sum_{k=n+p}^{\infty}\frac{k!(k-q)^{m}\left((p-q)!(k-q)-p!(p-q)^{m+1}(p-q-\alpha)\right)\Upsilon a_k}{(p-q)!(k-q)!} \end{array} \right\}}{\frac{(p-q)^{m+1}p!}{(p-q)!} - \sum_{k=n+p}^{\infty}\frac{k!(k-q)^{m}}{(k-q)!}\Upsilon a_k}$$

$$= \frac{\left\{ \begin{array}{l} \frac{p!(p-q)^{m+1}\left\{\left(-(p-q)(p-q)!+p!(p-q)^{m+1}(p-q-\alpha)\right)\right\}}{[(p-q)!]^{2}} \\ + \sum_{k=n+p}^{\infty}\frac{k!(k-q)^{m}\left((p-q)!(k-p+p-q)-p!(p-q)^{m+1}(p-q-\alpha)\right)\Upsilon a_k}{(p-q)!(k-q)!} \end{array} \right\}}{\frac{(p-q)^{m+1}p!}{(p-q)!} - \sum_{k=n+p}^{\infty}\frac{k!(k-q)^{m}}{(k-q)!}\Upsilon a_k}$$

$$= \frac{\left\{\begin{array}{l} \dfrac{p!(p-q)^{m+1}\left\{\left(-(p-q)(p-q)!+p!(p-q)^{m+1}(p-q-\alpha)\right)\right\}}{[(p-q)!]^2} \\ +\displaystyle\sum_{k=n+p}^{\infty}\frac{k!(k-q)^m\left((p-q)!(p-q)-p!(p-q)^{m+1}(p-q-\alpha)\right)\Upsilon a_k}{(p-q)!(k-q)!} \\ \qquad +\displaystyle\sum_{k=n+p}^{\infty}\frac{k!(k-q)^m(k-p)}{(k-q)!}\Upsilon a_k \end{array}\right\}}{\dfrac{(p-q)^{m+1}p!}{(p-q)!}-\displaystyle\sum_{k=n+p}^{\infty}\frac{k!(k-q)^m}{(k-q)!}\Upsilon a_k}$$

$$= \frac{\left\{\begin{array}{l} \dfrac{p!(p-q)^{m+1}\left(-(p-q)(p-q)!+p!(p-q)^{m+1}(p-q-\alpha)\right)}{[(p-q)!]^2} \\ +\left(\dfrac{(p-q)(p-q)!+p!(p-q)^{m+1}(p-q-\alpha)}{(p-q)!}\right)\displaystyle\sum_{k=n+p}^{\infty}\frac{k!(k-q)^m}{(k-q)!}\Upsilon a_k \\ +\displaystyle\sum_{k=n+p}^{\infty}\frac{k!(k-q)^m(k-\alpha-q+q+\alpha-p)}{(k-q)!}\Upsilon a_k \end{array}\right\}}{\dfrac{(p-q)^{m+1}p!}{(p-q)!}-\displaystyle\sum_{k=n+p}^{\infty}\frac{k!(k-q)^m}{(k-q)!}\Upsilon a_k}$$

$$= \frac{\left\{\dfrac{\left(-(p-q)(p-q)!+p!(p-q)^{m+1}(p-q-\alpha)\right)}{(p-q)!}\left(\dfrac{p!(p-q)^{m+1}}{(p-q)!}-\displaystyle\sum_{k=n+p}^{\infty}\frac{k!(k-q)^m}{(k-q)!}\Upsilon a_k\right)\right\}}{\dfrac{(p-q)^{m+1}p!}{(p-q)!}-\displaystyle\sum_{k=n+p}^{\infty}\frac{k!(k-q)^m}{(k-q)!}\Upsilon a_k}$$

$$+\frac{\displaystyle\sum_{k=n+p}^{\infty}\frac{k!(k-q)^m(k-q-\alpha)}{(k-q)!}\Upsilon a_k+\sum_{k=n+p}^{\infty}\frac{k!(k-q)^m(q-p+\alpha)}{(k-q)!}\Upsilon a_k}{\dfrac{(p-q)^{m+1}p!}{(p-q)!}-\displaystyle\sum_{k=n+p}^{\infty}\frac{k!(k-q)^m}{(k-q)!}\Upsilon a_k}$$

$$= \frac{-(p-q)(p-q)!+p!(p-q)^{m+1}(p-q-\alpha)}{(p-q)!}$$

$$+\frac{(q-p+\alpha)\displaystyle\sum_{k=n+p}^{\infty}\frac{k!(k-q)^m}{(k-q)!}\Upsilon a_k+\sum_{k=n+p}^{\infty}\frac{k!(k-q)^m(k-q-\alpha)}{(k-q)!}\Upsilon a_k}{\dfrac{(p-q)^{m+1}p!}{(p-q)!}-\displaystyle\sum_{k=n+p}^{\infty}\frac{k!(k-q)^m}{(k-q)!}\Upsilon a_k}$$

$$\leqslant \frac{-(p-q)(p-q)!+p!(p-q)^{m+1}(p-q-\alpha)}{(p-q)!}$$

$$+\frac{\dfrac{p!(p-q)^{m+1}(p-q-\alpha)}{(p-q)!}+(q-p+\alpha)\displaystyle\sum_{k=n+p}^{\infty}\frac{k!(k-q)^m}{(k-q)!}\Upsilon a_k}{\dfrac{(p-q)^{m+1}p!}{(p-q)!}-\displaystyle\sum_{k=n+p}^{\infty}\frac{k!(k-q)^m}{(k-q)!}\Upsilon a_k}$$

$$= \frac{-(p-q)(p-q)! + p!(p-q)^{m+1}(p-q-\alpha)}{(p-q)!}$$
$$= \frac{p!(p-q)^{m+1}(p-q-\alpha)}{(p-q)!} - \alpha$$

$$\left(\begin{array}{l} \lambda\,(0 \leqslant \lambda \leqslant 1);\ \alpha(0 \leqslant \alpha < p-q);\ \ p > q;\ p,n \in \mathbb{N};\ m,q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\,; \\ (p-q)! \leqslant p!(p-q)^m(p-q-\alpha) \end{array} \right)$$

provided that the inequality (2.1) is satisfied. Hence, *by the maximum modulus principle,* we infer that

$$f(z) \in T_q^m(n,p,\alpha,\lambda)$$

which evidently completes the proof of Theorem 2.1. □

REMARK 1. In its special case when $q = 0$, Theorem 2.1 yields a result given earlier by Kamali [3, p. 663, Theorem 2.1].

Finally, we note that the assertion (2.1) of Theorem 2.1 is sharp, the extremal function being

$$f(z) = z^p - \frac{p!(p-q)^{m+1}(p-q-\alpha)(n+p-q)!}{(p-q)!(p-q+\lambda n)(n+p)!(n+p-q)^m(n+p-q-\alpha)} z^{n+p}$$

$$(p,n \in \mathbb{N};\ m,q \in \mathbb{N}_0).$$

THEOREM 2.2. *Let the function* $f(z)$ *given by (1.1) be in the class* $T_q^m(n,p,\alpha,\lambda)$. *Then*

$$\sum_{k=n+p}^{\infty} \frac{k!}{(k-q)!} a_k \leqslant \frac{p!(p-q)^{m+1}(p-q-\alpha)}{(p-q)!(n+p-q)^m\,(p-q+\lambda n)\,(n+p-q-\alpha)} \tag{2.3}$$

and

$$\sum_{k=n+p}^{\infty} \frac{k!}{(k-q)!} k a_k \leqslant \frac{p!(p-q)^{m+1}(p-q-\alpha)(n+p)}{(p-q)!(n+p-q)^m(p-q+\lambda n)(n+p-q-\alpha)}. \tag{2.4}$$

Proof. By using Theorem 2.1, we find from inequality (2.1) that

$$(n+p-q)^m(p-q+\lambda n)(n+p-q-\alpha) \sum_{k=n+p}^{\infty} \frac{k!}{(k-q)!} a_k$$
$$\leqslant \sum_{k=n+p}^{\infty} \frac{k!(k-q)^m(k-q-\alpha)\Upsilon}{(k-q)!} a_k \leqslant \frac{p!(p-q)^{m+1}(p-q-\alpha)}{(p-q)!}$$

which immediately yields the first assertion (2.3) of Theorem 2.2.

On the other hand, by appealing to (2.1), we also have

$$(n+p-q)^m(p-q+\lambda n)\sum_{k=n+p}^{\infty}\frac{k!}{(k-q)!}ka_k$$
$$-(q+\alpha)(n+p-q)^m(p-q+\lambda n)\sum_{k=n+p}^{\infty}\frac{k!}{(k-q)!}a_k$$
$$\leqslant \frac{p!(p-q)^{m+1}(p-q-\alpha)}{(p-q)!}$$

that is,

$$(n+p-q)^m(p-q+\lambda n)\sum_{k=n+p}^{\infty}\frac{k!}{(k-q)!}ka_k$$
$$\leqslant \frac{p!(p-q)^{m+1}(p-q-\alpha)}{(p-q)!}+(q+\alpha)(p-q+\lambda n)(n+p-q)^m\sum_{k=n+p}^{\infty}\frac{k!}{(k-q)!}a_k$$

which, in view of the coefficient inequality (2.3) can be put in the form:

$$(p-q+\lambda n)(n+p-q)^m\sum_{k=n+p}^{\infty}\frac{k!}{(k-q)!}ka_k \leqslant \frac{p!(p-q)^{m+1}(p-q-\alpha)(n+p)}{(p-q)!(n+p-q-\alpha)}$$

or, equivalently,

$$\sum_{k=n+p}^{\infty}\frac{k!}{(k-q)!}ka_k \leqslant \frac{p!(p-q)^{m+1}(p-q-\alpha)(n+p)}{(p-q)!(p-q+\lambda n)(n+p-q-\alpha)(n+p-q)^m}.$$

REMARK 2. In its special case when $q=0$, Theorem 2.2 yields a result given earlier by Kamali [3, p. 666, Theorem 2.2].

Our main distortion inequalities for functions in the class $R_q^m(n,p,\alpha,\lambda;\mu)$ are given by Theorem 2.3 below.

THEOREM 2.3. *If* $f(z)\in T(n,p)$ *is in the class* $R_q^m(n,p,\alpha,\lambda;\mu)$, *then*

$$|f(z)| \leqslant |z|^p+\frac{p!(p-q)^{m+1}(p-q-\alpha)(n+p-q)!(p-q+\mu)(p-q+\mu+1)}{(n+p)!(p-q)!(p-q+\lambda n)(n+p-q)^m(n+p-q-\alpha)(n+p-q+\mu)}|z|^{n+p} \tag{2.5}$$

$$|f(z)| \geqslant |z|^p-\frac{p!(p-q)^{m+1}(p-q-\alpha)(n+p-q)!(p-q+\mu)(p-q+\mu+1)}{(n+p)!(p-q)!(p-q+\lambda n)(n+p-q)^m(n+p-q-\alpha)(n+p-q+\mu)}|z|^{n+p} \tag{2.6}$$

and (in general)

$$\left|f^{(t)}(z)\right| \leqslant \frac{p!}{(p-t)!}|z|^{p-t} \tag{2.7}$$
$$+\frac{p!(p-q+\mu)(p-q+\mu+1)(p-q)^{m+1}(p-q-\alpha)(n+p-q)!}{(n+p-t)!(p-q)!(p-q+\lambda n)(n+p-q)^m(n+p-q-\alpha)(n+p-q+\mu)}|z|^{n+p-t}$$

$$\left|f^{(t)}(z)\right| \geqslant \frac{p!}{(p-t)!}\,|z|^{p-t} - \frac{p!(p-q+\mu)(p-q+\mu+1)(p-q)^{m+1}(p-q-\alpha)(n+p-q)!}{(n+p-t)!(p-q)!(p-q+\lambda n)(n+p-q)^m(n+p-q-\alpha)(n+p-q+\mu)}\,|z|^{n+p-t}. \tag{2.8}$$

Proof. Suppose that $f(z) \in T(n,p)$ is given by (1.1). Also let the function $g(z) \in T_q^m(n,p,\alpha,\lambda)$ occurring in the nonhomogeneous differential equation (1.12), be given as in the definitions (1.3) or (1.4) with, of course,

$$b_k \geqslant 0 \quad (k = n+p, n+p+1, n+p+2, ...).$$

Then we readily see from (1.12) that

$$a_k = \frac{(p-q+\mu)(p-q+\mu+1)}{(k-q+\mu)(k-q+\mu+1)} b_k \quad (k = n+p, n+p+1, n+p+2, ...) \tag{2.9}$$

so that

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k = z^p - \sum_{k=n+p}^{\infty} \frac{(p-q+\mu)(p-q+\mu+1)}{(k-q+\mu)(k-q+\mu+1)} b_k z^k \tag{2.10}$$

and

$$|f(z)| \leqslant |z|^p + |z|^{n+p} \sum_{k=n+p}^{\infty} \frac{(p-q+\mu)(p-q+\mu+1)}{(k-q+\mu+1)(k-q+\mu)} b_k. \tag{2.11}$$

Next, since $g(z) \in T_q^m(n,p,\alpha,\lambda)$ the first assertion (2.3) of Theorem 2.2 yields the coefficient inequality:

$$b_k \leqslant \frac{p!(p-q)^{m+1}(p-q-\alpha)(n+p-q)!}{(n+p)!(p-q)!(p-q+\lambda n)(n+p-q)^m(n+p-q-\alpha)} \quad (k = n+p, n+p+1, ...). \tag{2.12}$$

which, in conjunction with (2.10), yields

$$|f(z)| \leqslant |z|^p + \left(\frac{p!(p-q)^{m+1}(p-q-\alpha)(n+p-q)!(p-q+\mu)(p-q+\mu+1)}{(n+p)!(p-q)!(p-q+\lambda n)(n+p-q)^m(n+p-q-\alpha)}\,|z|^{n+p}\right) \times \times \left(\sum_{k=n+p}^{\infty} \frac{1}{(k-q+\mu)(k-q+\mu+1)}\right). \tag{2.13}$$

Finally, in view of the following *telescopic* sum:

$$\sum_{k=n+p}^{\infty}\frac{1}{(k-q+\mu)(1+k-q+\mu)}=\sum_{k=n+p}^{\infty}\left(\frac{1}{k-q+\mu}-\frac{1}{1+k-q+\mu}\right)$$
$$=\lim_{\xi\to\infty}\sum_{k=n+p}^{\xi}\left(\frac{1}{k-q+\mu}-\frac{1}{1+k-q+\mu}\right)$$
$$=\lim_{\xi\to\infty}\left(\frac{1}{n+p-q+\mu}-\frac{1}{1+\xi-q+\mu}\right)$$
$$=\frac{1}{n+p-q+\mu} \tag{2.14}$$
$$(\mu\in\mathbb{R}-\{-n-p,-n-p-1,...\})$$

thefirst assertion (2.5) of Theorem 2.3 follows at once from (2.13).
Similarly, we can write

$$|f(z)|\geqslant|z|^p-|z|^{n+p}\sum_{k=n+p}^{\infty}\frac{(p-q+\mu)(p-q+\mu+1)}{(k-q+\mu)(k-q+\mu+1)}b_k. \tag{2.15}$$

Since $g(z)\in T_q^m(n,p,\alpha,\lambda)$, the first assertion (2.3) of Theorem 2.2 yields the coefficient inequality:

$$b_k\leqslant\frac{p!(p-q)^{m+1}(p-q-\alpha)(n+p-q)!}{(n+p-q)^m(n+p)!(p-q)!(p-q+\lambda n)(n+p-q-\alpha)}$$
$$(k=n+p,n+p+1,...) \tag{2.16}$$

which, in conjunction with (2.15), yields

$$|f(z)|\geqslant|z|^p-\left(\frac{p!(p-q)^{m+1}(p-q-\alpha)(n+p-q)!(p-q+\mu)(p-q+\mu+1)}{(n+p-q)^m(n+p)!(p-q)!(p-q+\lambda n)(n+p-q-\alpha)}|z|^{n+p}\right)\times$$
$$\times\left(\sum_{k=n+p}^{\infty}\frac{1}{(k-q+\mu)(k-q+\mu+1)}\right). \tag{2.17}$$

Finally, in view of the following *telescopic* sum:

$$\sum_{k=n+p}^{\infty}\frac{1}{(k-q+\mu)(1+k-q+\mu)}=\sum_{k=n+p}^{\infty}\left(\frac{1}{k-q+\mu}-\frac{1}{1+k-q+\mu}\right)$$
$$=\lim_{\xi\to\infty}\sum_{k=n+p}^{\xi}\left(\frac{1}{k-q+\mu}-\frac{1}{1+k-q+\mu}\right)$$
$$=\lim_{\xi\to\infty}\left(\frac{1}{n+p-q+\mu}-\frac{1}{1+\xi-q+\mu}\right)$$
$$=\frac{1}{n+p-q+\mu}$$
$$(\mu\in\mathbb{R}-\{-n-p,-n-p-1,...\})$$

the second assertion (2.6) of Theorem 2.3 follows at once from (2.17).

By setting $q = 0$ in inequalities (2.5) and (2.6), (or $q = 0$ and $t = 0$ in inequalities (2.7) and (2.8)) of Theorem 2.3, we arrive at Corollary 2.1.

COROLLARY 2.1. *If* $f(z) \in R_0^m(n,p,\alpha,\lambda;\mu)$, *then*

$$\left||f(z)| - |z|^p\right| \leqslant \frac{p^{m+1}(p-\alpha)(p+\mu)(p+\mu+1)}{(n+p)^m(p+\lambda n)(n+p-\alpha)(n+p+\mu)}|z|^{n+p} \quad (z \in U).$$

By letting $q = 0$ *and* $t = 1$ *in inequalities (2.7) and (2.8) of Theorem 2.3, we arrive at Corollary 2.2.*

COROLLARY 2.2. *If* $f(z) \in R_0^m(n,p,\alpha,\lambda;\mu)$, *then*

$$\left||f'(z)| - p|z|^{p-1}\right| \leqslant \frac{p^{m+1}(p-\alpha)(p+\mu)(p+\mu+1)}{(n+p)^{m-1}(p+\lambda n)(n+p-\alpha)(n+p+\mu)}|z|^{n+p-1} \quad (z \in U).$$

By setting $m = 0$, $\lambda = 0$, $q = 0$ *in Theorem 2.3, we arrive at Corollary 2.3.*

COROLLARY 2.3. *(see [3,9]). If the functions* f *and* g *satisfy the nonhomogeneous Cauchy-Euler diffential equation (1.12) with* $g \in S_n^*(p,\alpha)$, *then*

$$|z|^p - \frac{(p-\alpha)(p+\mu)(p+\mu+1)}{(n+p-\alpha)(n+p+\mu)}|z|^{n+p}$$
$$\leqslant |f(z)| \leqslant |z|^p + \frac{(p-\alpha)(p+\mu)(p+\mu+1)}{(n+p-\alpha)(n+p+\mu)}|z|^{n+p} \quad (z \in U).$$

By setting $m = 0$, $\lambda = 1$, $q = 0$ *in Theorem 2.3. We arrive at Corollary 2.4.*

COROLLARY 2.4. *(see [3,9]). If the functions* f *and* g *satisfy the nonhomogeneous Cauchy-Euler diffential equation (1.12) with* $g \in C_n(p,\alpha)$, *then*

$$|z|^p - \frac{p(p-\alpha)(p+\mu)(p+\mu+1)}{(n+p)(n+p-\alpha)(n+p+\mu)}|z|^{n+p}$$
$$\leqslant |f(z)| \leqslant |z|^p + \frac{p(p-\alpha)(p+\mu)(p+\mu+1)}{(n+p)(n+p-\alpha)(n+p+\mu)}|z|^{n+p} \quad (z \in U).$$

3. Neighborhoods for the classes $T_{\mathbf{q}}^{\mathbf{m}}(n,p,\alpha,\lambda)$ and $R_{\mathbf{q}}^{\mathbf{m}}(n,p,\alpha,\lambda;\mu)$

In this last section, we establish several inclusion relations for the normalized p-valently analytic functions classes $T_q^m(n,p,\alpha,\lambda)$ and $R_q^m(n,p,\alpha,\lambda;\mu)$ involving with the (n,δ)-neighborhood defined by (1.3) and (1.4).

THEOREM 3.1. *Let the function* $f(z) \in T(n,p)$ *be in the class* $T_q^m(n,p,\alpha,\lambda)$. *Then,*

$$T_q^m(n,p,\alpha,\lambda) \subset N_{n,p}^{\delta}(h^{(q)};f^{(q)}), \tag{3.1}$$

where $h(z)$ *is given by (1.5) and the parameter* δ *is given by*

$$\delta := \frac{p!(p-q)^{m+1}(p-q-\alpha)(n+p)}{(p-q)!(n+p-q)^m(n+p-q-\alpha)(p-q+\lambda n)}. \tag{3.2}$$

Proof. Assertion (3.1) would follow easily from the definition of $N_{n,p}^{\delta}(h^{(q)};f^{(q)})$, which is given by (1.5) with $g(z)$ replaced by $f(z)$ and the second assertion (2.4) of Theorem 2.2. □

THEOREM 3.2. *Let the function* $f(z) \in T(n,p)$ *be in the class* $R_q^m(n,p,\alpha,\lambda;\mu)$. *Then*

$$R_q^m(n,p,\alpha,\lambda;\mu) \subset N_{n,p}^{\delta}(g^{(q)};f^{(q)}), \tag{3.3}$$

where the function $g(z)$ *is given by (1.12) and the parameter* δ *is given by*

$$\delta := \frac{p!(p-q)^{m+1}(p-q-\alpha)(n+p)\,(n+(p-q+\mu)(p-q+\mu+2))}{(p-q)!(n+p-q-\alpha)(n+p-q)^m(p-q+\lambda n)(n+p-q+\mu)}. \tag{3.4}$$

Proof. Suppose that $f \in R_q^m(n,p,\alpha,\lambda;\mu)$. Then, upon substituting from (2.9) into the following coefficient inequality:

$$\sum_{k=n+p}^{\infty} \frac{k!}{(k-q)!}k\,|b_k - a_k| \leqslant \sum_{k=n+p}^{\infty} \frac{k!}{(k-q)!}kb_k + \sum_{k=n+p}^{\infty} \frac{k!}{(k-q)!}ka_k \quad (a_k \geqslant 0;\ b_k \geqslant 0), \tag{3.5}$$

we get

$$\begin{aligned}&\sum_{k=n+p}^{\infty} \frac{k!}{(k-q)!}k\,|b_k - a_k| \\ &\quad \leqslant \sum_{k=n+p}^{\infty} \frac{k!}{(k-q)!}kb_k + \sum_{k=n+p}^{\infty} \frac{(p-q+\mu)(p-q+\mu+1)}{(k-q+\mu)(1+k-q+\mu)}\frac{k!}{(k-q)!}kb_k.\end{aligned} \tag{3.6}$$

Next, since $g(z) \in T_q^m(n,p,\alpha,\lambda)$, the second assertion (2.4) of Theorem 2.2 yields that

$$\frac{k!}{(k-q)!}kb_k \leqslant \frac{p!(p-q)^{m+1}(p-q-\alpha)(n+p)}{(n+p-q)^m(p-q)!(n+p-q-\alpha)(p-q+\lambda n)} \quad (k = n+p, n+p+1, \ldots). \tag{3.7}$$

Finally, by making use of (2.4) as well as (3.7) on the right hand side of (3.6), we find that

$$\sum_{k=n+p}^{\infty} \frac{k!}{(k-q)!}k\,|b_k - a_k| \leqslant \frac{p!(p-q)^{m+1}(p-q-\alpha)(n+p)}{(p-q)!(n+p-q-\alpha)(n+p-q)^m(p-q+\lambda n)} \left(1 + \sum_{k=n+p}^{\infty} \frac{(p-q+\mu)(p-q+\mu+1)}{(k-q+\mu)(1+k-q+\mu)}\right) \tag{3.8}$$

which by virtue of the *telescopic* sum (2.14), immediately yields

$$\sum_{k=n+p}^{\infty} \frac{k!}{(k-q)!} k\,|b_k - a_k| \leqslant \frac{p!(p-q)^{m+1}(p-q-\alpha)(n+p)\,(n+(p-q+\mu)(p-q+\mu+2))}{(n+p-q)^m(p-q)!(n+p-q-\alpha)(p-q+\lambda n)(n+p-q+\mu)} =: \delta.$$

Thus, by definition (1.3) with $g(z)$ interchanged by $f(z), f \in N^{\delta}_{n,p}(g^{(q)}, f^{(q)})$. This evidently completes the proof of Theorem 3.2.

By putting $q = 0$ in Theorem 3.1, we have the following Corollary 3.1.

COROLLARY 3.1. *(see [3]). If the function* $f(z) \in T(n,p)$ *is in the class* $R^m_0(n,p,\alpha,\lambda;\mu)$. *Then*

$$R^m_q(n,p,\alpha,\lambda;\mu) \subset N^{\delta}_{n,p}(g;f),$$

where $g(z)$ *is given by (1.12) and the parameter* δ *is given by*

$$\delta := \frac{p^{m+1}(p-\alpha)}{(n+p)^{m-1}(n+p-\alpha)(p+\lambda n)}.$$

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