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COEFFICIENT INEQUALITIES FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS AND THEIR APPLICATIONS INVOLVING THE OWA–SRIVASTAVA OPERATOR OF FRACTIONAL CALCULUS

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Abstract. The purpose of the present paper is to derive several Fekete-Szegö type coefficient inequalities for certain subclasses of normalized analytic functions $f(z)$ defined in the open unit disk. Various applications of our main results involving (for example) the Owa-Srivastava operator of fractional calculus are also considered. Thus, as one of these applications of our result, we obtain the Fekete-Szegö type inequality for a class of normalized analytic functions, which is defined here by means of the Hadamard product (or convolution) and the Owa-Srivastava operator.

1. Introduction and Definitions

Let $\mathscr{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\Delta = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}. \tag{1.2}$$

Also let $\mathscr{S}$ be the subclass of $\mathscr{A}$ consisting of all univalent functions in Δ.

For functions f and g, analytic in Δ, we say that the function f is subordinate to g if there exists a Schwarz function $w(z)$, analytic in Δ with

$$w(0) = 0 \qquad \text{and} \qquad |w(z)| < 1 \quad (z \in \Delta),$$

such that

$$f(z) = g\big(w(z)\big) \qquad (z \in \Delta).$$

We denote this subordination by

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z) \quad (z \in \Delta).$$

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In particular, if the function g is univalent in Δ, the above subordination is equivalent to

$$f(0) = g(0) \qquad \text{and} \qquad f(\Delta) \subset g(\Delta).$$

Let $\phi(z)$ be an analytic function in Δ with

$$\phi(0) = 1, \ \ \phi'(0) > 0 \qquad \text{and} \qquad \Re\big(\phi(z)\big) > 0 \qquad (z \in \Delta),$$

which maps the open unit disk Δ onto a region starlike with respect to 1 and is symmetric with respect to the real axis. Then, by $\mathscr{S}^*(\phi)$ and $\mathscr{C}(\phi)$, respectively, we denote the subclasses of the normalized analytic function class $\mathscr{A}$, which satisfy the following subordination relations:

$$\frac{zf'(z)}{f(z)} \prec \phi(z) \quad (z \in \Delta) \qquad \text{and} \qquad 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z) \quad (z \in \Delta).$$

These function classes were introduced and studied by Ma and Minda [3]. In their particular case when

$$\phi(z) = \frac{1 + (1-2\alpha)z}{1-z} \qquad (z \in \Delta;\ 0 \leqq \alpha < 1),$$

these function classes would reduce, respectively, to the well-known classes $\mathscr{S}^*(\alpha)$ $(0 \leqq \alpha < 1)$ of starlike functions of order α in Δ and $\mathscr{C}(\alpha)$ $(0 \leqq \alpha < 1)$ of convex functions of order α in Δ. In the work by Ma and Minda [3], the Fekete-Szegö inequality for functions in the class $\mathscr{C}(\phi)$ was derived and, in view of the Alexander result relating the function classes $\mathscr{S}^*(\phi)$ and $\mathscr{C}(\phi)$, the Fekete-Szegö inequality for functions in the class $\mathscr{S}^*(\phi)$ was also deduced. For a brief history of the Fekete-Szegö problem for the starlike, convex, and various other subclasses of the normalized analytic function class $\mathscr{A}$, we refer the interested reader to the recent work by Srivastava *et al.* [13].

Motivated essentially by the aforementioned works, we prove the Fekete-Szegö type coefficient inequalities in Theorem 1 below for a more general class of normalized analytic functions which we introduce here in Definition 1. We also give several applications of our main results to certain interesting function classes which are defined by means of the Hadamard product (or convolution) and the Owa-Srivastava operator of fractional calculus (see Section 3). Some of the results obtained in this paper would generalize the results given in several earlier works (see, for example, [3], [8] and [12]).

We begin by introducing the following unification of the function classes $\mathscr{S}^*(\phi)$ and $\mathscr{C}(\phi)$:

DEFINITION 1. Let $\phi(z)$ be a univalent starlike function with respect to 1, which maps the open unit disk Δ onto a region in the right half-plane and is symmetric with respect to the real axis, with

$$\phi(0) = 1 \qquad \text{and} \qquad \phi'(0) > 0.$$

A function $f \in \mathscr{A}$ is said to be in the class $\mathscr{M}_{\alpha,\beta,\lambda}(\phi)$ if

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha} \left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]^{\beta} \prec \phi(z)$$

$$(0 < \beta \leqq 1;\ \ 0 \leqq \alpha \leqq 1;\ \ 0 \leqq \lambda \leqq 1).$$

We note that (see [3])

$$\mathscr{M}_{0,1,0}(\phi) \equiv \mathscr{S}^*(\phi) \qquad \text{and} \qquad \mathscr{M}_{0,\,1,\,1}(\phi) \equiv \mathscr{C}(\phi).$$

In order to prove one of our main results, we need the following lemma.

LEMMA 1. [3]. *If*

$$p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$$

is a function with positive real part in Δ, *then*

$$|c_2 - v c_1^2| \leqq \begin{cases} -4v + 2 & (v \leqq 0) \\ 2 & (0 \leqq v \leqq 1) \\ 4v - 2 & (v \geqq 1). \end{cases}$$

When $v < 0$ *or* $v > 1$, *the equality holds true if and only if* $p_1(z)$ *is* $\dfrac{1+z}{1-z}$ *or one of its rotations. If* $0 < v < 1$, *then the equality holds true if and only if* $p_1(z)$ *is* $\dfrac{1+z^2}{1-z^2}$ *or one of its rotations. If* $v = 0$, *the equality holds true if and only if*

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\eta\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\eta\right)\frac{1-z}{1+z} \qquad (0 \leqq \eta \leqq 1)$$

or one of its rotations. If $v = 1$, *the equality holds true if and only if* $p_1(z)$ *is the reciprocal of one of the functions such that the equality holds true in the case when* $v = 0$.

Although the above upper bound is sharp, in the case when $0 < v < 1$, it can be *further* improved as follows:

$$|c_2 - v c_1^2| + v|c_1|^2 \leqq 2 \qquad \left(0 < v \leqq \frac{1}{2}\right)$$

and

$$|c_2 - v c_1^2| + (1-v)|c_1|^2 \leqq 2 \qquad \left(\frac{1}{2} < v \leqq 1\right).$$

We also need the following result in our investigation.

LEMMA 2. [9]. *If*

$$p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$$

is a function with positive real part in Δ, *then*

$$|c_2 - v c_1^2| \leqq 2 \cdot \max\{1, |2v - 1|\}.$$

The result is sharp for the functions $p_1(z)$ *given by*

$$p_1(z) = \frac{1+z^2}{1-z^2} \qquad \textit{and} \qquad p_1(z) = \frac{1+z}{1-z}.$$

2. Fekete-Szegö Problem for the Function Class $\mathscr{M}_{\alpha,\beta,\lambda}(\phi)$

By making use of Lemma 1, we first prove the Fekete-Szegö type inequalities asserted by Theorem 1 below.

THEOREM 1. *Let*

$$0 \leqq \mu \leqq 1,\ 0 \leqq \alpha \leqq 1,\ 0 < \beta \leqq 1 \quad \text{and} \quad 0 \leqq \lambda \leqq 1.$$

Also let

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots,$$

where the coefficients B_n are real with

$$B_1 > 0 \qquad \text{and} \qquad B_2 \geqq 0.$$

If $f(z)$ given by (1.1) belongs to the function class $\mathscr{M}_{\alpha,\beta,\lambda}(\phi)$, then

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \dfrac{1}{4\xi}\left(2B_2 - \dfrac{B_1^2}{\rho^2}\,\gamma\right) & (\mu \leqq \sigma_1) \\ \dfrac{B_1}{2\xi} & (\sigma_1 \leqq \mu \leqq \sigma_2) \\ \dfrac{1}{4\xi}\left(-2B_2 + \dfrac{B_1^2}{\rho^2}\,\gamma\right) & (\mu \geqq \sigma_2), \end{cases}$$

where, for convenience,

$$\sigma_1 := \frac{2\rho^2(B_2 - B_1) - (\rho^2 - 3\tau)B_1^2}{4\xi B_1^2},$$

$$\sigma_2 := \frac{2\rho^2(B_2 + B_1) - (\rho^2 - 3\tau)B_1^2}{4\xi B_1^2},$$

$$\sigma_3 := \frac{2\rho^2 B_2 - (\rho^2 - 3\tau)B_1^2}{4\xi B_1^2},$$

$$\gamma := \rho^2 - 3\left[\alpha + \beta(1 + 3\lambda)\right] + 4\mu\rho, \tag{2.1}$$

$$\rho := \alpha + (1 + \lambda)\beta, \tag{2.2}$$

$$\xi := \alpha + (1 + 2\lambda)\beta, \tag{2.3}$$

and

$$\tau := \alpha + (1 + 3\lambda)\beta. \tag{2.4}$$

If $\sigma_1 \leqq \mu \leqq \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{\rho^2}{2\xi B_1}\left(1 - \frac{B_2}{B_1} + \frac{\gamma B_1}{2\rho^2}\right)|a_2|^2 \leqq \frac{B_1}{2\xi}.$$

Furthermore, if $\sigma_3 \leqq \mu \leqq \sigma_2$, *then*

$$|a_3 - \mu a_2^2| + \frac{\rho^2}{2\xi B_1}\left(1 + \frac{B_2}{B_1} - \frac{\gamma B_1}{2\rho^2}\right)|a_2|^2 \leqq \frac{B_1}{2\xi}.$$

Each of these results is sharp.

Proof. If $f(z) \in \mathscr{M}_{\alpha,\beta,\lambda}(\phi)$, then there exists a Schwarz function $w(z)$, analytic in Δ with

$$w(0) = 0 \qquad \text{and} \qquad |w(z)| < 1 \quad (z \in \Delta),$$

such that

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]^{\beta} = \phi\big(w(z)\big). \tag{2.5}$$

Define the function $p_1(z)$ by

$$p_1(z) := \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots . \tag{2.6}$$

Since $w(z)$ is a Schwarz function, we see that

$$\Re\big(p_1(z)\big) > 0 \quad (z \in \Delta) \qquad \text{and} \qquad p_1(0) = 1.$$

Now, defining the function $p(z)$ by

$$\begin{aligned} p(z) &:= \left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]^{\beta} \\ &= 1 + b_1 z + b_2 z^2 + \cdots , \end{aligned} \tag{2.7}$$

we find from (2.5) and (2.6) that

$$p(z) = \phi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right). \tag{2.8}$$

Thus, by using (2.6) in (2.8), we obtain

$$b_1 = \frac{1}{2}B_1 c_1 \qquad \text{and} \qquad b_2 = \frac{1}{2}B_1\left(c_2 - \frac{1}{2}c_1^2\right) + \frac{1}{4}B_2 c_1^2.$$

An easy computation would show that

$$\begin{aligned} &\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]^{\beta} \\ &\quad = 1 + [\alpha + (1+\lambda)\beta]\, a_2 z + 2\,[\alpha + (1+2\lambda)\beta]\, a_3 z^2 \\ &\qquad + \left(\frac{\alpha(\alpha-3)}{2} + \frac{\beta(\beta-1)}{2}(1+\lambda)^2 + \alpha\beta(1+\lambda) - \beta(1+3\lambda)\right) a_2^2 z^2 + \cdots , \end{aligned}$$

which, in view of (2.7), yields

$$b_1 = [\alpha + (1+\lambda)\beta]\, a_2$$

and

$$b_2 = 2\,[\alpha + (1+2\lambda)\beta]\, a_3 + \left(\frac{\alpha(\alpha-3)}{2} + \frac{\beta(\beta-1)}{2}(1+\lambda)^2 + \alpha(1+\lambda)\beta - (1+3\lambda)\beta\right) a_2^2.$$

Equivalently, we have

$$a_2 = \frac{B_1 c_1}{2\,[\alpha + (1+\lambda)\beta]}$$

and

$$a_3 = \frac{B_1}{4\,[\alpha + (1+2\lambda)\beta]}\left[c_2 - \frac{1}{2}\left(1 - \frac{B_2}{B_1} + B_1\Lambda_0\right) c_1^2\right],$$

where

$$\Lambda_0 = \left(\frac{\alpha(\alpha-3)}{2} + \frac{\beta(\beta-1)}{2}(1+\lambda)^2 + \alpha(1+\lambda)\beta - (1+3\lambda)\beta\right)\left(\frac{1}{[\alpha+(1+\lambda)\beta]^2}\right).$$

Therefore, we obtain

$$a_3 - \mu a_2^2 = \frac{B_1}{4\,[\alpha + (1+2\lambda)\beta]}\left(c_2 - v c_1^2\right), \tag{2.9}$$

where

$$v := \frac{1}{2}\left(1 - \frac{B_2}{B_1} + \frac{[\alpha + (1+\lambda)\beta]^2 + 4\mu[\alpha + (1+2\lambda)\beta] - 3\,[\alpha + (1+3\lambda)\beta]}{2\,[\alpha + (1+\lambda)\beta]^2}\, B_1\right).$$

The assertion of Theorem 1 now follows by an application of Lemma 1.

To show that the bounds asserted by Theorem 1 are sharp, we define the following functions:

$$K_{\phi_n}(z) \qquad (n \in \mathbb{N} \setminus \{1\};\ \mathbb{N} := \{1,2,3,\cdots\}),$$

with

$$K_{\phi_n}(0) = 0 = K'_{\phi_n}(0) - 1,$$

by

$$\left(\frac{zK'_{\phi_n}(z)}{K_{\phi_n}(z)}\right)^{\alpha}\left[(1-\lambda)\frac{zK'_{\phi_n}(z)}{K_{\phi_n}(z)} + \lambda\left(1 + \frac{zK''_{\phi_n}(z)}{K'_{\phi_n}(z)}\right)\right]^{\beta} = \phi(z^{n-1}),$$

and the functions $F_\eta(z)$ and $G_\eta(z)$ $(0 \leqq \eta \leqq 1)$, with

$$F_\eta(0) = 0 = F'_\eta(0) - 1 \qquad \text{and} \qquad G_\eta(0) = 0 = G'_\eta(0) - 1,$$

by

$$\left(\frac{zF'_\eta(z)}{F_\eta(z)}\right)^{\alpha}\left[(1-\lambda)\frac{zF'_\eta(z)}{F_\eta(z)} + \lambda\left(1 + \frac{zF''_\eta(z)}{F'_\eta(z)}\right)\right]^{\beta} = \phi\left(\frac{z(z+\eta)}{1+\eta z}\right)$$

and

$$\left(\frac{zG_\eta'(z)}{G_\eta(z)}\right)^\alpha \left[(1-\lambda)\frac{zG_\eta'(z)}{G_\eta(z)} + \lambda\left(1+\frac{zG_\eta''(z)}{G_\eta'(z)}\right)\right]^\beta = \phi\left(-\frac{z(z+\eta)}{1+\eta z}\right),$$

respectively. Then, clearly, the functions $K_{\phi_n}, F_\eta, G_\eta \in \mathscr{M}_{\alpha,\beta,\lambda}(\phi)$. We also write

$$K_\phi := K_{\phi_2}.$$

If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality in Theorem 1 holds true if and only if f is K_ϕ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds true if and only if f is K_{ϕ_3} or one of its rotations. If $\mu = \sigma_1$, then the equality holds true if and only if f is F_η or one of its rotations. If $\mu = \sigma_2$, then the equality holds true if and only if f is G_η or one of its rotations. □

By making use of Lemma 2, we immediately obtain the following Fekete-Szegö type inequality.

THEOREM 2. *Let*

$$0 \leqq \alpha \leqq 1,\ 0 < \beta \leqq 1 \quad \textit{and} \quad 0 \leqq \lambda \leqq 1.$$

Also let

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots,$$

where the coefficients B_n are real with

$$B_1 > 0 \qquad \textit{and} \qquad B_2 \geqq 0.$$

If $f \in \mathscr{M}_{\alpha,\beta,\lambda}(\phi)$, then

$$|a_3 - \mu a_2^2| = \left(\frac{B_1}{[\alpha+(1+2\lambda)\beta]}\right)\max\left\{1, \left|-\frac{B_2}{B_1} + \frac{\gamma}{2\,[\alpha+(1+\lambda)\beta]^2}\, B_1\right|\right\} \qquad (\mu \in \mathbb{C}),$$

where γ is defined as in (2.1). *The result is sharp.*

REMARK 1. The coefficient bounds for $|a_2|$ and $|a_3|$ are special cases of those asserted by Theorem 1.

REMARK 2. In its special case when $\lambda = 1$, Theorem 1 reduces to the result obtained in [8]. Moreover, by setting $\alpha = 0$ and $\beta = 1$, Theorem 1 reduces at once to the following result.

COROLLARY. *Let*

$$0 \leqq \mu \leqq 1 \qquad \textit{and} \qquad 0 \leqq \lambda \leqq 1.$$

Also let

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots,$$

where the coefficients B_n are real with

$$B_1 > 0 \qquad \textit{and} \qquad B_2 \geqq 0,$$

and suppose that

$$\sigma_1 := \frac{2(1+\lambda)^2(B_2 - B_1) - [(1+\lambda)^2 - 3(1+3\lambda)]B_1^2}{4(1+2\lambda)B_1^2},$$

$$\sigma_2 := \frac{2(1+\lambda)^2(B_2 + B_1) - [(1+\lambda)^2 - 3(1+3\lambda)]B_1^2}{4(1+2\lambda)B_1^2}$$

and

$$\sigma_3 := \frac{2(1+\lambda)^2 B_2 - [(1+\lambda)^2 - 3(1+3\lambda)]B_1^2}{4(1+2\lambda)B_1^2}.$$

If $f(z)$ given by (1.1) *belongs to the class $\mathscr{M}_{0,1,\lambda}(\phi)$, then*

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \dfrac{1}{4(1+2\lambda)}\left(2B_2 - \dfrac{B_1^2}{(1+\lambda)^2}\,\gamma_2\right) & (\mu \leqq \sigma_1) \\[2ex] \dfrac{1}{2(1+2\lambda)}\,B_1 & (\sigma_1 \leqq \mu \leqq \sigma_2) \\[2ex] \dfrac{1}{4(1+2\lambda)}\left(-2B_2 + \dfrac{B_1^2}{(1+\lambda)^2}\,\gamma_2\right) & (\mu \geqq \sigma_2), \end{cases}$$

where, for convenience,

$$\gamma_2 := (1+\lambda)^2 - 3(1+3\lambda) + 4\mu(1+2\lambda). \tag{2.10}$$

If $\sigma_1 \leqq \mu \leqq \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{(1+\lambda)^2}{2(1+2\lambda)B_1}\left(1 - \frac{B_2}{B_1} + \frac{\gamma_2 B_1}{2(1+\lambda)^2}\right)|a_2|^2 \leqq \frac{B_1}{2(1+2\lambda)}.$$

Furthermore, if $\sigma_3 \leqq \mu \leqq \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{(1+\lambda)^2}{2(1+2\lambda)B_1}\left(1 + \frac{B_2}{B_1} - \frac{\gamma_2 B_1}{2(1+\lambda)^2}\right)|a_2|^2 \leqq \frac{B_1}{2(1+2\lambda)}.$$

Each of these results is sharp.

REMARK 3. If, in Theorem 1, we set

$$\lambda = 1, \quad \alpha = 0 \qquad \text{and} \qquad \beta = 1,$$

we arrive at a known result due to Ma and Minda [3].

3. Applications to Analytic Functions Defined by Using Fractional Calculus Operators and Convolution

The subject of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during the past three decades or so. Two of the most recent works on this subject of widespread investigations include the rather comprehensive treatises on the theory and applications of fractional differential equations by Podlubny [7] and Kilbas *et al.* [2].

For the applications of the results given in the preceding sections, we first introduce the class $\mathcal{M}^{\delta}_{\alpha,\beta,\lambda}(\phi)$, which is defined by means of the Hadamard product (or convolution) and a certain operator of fractional calculus, known as the Owa-Srivastava operator (see, for details, [11] and [15]; see also [5], [6], and [14]).

DEFINITION 2. The fractional integral of order δ is defined, for a function $f(z)$, by

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\delta}} d\zeta \qquad (\delta > 0), \tag{3.1}$$

where the function $f(z)$ is analytic in a simply-connected domain of the complex z-plane containing the origin and the multiplicity of $(z-\zeta)^{\delta-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

DEFINITION 3. The fractional derivative of order δ is defined, for a function $f(z)$, by

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\delta}} d\zeta \qquad (0 \leqq \delta < 1), \tag{3.2}$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\delta}$ is removed, as in Definition 2.

DEFINITION 4. Under the hypotheses of Definition 3, the fractional derivative of order $n+\delta$ is defined, for a function $f(z)$, by

$$D_z^{n+\delta} f(z) = \frac{d^n}{dz^n} \left\{ D_z^{\delta} f(z) \right\} \qquad (0 \leqq \delta < 1;\ n \in \mathbb{N}_0). \tag{3.3}$$

Using Definitions 2, 3 and 4 of fractional derivatives and fractional integrals, Owa and Srivastava [6] introduced what is popularly referred to in the current literature as the Owa-Srivastava operator $\Omega^{\delta} : \mathcal{A} \to \mathcal{A}$ defined by

$$(\Omega^{\delta} f)(z) := \Gamma(2-\delta) z^{\delta} D_z^{\delta} f(z), \qquad (\delta \neq 2, 3, 4, \cdots). \tag{3.4}$$

In terms of the Owa-Srivastava operator Ω^{δ} defined by (3.4), we now introduce the function class $\mathcal{M}^{\delta}_{\alpha,\beta,\lambda}(\phi)$ in the following way:

$$\mathcal{M}^{\delta}_{\alpha,\beta,\lambda}(\phi) := \{ f : f \in \mathcal{A} \quad \text{and} \quad \Omega^{\delta} f \in \mathcal{M}_{\alpha,\beta,\lambda}(\phi) \}. \tag{3.5}$$

It is easily seen that the function class $\mathscr{M}^{\delta}_{\alpha,\beta,\lambda}(\phi)$ is a special case of the function class $\mathscr{M}^{g}_{\alpha,\beta,\lambda}(\phi)$ when

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}\, z^n. \tag{3.6}$$

Suppose now that

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n \qquad (g_n > 0).$$

Then, since

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{M}^{g}_{\alpha,\beta,\lambda}(\phi)$$

$$\Longleftrightarrow \quad (f * g)(z) = z + \sum_{n=2}^{\infty} g_n a_n z^n \in \mathscr{M}_{\alpha,\beta,\lambda}(\phi), \tag{3.7}$$

we can obtain the coefficient estimates for functions in the class $\mathscr{M}^{g}_{\alpha,\beta,\lambda}(\phi)$ from the corresponding estimates for functions in the class $\mathscr{M}_{\alpha,\beta,\lambda}(\phi)$. By applying Theorem 1 to the following Hadamard product (or convolution):

$$(f * g)(z) = z + g_2 a_2 z^2 + g_3 a_3 z^3 + \cdots ,$$

we get Theorem 3 below after an obvious change of the parameter μ.

THEOREM 3. *Let*

$$0 \leqq \mu \leqq 1, \ 0 \leqq \alpha \leqq 1, \ 0 < \beta \leqq 1 \quad \textit{and} \quad 0 \leqq \lambda \leqq 1.$$

Suppose also that

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots ,$$

where the coefficients B_n are real with

$$B_1 > 0, \ \ B_2 \geqq 0 \qquad \textit{and} \qquad B_n > 0 \quad (n \in \mathbb{N} \setminus \{1,2\}).$$

If $f(z)$ given by (1.1) belongs to the class $\mathscr{M}^{g}_{\alpha,\beta,\lambda}(\phi)$, then

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \dfrac{1}{4\xi g_3}\left(2B_2 - \dfrac{B_1^2}{\rho^2}\,\gamma_2\right) & (\mu \leqq \sigma_1) \\[2ex] \dfrac{B_1}{2\xi g_3} & (\sigma_1 \leqq \mu \leqq \sigma_2) \\[2ex] \dfrac{1}{4\xi g_3}\left(-2B_2 + \dfrac{B_1^2}{\rho^2}\,\gamma_2\right) & (\mu \geqq \sigma_2), \end{cases}$$

where, for convenience,

$$\sigma_1 := \frac{g_3}{g_2^2}\left(\frac{2\rho^2(B_2 - B_1) - (\rho^2 - 3\tau)B_1^2}{4\xi B_1^2}\right),$$

$$\sigma_2 := \frac{g_3}{g_2^2}\left(\frac{2\rho^2(B_2+B_1)-(\rho^2-3\tau)B_1^2}{4\xi B_1^2}\right),$$

and

$$\gamma_2 := \rho^2 - 3\tau + 4\mu\frac{g_3}{g_2^2}\xi, \tag{3.8}$$

and ρ, ξ *and* τ *are defined as in* (2.2), (2.3) *and* (2.4), *respectively. These results are sharp.*

Since, by (1.1) and the definition (3.4),

$$(\Omega^\delta f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}\, a_n z^n, \tag{3.9}$$

we readily obtain

$$g_2 := \frac{\Gamma(3)\Gamma(2-\delta)}{\Gamma(3-\delta)} = \frac{2}{2-\delta} \tag{3.10}$$

and

$$g_3 := \frac{\Gamma(4)\Gamma(2-\delta)}{\Gamma(4-\delta)} = \frac{6}{(2-\delta)(3-\delta)}. \tag{3.11}$$

For g_2 and g_3 given by (3.10) and (3.11), respectively, Theorem 3 reduces to the following interesting result.

THEOREM 4. *Let*

$$0 \leqq \mu \leqq 1,\ 0 \leqq \alpha \leqq 1,\ 0 < \beta \leqq 1 \quad \textit{and} \quad 0 \leqq \lambda \leqq 1.$$

Suppose also that

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots,$$

where the coefficients B_n *are real with* $B_1 > 0$ *and* $B_2 \geqq 0$. *If* $f(z)$ *given by* (1.1) *belongs to the function class* $\mathscr{M}^g_{\alpha,\beta,\lambda}(\phi)$, *then*

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \dfrac{(2-\delta)(3-\delta)}{24\xi}\left(2B_2 - \dfrac{B_1^2}{\rho^2}\,\gamma_3\right) & (\mu \leqq \sigma_1) \\[2ex] \dfrac{(2-\delta)(3-\delta)}{12\xi}\,B_1 & (\sigma_1 \leqq \mu \leqq \sigma_2) \\[2ex] \dfrac{(2-\delta)(3-\delta)}{24\xi}\left(-2B_2 + \dfrac{B_1^2}{\rho^2}\,\gamma_3\right) & (\mu \geqq \sigma_2), \end{cases}$$

where, for convenience,

$$\sigma_1 := \frac{2(3-\delta)}{3(2-\delta)}\,\frac{2\rho^2(B_2-B_1)-(\rho^2-3\tau)B_1^2}{4\xi B_1^2},$$

$$\sigma_2 := \frac{2(3-\delta)}{3(2-\delta)}\,\frac{2\rho^2(B_2+B_1)-(\rho^2-3\tau)B_1^2}{4\xi B_1^2},$$

and

$$\gamma_3 := \rho^2 - 3\tau + 4\mu\xi\,\frac{2(3-\delta)}{3(2-\delta)}, \tag{3.12}$$

and ρ, ξ *and* τ *are defined as in* (2.2), (2.3) *and* (2.4), *respectively.*

REMARK 4. In its special case when

$$\lambda = 0, \quad \beta = 1, \quad \alpha = 0, \quad B_1 = \frac{8}{\pi^2} \quad \text{and} \quad B_2 = \frac{16}{3\pi^2},$$

Theorem 4 coincides with the following result due to Srivastava *et al.* [12] for which $\Omega^\lambda f(z)$ is a parabolic starlike function ([1] and [10]).

THEOREM 5. (Srivastava and Mishra [12]) *Let* $0 \leqq \mu \leqq 1$. *Suppose also that*

$$\sigma_1 := \left(\frac{3-\delta}{2-\delta}\right)\left(\frac{1}{3} + \frac{5\pi^2}{72}\right) \qquad \textit{and} \qquad \sigma_2 := \left(\frac{3-\delta}{2-\delta}\right)\left(\frac{1}{3} - \frac{\pi^2}{72}\right).$$

If $f(z)$ *given by* (1.1) *belongs to the function class* $\mathscr{M}^{\delta}_{0,1,0}(\phi)$, *then*

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \frac{4}{3\pi^2}(3-\delta)(2-\delta)\left(\frac{12(2-\delta)\mu}{(3-\delta)\pi^2} - \frac{4}{\pi^2} - \frac{1}{3}\right) & (\mu \leqq \sigma_1) \\[2mm] \frac{2}{3\pi^2}(3-\delta)(2-\delta) & (\sigma_1 \leqq \mu \leqq \sigma_2) \\[2mm] \frac{4}{3\pi^2}(3-\delta)(2-\delta)\left(\frac{1}{3} + \frac{4}{\pi^2} - \frac{12(2-\delta)\mu}{(3-\delta)\pi^2}\right) & (\mu \geqq \sigma_2). \end{cases}$$

These results are sharp.

REMARK 5. For the following choices:

$$\lambda = 0, \quad \beta = 1, \quad \alpha = 0, \quad \delta = 1, \quad B_1 = \frac{8}{\pi^2} \quad \text{and} \quad B_2 = \frac{16}{3\pi^2},$$

Theorem 4 would coincide with the result obtained earlier by Ma and Minda [4].

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