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GENERALIZED SYSTEMS FOR RELAXED COCOERCIVE VARIATIONAL INEQUALITIES AND PROJECTION METHODS IN HILBERT SPACES

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Abstract. In this paper, we introduce a new algorithm for a generalized system for a relaxed cocoercive nonlinear inequality and an asymptotically nonexpansive mapping in Hilbert spaces by the convergence of projection methods. Our results include the previous results as special cases extend and improve the main results of [R.U. Verma, General convergence analysis for two-step projection methods and application to variational problems. Appl. Math. Lett. 18 (11) (2005), 1286-1292], [R.U. Verma, Generalized system for relaxed cocoercive variational inequalities and its projection methods, J. Optim. Theory Appl. 121 (1) (2004), 203-210], [R.U. Verma, Generalized class of partial relaxed monotonicity and its connections, Adv. Nonlinear Var. Inequal. 7 (2) (2004), 155-164], [N.H. Xiu, J.Z. Zhang, Local convergence analysis of projection type algorithms: Unified approach, J. Optim. Theory Appl. 115 (2002) 211-230], [N.H. Nie, Z. Liu, K.H. Kim, S.M. Kang, A system of nonlinear variational inequalities involving strong monotone and pseudocontractive mappings, Adv. Nonlinear Var. Inequal. 6 (2) (2003), 91-99], [S.S. Chang, H.W. Joseph Lee, C.K. Chan, Generalized system for relaxed cocoercive variational inequalities in Hilbert spaces, Appl. Math. Lett. 20 (3) (2007), 329-334] and many others.

1. Introduction and Preliminaries

Variational inequalities introduced by Stampacchia [6] in the early sixties have had a great impact and influence in the development of almost all branches of pure and applied sciences and have witnessed an explosive growth in theoretical advances, algorithmic development, see [1–11] and references therein. It combines novel theoretical and algorithmic advances with new domain of applications. Analysis of these problems requires a blend of technics from convex analysis, functional analysis and numerical analysis. As a result of interaction between different branches of mathematical and engineering sciences, we now have a variety of techniques to suggest and analyze various algorithms for solving variational inequalities and related optimization. It is well known that the variational inequalities are equivalent to the fixed point problems. This alternative equivalent formulation is very important from the numerical analysis point of view and has played a significant part in several numerical methods for solving

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variational inequalities and complementarity; see [1,3]. In particular, the solution of the variational inequalities can be computed using the iterative projection methods. It is well known that the convergence of the projection method requires the operator T to be strongly monotone and Lipschitz continuous. Gabay [4] has shown that the convergence of a projection method can be proved for cocoercive operators. Note that cocoercivity is a weaker condition than strong monotonicity. Recently, Verma [7] introduced a new system of nonlinear strongly monotone variational inequalities and studied the approximate solvability of this system based on a system of projection methods. Projection methods have been applied widely to problems arising especially from complementarity, convex quadratic programming, and variational problems. Additional research on the approximate solvability of a system of nonlinear variational inequalities is due to Nie et al. [5], Verma [10] and others.

In this paper, we consider, based on the projection method, the approximate solvability of a system of nonlinear relaxed cocoercive variational inequalities in the framework of Hilbert spaces. Solutions of the system of nonlinear relaxed cocoercive variational inequalities are also fixed points of an asymptotically nonexpansisve mapping. Our results obtained in this paper generalize the results of Chang et al. [2], Nie et al. [5], Verma [7–9], Xiu et al. [11] and some others.

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. Let C be a closed convex subset of H and let $A : C \to H$ be a nonlinear mapping. Let P_C be the projection of H onto the convex subset C. The classical variational inequality which denoted by $VI(C,A)$ is to find $u \in C$ such that

$$\langle Au, v-u \rangle \geqslant 0, \quad \forall v \in C. \tag{1.1}$$

We now recall some well-known concepts and results:

LEMMA 1.1. *For any $z \in H$, $u \in C$ satisfies the inequality:*

$$\langle u-z, v-u \rangle \geqslant 0, \quad \forall v \in C,$$

if and only if $u = P_C z$.

It is known that the projection operator P_C is nonexpansive and P_C satisfies the following:

$$\langle x-y, P_C x - P_C y \rangle \geqslant \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{1.2}$$

Moreover, $P_C x$ is characterized by the properties:

$$P_C x \in C, \quad \langle x - P_C x, P_C - y \rangle \geqslant 0, \quad \forall y \in C.$$

Using Lemma 1.1, one can show that the variational inequality (1.1) is equivalent to a fixed point problem.

LEMMA 1.2. *The point $u \in C$ is a solution of the variational inequality* (1.1) *if and only if $u \in C$ satisfies the relation $u = P_C(u - \lambda Au)$, where $\lambda > 0$ is a constant.*

It is clear from Lemma 1.2 that the variational inequalities and the fixed point problems are equivalent. This alternative equivalent formulation has played a significant role in the study of the variational inequalities and related optimization problems.

Recall the following definitions:

(1) A mapping A of C into H is called monotone if

$$\langle Au - Av, u - v\rangle \geqslant 0, \quad \forall u, v \in C.$$

(2) A is called v-strongly monotone if there exists a constant $v > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geqslant v\|x - y\|^2, \quad \forall x, y \in C.$$

This implies that

$$\|Ax - Ay\| \geqslant v\|x - y\|, \quad \forall x, y \in C,$$

that is, A is v-expansive and, when $v = 1$, it is expansive.

(3) A is said to be μ-cocoercive [12,14] if there exists a constant $\mu > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geqslant \mu\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Clearly, every μ-cocoercive mapping A is $\frac{1}{\mu}$-Lipschitz continuous.

(4) A is called relaxed u-cocoerceive if there exists a constant $u > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geqslant (-u)\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

(5) A is said to be relaxed (u, v)-cocoercive if there exist two constants $u, v > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geqslant (-u)\|Ax - Ay\|^2 + v\|x - y\|^2, \quad \forall x, y \in C.$$

For $u = 0$, A is v-strongly monotone. This class of mappings is more general that the class of strongly monotone mappings. It is easy to see that we have the following implication:

v-strongly monotonicity $\Rightarrow$ relaxed (u, v)-cocoercivity.

(6) $S : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leqslant \|x - y\|, \quad \forall x, y \in C.$$

(7) $S : C \to C$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \in [0, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|S^n x - S^n y\| \leqslant k_n\|x - y\|, \quad \forall x, y \in C,\ n \geqslant 0.$$

Next, we denote the fixed point of S by $F(S)$. We can characterize the problem. If $x^* \in F(S) \cap VI(C, A)$, then it follows from Lemma 2.2 that

$$x^* = S^n x^* = P_C[x^* - \rho T x^*] = S^n P_C[x^* - \rho T x^*],$$

where $\rho > 0$ is a constant.

This formulation is used to suggest the following iterative methods for finding a common element of two different sets of the fixed points of the asymptotically nonexpansive mappings and solutions of the variational inequalities.

Let $T : C \times C \times C \to H$ be a mapping. Consider a system (SNVI) of nonlinear variational inequality problems as follows:

Find $x^*, y^*, z^* \in C$ such that

$$\langle sT(y^*, z^*, x^*) + x^* - y^*, x - x^* \rangle \geqslant 0, \quad \forall x \in C, \; s > 0, \tag{1.3}$$

$$\langle tT(z^*, x^*, y^*) + y^* - z^*, x - x^* \rangle \geqslant 0, \quad \forall x \in C, \; t > 0, \tag{1.4}$$

$$\langle rT(x^*, y^*, z^*) + z^* - x^*, x - x^* \rangle \geqslant 0, \quad \forall x \in C, \; r > 0. \tag{1.5}$$

One can easily see the SNVI problems (1.3), (1.4) and (1.5) are equivalent to the following projection formulas:

$$x^* = P_C[y^* - sT(y^*, z^*, x^*)], \quad s > 0,$$

$$y^* = P_C[z^* - tT(z^*, x^*, y^*)], \quad t > 0,$$

$$z^* = P_C[x^* - rT(x^*, y^*, z^*)], \quad r > 0,$$

respectively, where P_C is the projection of H onto C.

Next, we consider some special classes of the SNVI problems (1.3), (1.4) and (1.5) as follows:

(I) If $r = 0$, then the SNVI problems (1.3), (1.4) and (1.5) collapse to the following SNVI:

Find $x^*, y^* \in C$ such that

$$\langle sT(y^*, x^*, x^*) + x^* - y^*, x - x^* \rangle \geqslant 0, \quad \forall x \in C, \; s > 0, \tag{1.6}$$

$$\langle tT(x^*, x^*, y^*) + y^* - x^*, x - x^* \rangle \geqslant 0, \quad \forall x \in C, \; t > 0. \tag{1.7}$$

(II) If $t = r = 0$, then the SNVI problems (1.3), (1.4) and (1.5) reduce to the following nonlinear variational inequality problem (NVI):

Find an $x^* \in C$ such that

$$\langle T(x^*, x^*, x^*), x - x^* \rangle \geqslant 0, \quad \forall x \in C. \tag{1.8}$$

(III) If C is a closed convex cone of H, then the SNVI problems (1.3), (1.4) and (1.5) are equivalent to the following system (SNC) of nonlinear complementarity problems:

Find $x^*, y^*, z^* \in C$ such that

$$T(x^*, y^*, z^*) \in C^*, \; T(y^*, z^*, x^*) \in C^*, \; T(z^*, x^*, y^*) \in C^*,$$

$$\langle sT(y^*, z^*, x^*) + x^* - y^*, x^* \rangle = 0, \quad s > 0, \tag{1.9}$$

$$\langle tT(z^*, x^*, y^*) + y^* - z^*, x^* \rangle = 0, \quad t > 0, \tag{1.10}$$

$$\langle rT(x^*, y^*, z^*) + z^* - x^*, x^* \rangle = 0, \quad r > 0, \tag{1.11}$$

where C^* is the polar cone to C defined by

$$C^* = \{f \in H : \langle f, x \rangle \geqslant 0, \quad \forall x \in C\}.$$

(IV) If $T : C \to H$ is a univariate mapping, then the SVNI problems (1.3), (1.4) and (1.5) are reduced to the following SNVI problems:

Find $x^*, y^* \in C$ such that

$$\langle sT(y^*) + x^* - y^*, x - x^* \rangle \geqslant 0, \quad \forall x \in C,\ s > 0, \tag{1.12}$$

$$\langle tT(z^*) + y^* - z^*, x - x^* \rangle \geqslant 0, \quad \forall x \in C,\ t > 0, \tag{1.13}$$

$$\langle rT(x^*) + z^* - x^*, x - x^* \rangle \geqslant 0, \quad \forall x \in C,\ r > 0. \tag{1.14}$$

One can easily get the SNVI problems (1.12), (1.13) and (1.14) are equivalent to the following projection formulas:

$$x^* = P_C[y^* - sT(y^*)], \quad s > 0, \tag{1.15}$$

$$y^* = P_C[z^* - tT(z^*)], \quad t > 0, \tag{1.16}$$

$$z^* = P_C[x^* - rT(x^*)], \quad r > 0. \tag{1.17}$$

2. Algorithms

In this section, we consider an introduction of the general three-step models for the projection methods and its special form can be applied to the convergence analysis for the projection methods in the context of the approximation solvability of the SNVI problems (1.3)–(1.5) and (1.12)–(1.14).

Algorithm 2.1. For any $x_0, y_0, z_0 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative processes:

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n S^n P_C[x_n - r_n T(x_n, y_n, z_n)], \\ y_n = (1-\beta_n)x_n + \beta_n S^n P_C[z_n - t_n T(z_n, x_n, y_n)], \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n S^n P_C[y_n - s_n T(y_n, z_n, x_n)], \end{cases} \tag{2.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0,1]$ for all $n \geqslant 0$ and S is an asymptotically nonexpansive mapping.

(I) If $T : C \to H$ is a univariate mapping, then the Algorithm 2.1 is reduced to the following:

Algorithm 2.2. For any $x_0, y_0, z_0 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative processes:

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n S^n P_C[x_n - r_n T(x_n)], \\ y_n = (1-\beta_n)x_n + \beta_n S^n P_C[z_n - t_n T(z_n)], \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n S^n P_C[y_n - s_n T(y_n)], \end{cases} \tag{2.2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0,1]$ for all $n \geqslant 0$ and S is an asymptotically nonexpansive mapping.

(II) If $\gamma_n = 1$ in Algorithm 2.1, then we have the following:

Algorithm 2.3. For any $x_0, y_0, z_0 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative processes:

$$\begin{cases} z_n = S^n P_C[x_n - r_n T(x_n, y_n, z_n)], \\ y_n = (1-\beta_n)x_n + \beta_n S^n P_C[z_n - t_n T(z_n, x_n, y_n)], \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n S^n P_C[y_n - s_n T(y_n, z_n, x_n)], \end{cases} \tag{2.3}$$

where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0,1]$ for all $n \geqslant 0$ and S is an asymptotically nonexpansive mapping.

(III) If $\beta_n = \gamma_n = 1$ in Algorithm 2.1, then we have the following:

Algorithm 2.4. For any $x_0, y_0, z_0 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative processes:

$$\begin{cases} z_n = S^n P_C[x_n - r_n T(x_n, y_n, z_n)], \\ y_n = S^n P_C[z_n - t_n T(z_n, x_n, y_n)], \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n S^n P_C[y_n - s_n ST(y_n, z_n, x_n)], \end{cases} \tag{2.4}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ for all $n \geqslant 0$ and S is an asymptotically nonexpansive mapping.

(IV) If $r_n = t_n = 0$ in Algorithm 2.1, then we have the following:

Algorithm 2.5. For any $x_0 \in C$, compute the sequence $\{x_n\}$ by the iterative process:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n S^n P_C[x_n - s_n T(x_n, x_n, x_n)], \tag{2.5}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ for all $n \geqslant 0$ and S is an asymptotically nonexpansive mapping.

In order to prove our main results, we need the following lemmas and definitions.

LEMMA 2.1.. *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leqslant (1-\lambda_n)a_n + b_n + c_n, \quad \forall n \geqslant n_0,$$

where n_0 is some nonnegative integer, $\{\lambda_n\}$ is a sequence in $(0,1)$ with $\sum_{n=1}^{\infty} \lambda_n = \infty$, $b_n = \circ(\lambda_n)$ and $\sum_{n=0}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} a_n = 0$.

DEFINITION 2.1. A mapping $T : C \times C \times C \to H$ is said to be relaxed (u,v)-cocoercive if there exist constants $u, v > 0$ such that, for all $x, x' \in C$,

$$\begin{aligned} &\langle T(x,y,z) - T(x',y',z'), x - x' \rangle \\ &\qquad \geqslant (-u)\|T(x,y,z) - T(x',y',z')\|^2 + v\|x - x'\|^2, \quad \forall y, y', z, z' \in C. \end{aligned}$$

DEFINITION 2.2. A mapping $T : C \times C \times C \to H$ is said to be μ-Lipschitz continuous in the first variable if there exists a constant $\mu > 0$ such that, for all $x, x' \in C$,

$$\|T(x,y,z) - T(x',y',z')\| \leqslant \mu\|x - x'\|, \quad \forall y, y', z, z' \in C.$$

3. Main results

THEOREM 3.1. *Let C be a closed convex subset of a real Hilbert space H. Let $T : C\times C\times C \to H$ be a relaxed (u,v)-cocoerceive and μ-Lipschitz continuous mapping in the first variable and $S : C \to C$ be an asymptotically nonexpansive mapping. Suppose that $x^*, y^*, z^* \in C$ are solutions of the SNVI problems (1.3)–(1.5), $x^*, y^*, z^* \in F(S)$ and $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ are the sequences generated by Algorithm 2.1. If $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[0,1]$ satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(ii) $\sum_{n=0}^{\infty}(1-\beta_n) < \infty$ *and* $\sum_{n=0}^{\infty}(1-\gamma_n) < \infty$;

(iii) $0 < s_n, t_n, r_n < \frac{2(v-u\mu^2)}{\mu^2}$;

(iv) $v > u\mu^2$,

then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converges strongly to x^*, y^* and z^*, respectively.

Proof. Since x^*, y^* and z^* are the common elements in the set of solutions of the SNVI problems (1.3)–(1.5) and the set of fixed points of S, we have

$$\begin{cases} x^* = S^nP_C[y^* - s_nT(y^*,z^*,x^*)], & s>0,\\ y^* = S^nP_C[z^* - t_nT(z^*,x^*,y^*)], & t>0,\\ z^* = S^nP_C[x^* - r_nT(x^*,y^*,z^*)], & r>0. \end{cases}$$

Observing (2.1), we obtain

$$\begin{aligned} &\|x_{n+1}-x^*\| \\ &= \|(1-\alpha_n)x_n + \alpha_nS^nP_C[y_n - s_nT(y_n,z_n,x_n)] - x^*\| \\ &\leqslant (1-\alpha_n)\|x_n-x^*\| + \alpha_nk_n\|y_n-y^*-s_n[T(y_n,z_n,x_n)-T(y^*,z^*,x^*)]\|. \end{aligned} \tag{3.1}$$

By the assumption that T is relaxed (u,v)-cocoercive and μ-Lipschitz continuous in the first variable, we obtain

$$\begin{aligned} &\|y_n-y^*-s_n[T(y_n,z_n,x_n)-T(y^*,z^*,x^*)]\|^2 \\ &= \|y_n-y^*\| - 2s_n\langle y_n-y^*, T(y_n,z_n,x_n)-T(y^*,z^*,x^*)\rangle \\ &\quad + s_n^2\|T(y_n,z_n,x_n)-T(y^*,z^*,x^*)\|^2 \\ &\leqslant \|y_n-y^*\| - 2s_n[-u\|T(y_n,z_n,x_n)-T(y^*,z^*,x^*)\|^2 + v\|y_n-y^*\|^2] \\ &\quad + s_n^2\mu^2\|y_n-y^*\|^2 \\ &\leqslant \|y_n-y^*\| + 2s_nu\mu^2\|y_n-y^*\|^2 - 2s_nv\|y_n-y^*\|^2 + s_n^2\mu^2\|y_n-y^*\|^2 \\ &= \theta_{1n}^2\|y_n-y^*\|^2, \end{aligned} \tag{3.2}$$

where $\theta_{1n}^2 = 1 + s_n^2\mu^2 - 2s_nv + 2s_nu\mu^2$. From the conditions (iii) and (iv), we know

$\theta_{1n} < 1$. Substituting (3.2) into (3.1) yields that

$$\|x_{n+1} - x^*\| \leqslant (1-\alpha_n)\|x_n - x^*\| + k_n\alpha_n\theta_{1n}\|y_n - y^*\|. \tag{3.3}$$

Now, we estimate

$$\begin{aligned} &\|y_n - y^*\| \\ &= \|(1-\beta_n)x_n + \beta_n S^n P_C[z_n - t_n T(z_n, x_n, y_n)] - y^*\| \\ &\leqslant (1-\beta_n)\|x_n - y^*\| + k_n\beta_n\|z_n - z^* - t_n[T(z_n, x_n, y_n) - T(z^*, x^*, y^*)]\|. \end{aligned} \tag{3.4}$$

By the assumption that T is relaxed (u,v)-cocoercive and μ-Lipschitz continuous in the first variable, we obtain

$$\begin{aligned} &\|z_n - z^* - t_n[T(z_n, x_n, y_n) - T(z^*, x^*, y^*)]\|^2 \\ &= \|z_n - z^*\|^2 - 2t_n\langle z_n - z^*, T(z_n, x_n, y_n) - T(z^*, x^*, y^*)\rangle \\ &\quad + t_n^2\|T(z_n, x_n, y_n) - T(z^*, x^*, y^*)\|^2 \\ &\leqslant \|z_n - z^*\|^2 - 2t_n[-u\|T(z_n, x_n, y_n) - T(z^*, x^*, y^*)\|^2 + v\|z_n - z^*\|^2] \\ &\quad + t_n^2\mu^2\|z_n - z^*\|^2 \\ &\leqslant \|z_n - z^*\|^2 + 2t_n u\mu^2\|z_n - z^*\|^2 - 2t_n v\|z_n - z^*\|^2 + t_n^2\mu^2\|z_n - z^*\|^2 \\ &\leqslant \theta_{2n}^2\|z_n - z^*\|^2, \end{aligned} \tag{3.5}$$

where $\theta_{2n}^2 = 1 + t_n^2\mu^2 - 2t_n v + 2t_n u\mu^2$. From the conditions (iii) and (iv), we know $\theta_{2n} < 1$. Substituting (3.5) into (3.4) yields that

$$\|y_n - y^*\| \leqslant (1-\beta_n)\|x_n - x^*\| + (1-\beta_n)\|x^* - y^*\| + k_n\beta_n\theta_{2n}\|z_n - z^*\| \tag{3.6}$$

Similarly, Substituting (3.6) into (3.3), we have

$$\begin{aligned} &\|x_{n+1} - x^*\| \\ &\quad\leqslant (1-\alpha_n)\|x_n - x^*\| \\ &\qquad + k_n\alpha_n\theta_{1n}[(1-\beta_n)\|x_n - x^*\| + (1-\beta_n)\|x^* - y^*\| + k_n\beta_n\theta_{2n}\|z_n - z^*\|] \\ &\quad= (1-\alpha_n + k_n\alpha_n\theta_{1n}(1-\beta_n))\|x_n - x^*\| \\ &\qquad + k_n\alpha_n\theta_{1n}[(1-\beta_n)\|x^* - y^*\| + k_n\beta_n\theta_{2n}\|z_n - z^*\|]. \end{aligned} \tag{3.7}$$

Next, we show that

$$\begin{aligned} \|z_n - z^*\| &= \|(1-\gamma_n)x_n + \gamma_n S^n P_C[x_n - r_n T(x_n, y_n, z_n)] - z^*\| \\ &\leqslant (1-\gamma_n)\|x_n - z^*\| + k_n\gamma_n\|x_n - x^* - r_n[T(x_n, y_n, z_n) - T(x^*, y^*, z^*)]\|. \end{aligned} \tag{3.8}$$

By the assumption that T is relaxed (u,v)-cocoercive and μ-Lipschitz continuous in the first variable, we obtain

$$\begin{aligned}
&\|x_n - x^* - r_n[T(x_n,y_n,z_n) - T(x^*,y^*,z^*)]\|^2 \\
&= \|x_n - x^*\| - 2r_n\langle x_n - x^*, T(x_n,y_n,z_n) - T(x^*,y^*,z^*)\rangle \\
&\quad + r_n^2\|T(x_n,y_n,z_n) - T(x^*,y^*,z^*)\|^2 \\
&\leqslant \|x_n - x^*\| - 2r_n[-u\|T(x_n,y_n,z_n) - T(x^*,y^*,z^*)\|^2 + v\|x_n - x^*\|^2] \\
&\quad + r_n^2\mu^2\|x_n - x^*\|^2 \\
&\leqslant \|x_n - x^*\| + 2r_n u\mu^2\|x_n - x^*\|^2 - 2r_n v\|x_n - x^*\|^2 + r_n^2\mu^2\|x_n - x^*\|^2 \\
&= \theta_{3n}^2\|x_n - x^*\|^2,
\end{aligned} \tag{3.9}$$

where $\theta_{3n}^2 = 1 + r_n^2\mu^2 - 2r_n v + 2r_n u\mu^2$. From the conditions (iii) and (iv), we know $\theta_{3n} < 1$. Substituting (3.9) into (3.8), we obtain

$$\begin{aligned}
&\|z_n - z^*\| \\
&\leqslant (1-\gamma_n)\|x_n - z^*\| + k_n\gamma_n\theta_{3n}\|x_n - x^*\| \\
&\leqslant (1-\gamma_n)\|x_n - x^*\| + (1-\gamma_n)\|x^* - z^*\| + k_n\gamma_n\theta_{3n}\|x_n - x^*\| \\
&\leqslant k_n\|x_n - x^*\| + (1-\gamma_n)\|x^* - z^*\|.
\end{aligned} \tag{3.10}$$

Similarly, substituting (3.10) into (3.7) yields that

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&\leqslant (1-\alpha_n + k_n\alpha_n\theta_{1n}(1-\beta_n))\|x_n - x^*\| + k_n\alpha_n\theta_{1n}(1-\beta_n)\|x^* - y^*\| \\
&\quad + k_n\alpha_n\beta_n\theta_{1n}\theta_{2n}[\|x_n - x^*\| + (1-\gamma_n)\|x^* - z^*\|] \\
&\leqslant (1 - \alpha_n(1 - k_n\theta_{1n}(1-\beta_n) - k_n\beta_n\theta_{1n}\theta_{2n}))\|x_n - x^*\| \\
&\quad + M(1-\beta_n)\|x^* - y^*\| + M(1-\gamma_n)\|x^* - z^*\|,
\end{aligned} \tag{3.11}$$

where M is an appropriate constant such that $M \geqslant \sup\{k_n\}_{n\geqslant 0}$. Applying Lemma 2.1 into (3.11), we can get the desired conclusion easily. This completes the proof. □

From Theorem 3.1, we can get the following results immediately:

THEOREM 3.2. *Let C be a closed convex subset of a real Hilbert space H. Let $T: C \to H$ be a relaxed (u,v)-cocoerceive and μ-Lipschitz continuous mapping and $S: C \to C$ be an asymptotically nonexpansive mapping. Suppose that $x^*, y^*, z^* \in C$ are solutions of the SNVI problems (1.12)–(1.14), $x^*, y^*, z^* \in F(S)$ and $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ are the sequences generated by Algorithm 2.2. If $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[0,1]$ satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty}\alpha_n = \infty$;
(ii) $\sum_{n=0}^{\infty}(1-\beta_n) < \infty$ *and* $\sum_{n=0}^{\infty}(1-\gamma_n) < \infty$;
(iii) $0 < s_n, t_n, r_n < \frac{2(v-u\mu^2)}{\mu^2}$;
(iv) $v > u\mu^2$,

then the sequences $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *converge strongly to* x^*, y^* *and* z^*, *respectively.*

REMARK 1. Theorem 3.1 and Theorem 3.2 extends and improves the main results in Chang et al. [2] and Verma [7–9].

THEOREM 3.3. *Let* C *be a closed convex subset of a real Hilbert space* H. *Let* $T : C \to H$ *be a relaxed* (u,v)*-cocoerceive and* μ*-Lipschitz continuous mapping and* $S : C \to C$ *be an asymptotically nonexpansive mapping. Suppose that* $x^*, y^*, z^* \in C$ *are solutions of the SNVI problems* $(1.3)-(1.5)$, $x^*, y^*, z^* \in F(S)$ *and* $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ *are the sequences generated by Algorithm 2.3. If* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are three sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=0}^{\infty}(1-\beta_n) < \infty$;
(iii) $0 < s_n, t_n, r_n < \frac{2(v-u\mu^2)}{\mu^2}$;
(iv) $v > u\mu^2$,

then the sequences $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *converge strongly to* x^*, y^* *and* z^*, *respectively.*

THEOREM 3.4. *Let* C *be a closed convex subset of a real Hilbert space* H. *Let* $T : C \times C \times C \to H$ *be a relaxed* (u,v)*-cocoerceive and* μ*-Lipschitz continuous mapping in the first variable and* $S : C \to C$ *be an asymptotically nonexpansive mapping. Suppose that* $x^*, y^*, z^* \in C$ *are solutions of the SNVI problems* $(1.3)-(1.5)$, $x^*, y^*, z^* \in F(S)$ *and* $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ *are the sequences generated by Algorithm 2.4. If* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are three sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < s_n, t_n, r_n < \frac{2(v-u\mu^2)}{\mu^2}$;
(iii) $v > u\mu^2$,

then the sequences $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *converge strongly to* x^*, y^* *and* z^*, *respectively.*

THEOREM 3.5. *Let* C *be a closed convex subset of a real Hilbert space* H. *Let* $T : C \to H$ *be a relaxed* (u,v)*-cocoerceive and* μ*-Lipschitz continuous mapping in the first variable and* $S : C \to C$ *be an asymptotically nonexpansive mapping. Suppose that* $x^* \in C$ *is a solution of the NVI problem* (1.8), $x^* \in F(S)$ *and* $\{x_n\}$ *is a sequence generated by Algorithm 2.5. If* $\{\alpha_n\}$ *is a sequence in* $[0,1]$ *satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < s_n, t_n, r_n < \frac{2(v-u\mu^2)}{\mu^2}$;
(iii) $v > u\mu^2$,

then the sequence $\{x_n\}$ *converges strongly to* x^*.

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