Mathematical
Inequalities
& Applications
Volume 12, Number 2 (2009), 403–411

REVERSE INEQUALITIES OF ERDÖS–MORDELL TYPE

MEA BOMBARDELLI AND SHANHE WU

(*Communicated by V. Volenec*)

Abstract. This paper deals with the reverse inequalities of Erdös-Mordell type. Our result contains as special case the following reverse Erdös-Mordell inequality:

$$R_1+R_2+R_3<\sqrt{2}(\rho_1+\rho_2+\rho_3),$$

where R_i and ρ_i (i=1,2,3) denote respectively the distances from an interior point Q of $\triangle A_1A_2A_3$ to the vertexes A_1, A_2, A_3 and to the circumcenters of $\triangle A_2QA_3$, $\triangle A_3QA_1$, $\triangle A_1QA_2$. Some other closely related inequalities are also considered.

1. Introduction

For a given triangle $A_1A_2A_3$, let A_i and a_i (i=1,2,3) denote respectively the vertices and its opposite sides. Let R_i and r_i (i=1,2,3) represent respectively the distances from an interior point Q of $\triangle A_1A_2A_3$ to the vertex A_i and to the side opposite to A_i .

In 1935, P. Erdös proposed [1] the following conjectured inequality as an *Open Problem*:

$$R_1+R_2+R_3 \geqslant 2(r_1+r_2+r_3). \tag{1}$$

Inequality (1) was first proved by Mordell and Barrow in 1937 [2], and since then, this inequality is known as the Erdös–Mordell's inequality. Over the past years, the Erdös–Mordell's inequality has received considerable attention from researchers in the fields of geometry, and has drawn a large number of research papers involving its new proofs, various generalizations, variations and applications etc. Some related results with historical comments on the Erdös–Mordell's inequality can be found in [3] to [28]. We recall here some improved forms of the Erdös–Mordell's inequality involving weights and exponents.

In 2001, Dar and Gueron [29] gave us for positive numbers $\lambda_1, \lambda_2, \lambda_3$ the following weighted Erdös–Mordell's inequality:

$$\lambda_1R_1+\lambda_2R_2+\lambda_3R_3 \geqslant 2\sqrt{\lambda_1\lambda_2\lambda_3}\left(\frac{r_1}{\sqrt{\lambda_1}}+\frac{r_2}{\sqrt{\lambda_2}}+\frac{r_3}{\sqrt{\lambda_3}}\right). \tag{2}$$

Mathematics subject classification (2000): 26D05, 26D15, 51M16.

Keywords and phrases: Erdös-Mordell's inequality, Oppenheim's inequality, triangle, interior point, weight, exponent.

Paper MIA-12-31

In 2004, Janous [30] generalized Dar-Gueron's inequality (2) by introducing an exponential parameter, as follows

$$\lambda_1 R_1^t + \lambda_2 R_2^t + \lambda_3 R_3^t \geqslant 2^{\min\{t,1\}} \sqrt{\lambda_1 \lambda_2 \lambda_3} \left(\frac{r_1^t}{\sqrt{\lambda_1}} + \frac{r_2^t}{\sqrt{\lambda_2}} + \frac{r_3^t}{\sqrt{\lambda_3}} \right), \tag{3}$$

where $\lambda_1, \lambda_2, \lambda_3$ and t are positive numbers.

In a recent paper [31], the second author sharpened Janous's inequality (3) in the following form:

$$\lambda_1 R_1^t + \lambda_2 R_2^t + \lambda_3 R_3^t \geqslant 2^{\min\{t,1\}} \sqrt{\lambda_1 \lambda_2 \lambda_3} \left(\frac{w_1^t}{\sqrt{\lambda_1}} + \frac{w_2^t}{\sqrt{\lambda_2}} + \frac{w_3^t}{\sqrt{\lambda_3}} \right), \tag{4}$$

where $\lambda_1, \lambda_2, \lambda_3$ and t are positive numbers, w_1, w_2, w_3 denote respectively the lengths of the bisectors of $\angle A_2QA_3$, $\angle A_3QA_1$, $\angle A_1QA_2$ from Q to its intersection with the sides of $\triangle A_1A_2A_3$.

The purpose of this paper is to establish a new class of inequalities of Erdös–Mordell type, we show that several interesting inequalities including the reverse Erdös–Mordell's inequality and reverse Oppenheim's inequality can be obtained as direct consequences of our result.

Our main result is stated in the following theorem:

THEOREM 1. *Suppose Q is an interior point of $\triangle A_1A_2A_3$, the bisectors of $\angle A_2QA_3$, $\angle A_3QA_1$, $\angle A_1QA_2$ intersect respectively the circumcircles of $\triangle A_2QA_3$, $\triangle A_3QA_1$, $\triangle A_1QA_2$ in the points A_1', A_2', A_3'. Let $QA_1 = R_1$, $QA_2 = R_2$, $QA_3 = R_3$, $QA_1' = \ell_1$, $QA_2' = \ell_2$, $QA_3' = \ell_3$. Then for $\lambda_i > 0$ $(i = 1,2,3)$ and $t > 0$, we have the inequality*

$$\lambda_1 R_1^t + \lambda_2 R_2^t + \lambda_3 R_3^t < 2^{-\min\{\frac{t}{2}, 1-\frac{t}{2}\}} (\lambda_1 \lambda_2 \lambda_3) \left(\frac{\ell_1^t}{\lambda_1^2} + \frac{\ell_2^t}{\lambda_2^2} + \frac{\ell_3^t}{\lambda_3^2} \right). \tag{5}$$

For $\lambda_i > 0$ $(i = 1,2,3)$ and $t < 0$, we have the inequality

$$\lambda_1 R_1^t + \lambda_2 R_2^t + \lambda_3 R_3^t \geqslant 2^{\min\{-t,1\}} \sqrt{\lambda_1 \lambda_2 \lambda_3} \left(\frac{\ell_1^t}{\sqrt{\lambda_1}} + \frac{\ell_2^t}{\sqrt{\lambda_2}} + \frac{\ell_3^t}{\sqrt{\lambda_3}} \right). \tag{6}$$

When $t = -1$, equality holds in (6) if and only if $a_1 : a_2 : a_3 = \sqrt{\lambda_1} : \sqrt{\lambda_2} : \sqrt{\lambda_3}$ and Q is the circumcenter of $\triangle A_1A_2A_3$; when $-1 < t < 0$, equality holds in (6) if and only if $\triangle A_1A_2A_3$ is equilateral, Q is its center and $\lambda_1 = \lambda_2 = \lambda_3$; when $t < -1$, (6) is a strict inequality.

2. Lemmas

In order to prove Theorem 1, we need the following lemmas.

LEMMA 1. *Let Q be an interior point of $\triangle A_1A_2A_3$, and let $\angle A_2QA_3 = 2\alpha_1$, $\angle A_3QA_1 = 2\alpha_2$, $\angle A_1QA_2 = 2\alpha_3$. Then, under the definitions of R_i and ℓ_i ($i = 1, 2, 3$) in Theorem* 1*, we have the following identities*

$$\ell_1 = \frac{R_2+R_3}{2\cos\alpha_1}, \qquad \ell_2 = \frac{R_3+R_1}{2\cos\alpha_2}, \qquad \ell_3 = \frac{R_1+R_2}{2\cos\alpha_3}. \tag{7}$$

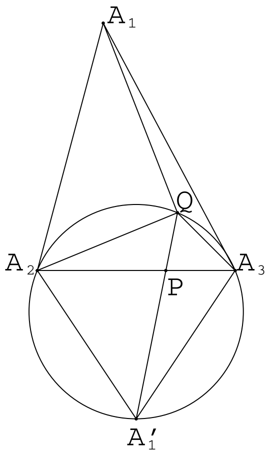

Figure 1: Lemma 1.

Proof. Using Ptolemy's equality for cyclic quadrangle $QA_2A_1'A_3$ (see Figure 1) :

$$A_2A_3 \cdot QA_1' = QA_2 \cdot A_3A_1' + QA_3 \cdot A_2A_1',$$

we obtain

$$QA_1' = A_2Q \cdot \left(\frac{A_3A_1'}{A_2A_3}\right) + A_3Q \cdot \left(\frac{A_2A_1'}{A_2A_3}\right). \tag{8}$$

On the other hand, by the law of sine, we have

$$\frac{A_3A_1'}{A_2A_3} = \frac{\sin\angle A_3A_2A_1'}{\sin\angle A_2A_1'A_3} = \frac{\sin\alpha_1}{\sin(\pi-2\alpha_1)} = \frac{1}{2\cos\alpha_1},$$

$$\frac{A_2A_1'}{A_2A_3} = \frac{\sin\angle A_2A_3A_1'}{\sin\angle A_2A_1'A_3} = \frac{\sin\alpha_1}{\sin(\pi-2\alpha_1)} = \frac{1}{2\cos\alpha_1},$$

applying the above identities to (8) yields

$$\ell_1 = \frac{R_2+R_3}{2\cos\alpha_1}.$$

Similarly, the second and third identities in (7) can be proved. The proof of Lemma 1 is complete. □

LEMMA 2. (Power means inequality [31],[32]). *If* $a_i > 0$ $(i = 1,2,\dots,n)$ *and* $\lambda > 0$*, then*

$$\left(\sum_{i=1}^{n} a_i\right)^{\lambda} \geqslant n^{\min\{\lambda-1,\,0\}} \left(\sum_{i=1}^{n} a_i^{\lambda}\right), \tag{9}$$

with equality holding if and only if $\lambda = 1$*, or* $a_1 = a_2 = \cdots = a_n$ *for the case of* $0 < \lambda < 1$.

If $a_i > 0,\ \mu_i > 0, (i = 1,2,\dots,n)$ *and* $0 < \lambda \leqslant 1$*, then*

$$\sum_{i=1}^{n} \mu_i a_i^{\lambda} \leqslant \left(\sum_{i=1}^{n} \mu_i\right)^{1-\lambda} \left(\sum_{i=1}^{n} \mu_i a_i\right)^{\lambda}, \tag{10}$$

with equality holding if and only if $\lambda = 1$*, or* $a_1 = a_2 = \cdots = a_n$ *for the case of* $0 < \lambda < 1$.

LEMMA 3. *Let* $x_i > 0,\ 0 < \varphi_i < \frac{\pi}{2}$ $(i = 1,2,3)$ *and* $\varphi_1 + \varphi_2 + \varphi_3 = \pi$*. Then for* $\lambda > 0$ *the following inequality holds true*

$$x_2x_3\cos^{\lambda}\varphi_1 + x_3x_1\cos^{\lambda}\varphi_2 + x_1x_2\cos^{\lambda}\varphi_3 \leqslant 2^{-\min\{\lambda,1\}}\left(x_1^2 + x_2^2 + x_3^2\right). \tag{11}$$

Equality holds in (11) if and only if $x_2x_3\sin\varphi_1 = x_3x_1\sin\varphi_2 = x_1x_2\sin\varphi_3$ *for the case of* $\lambda = 1$*, or* $x_1 = x_2 = x_3$ *and* $\varphi_1 = \varphi_2 = \varphi_3$ *for the case of* $0 < \lambda < 1$.

Proof. Case (I): When $0 < \lambda \leqslant 1$. It follows from Lemma 2 and the arithmetic-geometric means inequality that

$$\begin{aligned}
&x_2x_3\cos^{\lambda}\varphi_1 + x_3x_1\cos^{\lambda}\varphi_2 + x_1x_2\cos^{\lambda}\varphi_3 \\
&\qquad \leqslant (x_2x_3 + x_3x_1 + x_1x_2)^{1-\lambda}(x_2x_3\cos\varphi_1 + x_3x_1\cos\varphi_2 + x_1x_2\cos\varphi_3)^{\lambda} \\
&\qquad \leqslant (x_1^2 + x_2^2 + x_3^2)^{1-\lambda}(x_2x_3\cos\varphi_1 + x_3x_1\cos\varphi_2 + x_1x_2\cos\varphi_3)^{\lambda} \\
&\qquad \leqslant 2^{-\lambda}(x_1^2 + x_2^2 + x_3^2).
\end{aligned}$$

The latter inequality follows from the well-known Wolstenholme's inequality (see [4, p. 421])

$$x_1^2 + x_2^2 + x_3^2 \geqslant 2x_2x_3\cos\varphi_1 + 2x_3x_1\cos\varphi_2 + 2x_1x_2\cos\varphi_3, \tag{12}$$

where $x_i > 0,\ 0 < \varphi_i < \frac{\pi}{2}$ $(i = 1,2,3)$ and $\varphi_1 + \varphi_2 + \varphi_3 = \pi$. Furthermore, the equality holds in (12) if and only if $x_2x_3\sin\varphi_1 = x_3x_1\sin\varphi_2 = x_1x_2\sin\varphi_3$.

Case (II): When $\lambda > 1$. we have

$$x_2x_3\cos^{\lambda}\varphi_1 + x_3x_1\cos^{\lambda}\varphi_2 + x_1x_2\cos^{\lambda}\varphi_3 < x_2x_3\cos\varphi_1 + x_3x_1\cos\varphi_2 + x_1x_2\cos\varphi_3,$$

Now, using the Wolstenholme's inequality (12) leads us to

$$x_2x_3\cos^{\lambda}\varphi_1 + x_3x_1\cos^{\lambda}\varphi_2 + x_1x_2\cos^{\lambda}\varphi_3 < 2^{-1}(x_1^2 + x_2^2 + x_3^2).$$

The Lemma 3 is proved. □

3. Proof the main result (Theorem 1)

In our proof of Theorem 1, we consider the following two cases.

Case (I): When $t>0$. By applying Lemma 1 and Lemma 2, we obtain

$$\begin{aligned}\frac{\ell_1^t}{\lambda_1^2}+\frac{\ell_2^t}{\lambda_2^2}+\frac{\ell_3^t}{\lambda_3^2}&=\frac{1}{\lambda_1^2}\left(\frac{R_2+R_3}{2\cos\alpha_1}\right)^t+\frac{1}{\lambda_2^2}\left(\frac{R_3+R_1}{2\cos\alpha_2}\right)^t+\frac{1}{\lambda_3^2}\left(\frac{R_1+R_2}{2\cos\alpha_3}\right)^t\\ &\geqslant 2^{\min\{t-1,0\}-t}\left(\frac{R_2^t+R_3^t}{\lambda_1^2\cos^t\alpha_1}+\frac{R_3^t+R_1^t}{\lambda_2^2\cos^t\alpha_2}+\frac{R_1^t+R_2^t}{\lambda_3^2\cos^t\alpha_3}\right)\\ &=2^{\min\{-t,-1\}}\left[\left(\frac{\lambda_2^{-2}}{\cos^t\alpha_2}+\frac{\lambda_3^{-2}}{\cos^t\alpha_3}\right)R_1^t\right.\\ &\quad\left.+\left(\frac{\lambda_3^{-2}}{\cos^t\alpha_3}+\frac{\lambda_1^{-2}}{\cos^t\alpha_1}\right)R_2^t+\left(\frac{\lambda_1^{-2}}{\cos^t\alpha_1}+\frac{\lambda_2^{-2}}{\cos^t\alpha_2}\right)R_3^t\right].\end{aligned}\tag{13}$$

On the other hand, it follows from the arithmetic-geometric means inequality that

$$\frac{\lambda_1^{-2}}{\cos^t\alpha_1}+\frac{\lambda_2^{-2}}{\cos^t\alpha_2}\geqslant\frac{2\lambda_1^{-1}\lambda_2^{-1}}{(\cos\alpha_1\cos\alpha_2)^{\frac{t}{2}}}.\tag{14}$$

From $\alpha_1+\alpha_2+\alpha_3=\pi$, $0<\alpha_1<\frac{\pi}{2}$, $0<\alpha_2<\frac{\pi}{2}$ and $0<\alpha_3<\frac{\pi}{2}$, we conclude that

$$\frac{\pi}{2}<\alpha_1+\alpha_2<\pi,$$

we thus have

$$\begin{aligned}\cos\alpha_1\cos\alpha_2&=\frac{1}{2}\left[\cos(\alpha_1-\alpha_2)+\cos(\alpha_1+\alpha_2)\right]\\ &\leqslant\frac{1}{2}\left[1+\cos(\alpha_1+\alpha_2)\right]<\frac{1}{2}.\end{aligned}\tag{15}$$

Combining inequalities (14) and (15) gives

$$\frac{\lambda_1^{-2}}{\cos^t\alpha_1}+\frac{\lambda_2^{-2}}{\cos^t\alpha_2}>\frac{2^{\frac{2+t}{2}}}{\lambda_1\lambda_2}.$$

Similarly to the above, we can obtain

$$\frac{\lambda_2^{-2}}{\cos^t\alpha_2}+\frac{\lambda_3^{-2}}{\cos^t\alpha_3}>\frac{2^{\frac{2+t}{2}}}{\lambda_2\lambda_3},$$

and

$$\frac{\lambda_3^{-2}}{\cos^t\alpha_3}+\frac{\lambda_1^{-2}}{\cos^t\alpha_1}>\frac{2^{\frac{2+t}{2}}}{\lambda_3\lambda_1}.$$

Applying the above inequalities to (13) gives

$$\frac{\ell_1^t}{\lambda_1^2}+\frac{\ell_2^t}{\lambda_2^2}+\frac{\ell_3^t}{\lambda_3^2} > 2^{\min\{\frac{t}{2},1-\frac{t}{2}\}}\left(\frac{1}{\lambda_2\lambda_3}R_1^t+\frac{1}{\lambda_3\lambda_1}R_2^t+\frac{1}{\lambda_1\lambda_2}R_3^t\right),$$

which leads to the desired inequality (5).

Case (II): When $t<0$. It follows from Lemma 1 that

$$\begin{aligned}&\sqrt{\lambda_1\lambda_2\lambda_3}\left(\frac{\ell_1^t}{\sqrt{\lambda_1}}+\frac{\ell_2^t}{\sqrt{\lambda_2}}+\frac{\ell_3^t}{\sqrt{\lambda_3}}\right)\\&\quad=\sqrt{\lambda_2\lambda_3}\left(\frac{2\cos\alpha_1}{R_2+R_3}\right)^{-t}+\sqrt{\lambda_3\lambda_1}\left(\frac{2\cos\alpha_2}{R_3+R_1}\right)^{-t}+\sqrt{\lambda_1\lambda_2}\left(\frac{2\cos\alpha_3}{R_1+R_2}\right)^{-t}.\end{aligned}\tag{16}$$

Since $-t>0$, by using the arithmetic-geometric means inequality and the inequality given by Lemma 3, we obtain

$$\begin{aligned}&\sqrt{\lambda_2\lambda_3}\left(\frac{2\cos\alpha_1}{R_2+R_3}\right)^{-t}+\sqrt{\lambda_3\lambda_1}\left(\frac{2\cos\alpha_2}{R_3+R_1}\right)^{-t}+\sqrt{\lambda_1\lambda_2}\left(\frac{2\cos\alpha_3}{R_1+R_2}\right)^{-t}\\&\leqslant\sqrt{\lambda_2\lambda_3R_2^tR_3^t}\left(\cos^{-t}\alpha_1\right)+\sqrt{\lambda_3\lambda_1R_3^tR_1^t}\left(\cos^{-t}\alpha_2\right)+\sqrt{\lambda_1\lambda_2R_1^tR_2^t}\left(\cos^{-t}\alpha_3\right)\\&\leqslant 2^{-\min\{-t,1\}}\left(\lambda_1R_1^t+\lambda_2R_2^t+\lambda_3R_3^t\right).\end{aligned}\tag{17}$$

Combining inequalities (16) and (17) yields the inequality (6). The condition of equality in (6) can be deduced from the condition of equality in the arithmetic-geometric means inequality and inequality (11). This completes the proof of Theorem 1. □

4. Applications of Theorem 1

In this section, we show some consequences of Theorem 1. Putting $\lambda_1=\lambda_2=\lambda_3=1$ in (5), we obtain

COROLLARY 1. *For real numbers $t>0$, the following inequality holds true*

$$R_1^t+R_2^t+R_3^t<2^{-\min\{\frac{t}{2},1-\frac{t}{2}\}}\left(\ell_1^t+\ell_2^t+\ell_3^t\right).\tag{18}$$

For real numbers $t<0$, the following inequality holds true

$$R_1^t+R_2^t+R_3^t\geqslant 2^{\min\{-t,1\}}\left(\ell_1^t+\ell_2^t+\ell_3^t\right).\tag{19}$$

Choosing in particular $t=1$ in (18) yields the following reverse Erdös-Mordell's inequality:

COROLLARY 2.

$$R_1+R_2+R_3<\frac{1}{\sqrt{2}}\left(\ell_1+\ell_2+\ell_3\right).\tag{20}$$

REMARK. It is worth noticing that the coefficient $\frac{1}{\sqrt{2}}$ in (20) is best possible in the sense that it cannot be replaced by a smaller constant. In order to prove this statement we consider the inequality (20) in a general form as

$$R_1+R_2+R_3<\mu\,(\ell_1+\ell_2+\ell_3),$$

i.e.,

$$R_1+R_2+R_3<\mu\left(\frac{R_2+R_3}{2\cos\alpha_1}+\frac{R_3+R_1}{2\cos\alpha_2}+\frac{R_1+R_2}{2\cos\alpha_3}\right). \tag{21}$$

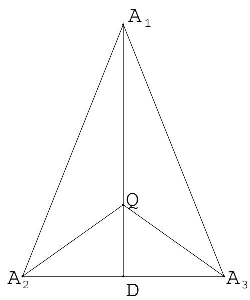

Figure 2: Remark

We construct an isosceles triangle $A_1A_2A_3$ (see Figure 2): let A_1D be the altitude from the vertex A_1 to the side A_2A_3, and let $A_1A_2=A_1A_3=1$, $\angle DA_1A_2=\angle DA_1A_3=\beta$, $\angle DA_2Q=\angle DA_3Q=\sqrt{\beta}$ $(0<\beta<\frac{\pi}{6})$. Then we have

$$R_1=QA_1=\frac{\cos(\beta+\sqrt{\beta})}{\cos\sqrt{\beta}},\quad R_2=QA_2=\frac{\sin\beta}{\cos\sqrt{\beta}},\quad R_3=QA_3=\frac{\sin\beta}{\cos\sqrt{\beta}},$$

$$\alpha_1=\frac{\pi}{2}-\sqrt{\beta},\quad \alpha_2=\frac{\pi}{4}+\frac{\sqrt{\beta}}{2},\quad \alpha_3=\frac{\pi}{4}+\frac{\sqrt{\beta}}{2}.$$

Now, substituting the above identities into (21) with a simple calculation yields that

$$\cos(\beta+\sqrt{\beta})+2\sin\beta\;<\;\mu\left[\frac{\sin\beta}{\sin\sqrt{\beta}}+\frac{\sin\beta+\cos(\beta+\sqrt{\beta})}{\cos(\frac{\pi}{4}+\frac{\sqrt{\beta}}{2})}\right]. \tag{22}$$

In (22), passing the limit as $\beta\to 0$, we find that $\mu>\frac{1}{\sqrt{2}}$. Thus the best possible values for μ in (21) is that $\mu=\frac{1}{\sqrt{2}}$, which implies that $\frac{1}{\sqrt{2}}$ is the best possible coefficient in (20).

Putting in the inequality (19) $t=-1$, we obtain the following result

COROLLARY 3.

$$\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\;\geqslant\;2\left(\frac{1}{\ell_1}+\frac{1}{\ell_2}+\frac{1}{\ell_3}\right). \tag{23}$$

It provides a reverse version of the following inequality of Oppenheim [6]:

$$\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3} \leqslant \frac{1}{2}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right). \tag{24}$$

In addition, let ρ_1, ρ_2, ρ_3 denote the distances from Q to the circumcenters of $\triangle A_2QA_3$, $\triangle A_3QA_1$, $\triangle A_1QA_2$ respectively. Then, from inequalities (20) and (23), the following reverse inequalities of Erdös-Mordell and Oppenheim type are derived:

COROLLARY 4.

$$R_1+R_2+R_3 < \sqrt{2}\,(\rho_1+\rho_2+\rho_3), \tag{25}$$

$$\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3} \geqslant \frac{1}{\rho_1}+\frac{1}{\rho_2}+\frac{1}{\rho_3}. \tag{26}$$

Acknowledgments. The research was supported, in part, by the Natural Science Foundation of Fujian province of China under Grant S0850023, and, in part, by the Science Foundation of Project of Fujian Province Education Department of China under Grant JA08231.

REFERENCES

[1] P. ERDÖS, *Problem 3740,* Amer. Math. Monthly, **42** (1935) 396.
[2] L. J. MORDELL, D. F. BARROW, *Solution of Problem 3740,* Amer. Math. Monthly, **44** (1937) 252–254.
[3] O. BOTTEMA, R. Z. DJORDJEVIĆ, R. R. JANIĆ, D. S. MITRINOVIĆ AND P. M. VASIĆ, *Geometric Inequalities.* Wolters-Noordhoff, Groningen, 1969.
[4] D. S. MITRINOVIĆ, J. E. PEČARIĆ AND V. VOLENEC, *Recent Advances in Geometric Inequalities.* Kluwer Academic Publishers, Dordrecht, Netherlands, 1989.
[5] D. S. MITRINOVIĆ, J. E. PEČARIĆ, V. VOLENEC AND J. CHEN, *Addenda to the Monograph: Recent Advances in Geometric Inequalities(I).* Journal of Ningbo University, **4**(2) (1991).
[6] A. OPPENHEIM, *The Erdös inequality and other inequalities for a triangle,* Amer. Math. Monthly, **68** (1961) 226–230.
[7] V. KOMORNIK, *A short proof of the Erdös–Mordell Theorem,* Amer. Math. Monthly, **104** (1997) 57–60.
[8] R. A. SATNOIANU, *Erdös–Mordell type inequality in a triangle,* Amer. Math. Monthly, **110** (2003) 727–729.
[9] H. C. LENHARD, *Verallgemeinerung und Verschärfung der Erdös-Mordellschen Ungleichung für polygone,* Arch. Math., **12** (1961) 311–314.
[10] S. GUERON, I. SHAFRIR, *A weighted Erdös–Mordell inequality for polygons,* Amer. Math. Monthly, **112** (2005) 257–263.
[11] D. S. MITRINOVIĆ, J. E. PEČARIĆ, *On the Erdös–Mordell's inequality for a polygon,* J. College Arts Sci., Chiba Univ., B-19 (1986) 3–6.
[12] A. AVEZ, *A short proof of a theorem of Erdös and Mordell,* Amer. Math. Monthly, **100** (1993) 60–62.
[13] L. BANKOFF, *An elementary proof of the Erdös–Mordell theorem,* Amer. Math. Monthly, **65** (1958) 521.
[14] N. DERGIADES, *Signed distances and the Erdös–Mordell inequality,* Forum Geom., **4** (2004) 67–68.
[15] D. K. KAZARINOFF, *A simple proof of the Erdös–Mordell inequality for triangles,* Michigan Math. J., **4** (1957) 97–98.

[16] N. D. KAZARINOFF, *D. K. Kazarinoff's inequality for tetrahedra,* Michigan Math. J., **4** (1957) 99–104.
[17] N. D. KAZARINOFF, *Geometric Inequalities,* New York: Random House, 1961, pp. 78-87.
[18] H. EHRET, *An approach to trigonometric inequalities,* Amer. Math. Monthly, **77** (1970) 254–257.
[19] H. LEE, *Another proof of the Erdös–Mordell theorem,* Forum Geom., **1** (2001) 7–8.
[20] G. STEENSHOLT, *Note on an elementary property of triangles,* Amer. Math. Monthly, **63** (1956) 571–572.
[21] C. ALSINA, R. B. NELSEN, *A Visual Proof of the Erdös–Mordell inequality,* Forum Geom., **7** (2007) 99–102.
[22] L. CARLITZ, *Some inequalities for a triangle,* Amer. Math. Monthly, **71** (1964) 881–885.
[23] H. BRABANT, *The Erdös–Mordell inequality again,* Nieuw Tijdschr. Wisk., **46** (1958/1959) 87.
[24] H. S. M. COXETER, *Introduction to Geometry, 2nd ed.* New York: Wiley, 1969, pp. 9.
[25] L. FEJES TÓTH, *Lagerungen in der Ebene, auf der Kugel und im Raum*, 2nd ed. Berlin: Springer-Verlag, 1953.
[26] D. G. KONTOGIANNIS, *Equalities and Inequalities in the Triangle.* Athens: Ekpaideutikis, 1996, pp. 127–128.
[27] L. J. MORDELL, *On Geometric Problems of Erdös and Oppenheim.* Math. Gaz. 46 (1962) 213-215.
[28] G. R. VELDKAMP, *The Erdös–Mordell inequality.* Nieuw Tijdschr. Wisk., **45** (1957/1958) 193–196.
[29] S. DAR, S. GUERON, *A weighted Erdös–Mordell inequality*, Amer. Math. Monthly, **108** (2001) 165–168.
[30] W. JANOUS, *Further inequalities of Erdös–Mordell type*, Forum Geom., **4** (2004) 203–206.
[31] S. WU, L. DEBNATH, *Generalization of the Wolstenholme cyclic inequality and its application*, Computers & Math. Appl. **53**(1) (2007) 104–114.
[32] S. WU, *Generalization and sharpness of the power means inequality and their applications*, J. Math. Anal. Appl., **312** (2), (2005) 637–652.

(Received July 18, 2007)

Mea Bombardelli
Department of Mathematics
University of Zagreb
Bijenička cesta 30
HR-10000 Zagreb
Croatia
e-mail: Mea.Bombardelli@math.hr

Shanhe Wu
Department of Mathematics
Longyan College
Longyan Fujian 364012
People's Republic of China
e-mail: wushanhe@yahoo.com.cn