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GENERALIZATIONS OF SOME CLASSICAL RESULTS UNDER MVBV CONDITION

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Abstract. The present paper applies MVBV condition to generalize certain classical results for asymptotic relation for a certain trigonometric series, and for an important trigonometric inequality. The new results contain all previous corresponding results. By citing a theorem in [7], we also show that MVBV condition is the ultimate generalization for the trigonometric inequality.

1. Introduction

A great number of interesting results in Fourier analysis are established by assuming monotonicity of coefficients, therefore, how to generalize monotonicity condition has become an important subject. Recently, Le and Zhou [2] proposed a new natural unified condition to generalize all the monotonicity, various quasi-monotonicity conditions and RBV condition (first raised by Leindler [4]). The GBV condition they introduced is the following

DEFINITION 1. Let $\mathbf{C} = \{C_n\}_{n=1}^{\infty}$ be a complex sequence satisfying $C_n \in \{z : |\arg z| \leqslant \theta_0\}$ for some $\theta_0 \in [0, \pi/2)$, and there exists a natural number l_0 such for any m that

$$\sum_{n=m}^{2m} |\Delta C_n| \leqslant M(\mathbf{C}) \max_{m \leqslant n < m+l_0} |C_n|, \tag{1}$$

where $\Delta C_n := C_n - C_{n+1}$, $M(\mathbf{C})$ is a positive constant only depending upon the sequence $\mathbf{C}$, then we say $\mathbf{C} = \{C_n\}_{n=1}^{\infty}$ is a Group Bounded Variation Sequence, denoted by $\{C_n\}_{n=1}^{\infty} \in GBVS$.

The following two results are established by using the GBV condition to generalize the quasimonotonicity and RBV condition.

THEOREM 1.1. (Le and Zhou [3]) *If a positive sequence* $\mathbf{C} = \{C_n\}_{n=1}^{\infty}$ *satisfies*

$$nC_n \leqslant K \qquad n = 1, 2, \cdots, \tag{2}$$

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where K is a positive constant, and the GBV condition (1) $(\mathbf{C} \in GBVS)$*, suppose* $\{n_m\}$ *satisfies*

$$\sum_{j=m}^{\infty} \frac{1}{n_j} \leqslant \frac{A}{n_m}, \qquad m = 1, 2, \cdots, A > 1, \tag{3}$$

where $\{n_m\}$ *is a subsequence of natural numbers satisfying* $1 = n_1 < n_2 < n_3 < \cdots$*, then for any* x *we have*

$$\sum_{j=1}^{\infty} \left| \sum_{k=n_j}^{n_{j+1}-1} C_k \sin kx \right| \leqslant K_1(\mathbf{C})A,$$

where $K_1(\mathbf{C})$ *is a positive constant depending upon* $\{C_n\}$ *only.*

REMARK. Theorem 1.1 is a generalization to an important inequality

$$\sup_{m \geqslant 1} \left| \sum_{k=1}^{m} \frac{\sin kx}{k} \right| \leqslant 3\sqrt{\pi}$$

for all x. For the history and references, readers could check [3] and the references therein.

THEOREM 1.2. (Zhou and Le [6]) *Let* $\mathbf{C} = \{C_n\}_{n=0}^{\infty}$ *be a complex sequence satisfying the GBV condition* (1)*, and* $\omega(t)$ *be a nondecreasing function on* $[0, \infty)$ *with* $0 = \omega(0) < \omega(a) \leqslant \omega(a+b) \leqslant \omega(a) + \omega(b)$ *for* $a, b > 0$*, and* $\lim_{t \to 0+} \omega(t) = \omega(0)$ *which also satisfies the following two equivalence conditions*

$$t \int_t^1 \frac{\omega(u)}{u^2} du = O(\omega(t)), \int_0^t \frac{\omega(u)}{u} du = O(\omega(t)). \tag{4}$$

If $C_n / \omega(n^{-1}) \to A$ *as* $n \to \infty$*, then*

$$f(x) = \sum_{n=0}^{\infty} C_n e^{inx} \sim A \sum_{n=1}^{\infty} \omega(n^{-1}) e^{inx}$$

as $x \to 0$.

REMARK. Theorem 1.2 is a generalization of a classical asymptotic result of Hardy and the related references could be found in [6].

Very recently, S. P. Zhou, P. Zhou and Yu ([7]) gives a further generalization to monotonicity called as MVBV (Mean Value Bounded Variation) condition. They proved this condition is the weakest one to generalize monotonicity (i.e., GBVS $\subsetneq$ MVBVS) and cannot be weakened further in uniform convergence for trigonometric series.

DEFINITION 2. A complex sequence $\mathbf{C} := \{C_n\}_{n=1}^{\infty}$ is said to be a Mean Value Bounded Variation Sequence ($\{C_n\}_{n=1}^{\infty} \in MVBVS$) if $C_n \in \{z : |\arg z| \leqslant \theta_0\}$ for some $\theta_0 \in [0, \pi/2)$, and there is a $\lambda \geqslant 2$ such that

$$\sum_{k=n}^{2n} |\Delta C_n| \leqslant \frac{M(\mathbf{C})}{n} \sum_{k=[\lambda^{-1}n]}^{[\lambda n]} |C_k|, \tag{5}$$

where $M(\mathbf{C})$ is a positive constant only depending upon the sequence $\mathbf{C}$.

REMARK. There are some related independent antecedents from Leindler (see [8]–[10]) to the above mentioned ultimate MVBV condition.

In this paper, we establish the following corresponding theorems under MVBV condition.

THEOREM 1. *Let $\mathbf{C} = \{C_n\}_{n=0}^{\infty} \in MVBVS$ be a positive sequence satisfying (2) and (5), If $\{n_m\}$ satisfies (3), then for any x we have*

$$\sum_{j=1}^{\infty} \left| \sum_{k=n_j}^{n_{j+1}-1} C_k \sin kx \right| \leqslant K_1(C)A.$$

THEOREM 2. *Let $\mathbf{C} = \{C_n\}_{n=0}^{\infty} \in MVBVS$ be a complex sequence satisfying (5), and $\omega(t)$ be as the same in Theorem 1.2. If $C_n/\omega(n^{-1}) \to A$ as $n \to \infty$, then*

$$f(x) = \sum_{n=0}^{\infty} C_n e^{inx} \sim A \sum_{n=1}^{\infty} \omega(n^{-1}) e^{inx}$$

as $x \to 0$.

Throughout the paper, we always use M or $M(x)$ to indicate a positive constant or a positive constant depending upon x only, their value may vary in different occurrences.

2. Proofs

Proof of Theorem 1. The cases $x = 0$ or $x = \pi$ are trivial. Let $x \in (0, \pi)$, select n and i in turn such that $\frac{\pi}{n+1} \leqslant x < \frac{\pi}{n}$, $n_i \leqslant n < n_{i+1}$. Since $kC_k \leqslant K$, we have

$$\begin{aligned}\sum_{j=1}^{i-1} \left| \sum_{k=n_j}^{n_{j+1}-1} C_k \sin kx \right| + \left| \sum_{k=n_i}^{n} C_k \sin kx \right| &\leqslant \sum_{j=1}^{i-1} \sum_{k=n_j}^{n_{j+1}-1} C_k \sin kx + \sum_{k=n_i}^{n} C_k \sin kx \\ &\leqslant \sum_{j=1}^{n} C_k kx \leqslant Knx < K\pi.\end{aligned}$$

We know that

$$\left| \sum_{k=p}^{q} \sin kx \right| \leqslant \frac{A^*}{x}, \ q \geqslant p \geqslant 0.$$

Write

$$I := \left|\sum_{k=n+1}^{n_{i+1}-1} C_k \sin kx\right| + \sum_{j=i+1}^{\infty}\left|\sum_{k=n_j}^{n_{j+1}-1} C_k \sin kx\right|,$$

by using Abel's transformation we see that

$$\begin{aligned} I &= \left|\sum_{k=n+1}^{n_{i+1}-2} \triangle C_k \sum_{j=n+1}^{k} \sin jx + C_{n_{i+1}-1}\sum_{j=n+1}^{n_{i+1}-1} \sin jx\right| \\ &\quad + \sum_{j=i+1}^{\infty}\left|\sum_{k=n_j}^{n_{j+1}-2} \triangle C_k \sum_{l=n_j}^{k} \sin lx + C_{n_{j+1}-1}\sum_{l=n_j}^{n_{j+1}-1} \sin lx\right|, \end{aligned}$$

hence

$$\begin{aligned} I &\leqslant \frac{A^*}{x}\left(\sum_{k=n+1}^{n_{i+1}-2} |\Delta C_k| + C_{n_{i+1}-1}\right) + \frac{A^*}{x}\sum_{j=i+1}^{\infty}\left(\sum_{k=n_j}^{n_{j+1}-2} |\triangle C_k| + C_{n_{j+1}-1}\right) \\ &=: I_1 + I_2. \end{aligned}$$

Noticing that $nC_n \leqslant K$ implies that $C_n \to 0$, $n \to \infty$, we get

$$C_{n_{i+1}-1} = \sum_{k=n_{i+1}-1}^{\infty} \Delta C_k \leqslant \sum_{k=n_{i+1}-1}^{\infty} |\Delta C_k| \leqslant \sum_{k=n}^{\infty} |\Delta C_k|.$$

We first estimate I_1. According to the condition, one has

$$\begin{aligned} I_1 &\leqslant \frac{2A^*}{x}\sum_{j=0}^{\infty}\sum_{k=2^j n}^{2^{j+1}n} |\triangle C_k| \\ &\leqslant \frac{2A^*}{x}\frac{M(C)}{n}\sum_{j=0}^{\infty}\frac{1}{2^j}\sum_{k=[\lambda^{-1}2^j n]}^{[\lambda 2^j n]} C_k \\ &\leqslant \frac{2A^*}{x}\frac{M(C)K}{n}\sum_{j=0}^{\infty}\frac{1}{2^j}\sum_{k=[\lambda^{-1}2^j n]}^{[\lambda 2^j n]} \frac{1}{k} \\ &\leqslant \frac{2A^*}{x}\frac{M(C,\lambda)K}{n}\sum_{j=0}^{\infty}\frac{1}{2^j} \leqslant 2A^* M(C,\lambda)K. \end{aligned}$$

For I_2, a similar argument leads to that

$$C_{n_{j+1}-1} \leqslant \sum_{k=n_{j+1}-1}^{\infty} |\Delta C_k| \leqslant \sum_{k=n_j}^{\infty} |\Delta C_k|,$$

therefore

$$
\begin{aligned}
I_2 &\leqslant \frac{2A^*}{x}\sum_{j=i+1}^{\infty}\sum_{l=0}^{\infty}\sum_{k=2^l n_j}^{2^{l+1}n_j}|\triangle C_k| \\
&\leqslant \frac{2A^*}{x}M(C)\sum_{j=i+1}^{\infty}\sum_{l=0}^{\infty}\frac{1}{2^l n_j}\sum_{k=[\lambda^{-1}2^l n_j]}^{[\lambda 2^l n_j]} C_k \\
&\leqslant \frac{2A^*}{x}M(C)K\sum_{j=i+1}^{\infty}\sum_{l=0}^{\infty}\frac{1}{2^l n_j}\sum_{k=[\lambda^{-1}2^l n_j]}^{[\lambda 2^l n_j]}\frac{1}{k} \\
&\leqslant \frac{2A^*}{x}M(C,\lambda)K\sum_{j=i+1}^{\infty}\frac{1}{n_j}\sum_{l=0}^{\infty}\frac{1}{2^l} \\
&\leqslant \frac{2A^*}{x}M(C,\lambda)K\sum_{j=i+1}^{\infty}\frac{1}{n_j} \\
&\leqslant \frac{2A^*}{x}M(C,\lambda)K\frac{A}{n_{i+1}} \leqslant 2A^*M(C,\lambda)KA.
\end{aligned}
$$

Combining the above estimates, we have proved the required inequality. □

To prove Theorem 2, we will use the following two lemmas.

LEMMA 1. (Borwein and Zhou [5, Lemma 6.2]) *Let* $\omega(t)$ *be a modulus of continuity satisfying* (4). *Then there exists a constant* $0 < M_1 < 1$ *such that for all* t, $0 < t \leqslant 1$,

$$\frac{\omega(M_1 t)}{\omega(t)} \leqslant \frac{1}{2}.$$

LEMMA 2. (Xie and Zhou [6, Lemma 3]) *Let* $\omega(t)$ *be a modulus of continuity satisfying* (4). *Then there is a positive constant* M_0 *depending upon* $\omega(t)$ *only such that*

$$\left|\sum_{n=1}^{\infty}\omega(n^{-1})e^{inx}\right| \geqslant M_0\frac{\omega(|x|)}{|x|}$$

as $x \to 0$.

Proof of Theorem 2. Given $\varepsilon > 0$. Suppose $x > 0$ without loss of generality. Write

$$
\begin{aligned}
f(x) &= A\sum_{n=1}^{\infty}\omega(n^{-1})e^{inx} + \Big(C_0 + \sum_{n=1}^{N}(C_n - A\omega(n^{-1}))e^{inx}\Big) \\
&\quad + \sum_{n=N+1}^{\infty} C_n e^{inx} - A\sum_{n=N+1}^{\infty}\omega(n^{-1})e^{inx} \\
&=: I_1 + I_2 + I_3 + I_4,
\end{aligned}
$$

where we take $N = [1/(\varepsilon x)]$, and $[x]$ is the greatest integer not exceeding x. Since $C_n/\omega(n^{-1}) \to A$ as $n \to \infty$, there exists an N_0 such that

$$|C_n - A\omega(n^{-1})| \leqslant \varepsilon^2\omega(n^{-1})$$

for all $n > N_0$, it follows that

$$I_2 \leqslant |C_0| + \left(\sum_{n=1}^{N_0} + \sum_{N_0+1}^{N}\right) |C_n - A\omega(n^{-1})||e^{inx}| \leqslant MN_0 + M\varepsilon^2 \sum_{n=1}^{N} \omega(n^{-1}).$$

By using the first equivalence condition of $\omega(x)$, we have

$$\sum_{n=1}^{N} \omega(n^{-1}) \leqslant M \int_{1/N}^{1} \frac{\omega(u)}{u^2} du \leqslant MN\omega(N^{-1}).$$

Note that $\omega(x)/x \to \infty$ as $x \to 0+$ also from (4), we have

$$|I_2| \leqslant MN_0 + M\varepsilon^2 N\omega(N^{-1}) \leqslant M\varepsilon x^{-1}\omega(x).$$

Next, we estimate I_3. By Abel's transformation and condition (5),

$$\begin{aligned}
|I_3| &= \left|\sum_{k=N+1}^{\infty} \Delta C_k \sum_{v=1}^{k} e^{ivx} - C_{N+1} \sum_{v=1}^{N} e^{ivx}\right| \\
&\leqslant \sum_{k=N+1}^{\infty} |\Delta C_k| \left|\sum_{v=1}^{k} e^{ivx}\right| + |C_{N+1}| \left|\sum_{v=1}^{N} e^{ivx}\right| \\
&\leqslant Mx^{-1} \sum_{k=N+1}^{\infty} |\Delta C_k| + Mx^{-1} |C_{N+1}| \\
&= Mx^{-1} \sum_{j=0}^{\infty} \sum_{2^j N \leqslant k < 2^{j+1} N} |\Delta C_k| + Mx^{-1} |C_{N+1}| \\
&\leqslant M(\mathbf{C})x^{-1} \sum_{j=0}^{\infty} \frac{1}{2^j N} \sum_{k=[\lambda^{-1} 2^j N]}^{[\lambda 2^j N]} |C_k| + Mx^{-1} |C_{N+1}| \\
&\leqslant M(\mathbf{C})x^{-1} \left(\omega(N^{-1}) + \sum_{j=0}^{\infty} \frac{1}{2^j N} \sum_{k=[\lambda^{-1} 2^j N]}^{[\lambda 2^j N]} \omega(k^{-1})\right) \\
&\leqslant M(\mathbf{C})x^{-1} \left(\omega(N^{-1}) + \sum_{j=0}^{\infty} \omega(\frac{\lambda}{2^j N})\right).
\end{aligned}$$

Applying Lemma 1 and noting that $N = [1/(\varepsilon x)]$, we have

$$\omega\left(\frac{\lambda}{2^j N}\right) \leqslant \left(\frac{1}{2}\right)^{\log(2^{-j}\lambda)/\log M_1} \omega(N^{-1}) \leqslant M_2 \left(\frac{1}{2}\right)^{M_3 j} \omega(N^{-1}),$$

where $M_2 = \left(\frac{1}{2}\right)^{\log\lambda/\log M_1}$, $M_3 = -\frac{\log 2}{\log M_1} > 0$, M_1 is the constant appeared in Lemma 1. By Lemma 1 again,

$$\omega(N^{-1}) \leqslant \left(\frac{1}{2}\right)^{\log\varepsilon/\log M_1} \omega(x),$$

hence

$$|I_3| \leqslant M(\mathbf{C},\lambda)x^{-1}\varepsilon_0\omega(x),$$

where $\varepsilon_0 := \left(\frac{1}{2}\right)^{\log\varepsilon/\log M_1}$, $\log\varepsilon/\log M_1 > 0$.

Since $\omega(n^{-1})$ is decreasing, we deal with I_4 similarly (but more easily) to obtain that

$$|I_4| \leqslant M(\mathbf{C},\lambda)\varepsilon_0 x^{-1}\omega(x).$$

Summing up all the above estimates, we obtain that

$$\left| f(x) - A\sum_{n=1}^{\infty}\omega(n^{-1})e^{inx}\right| \leqslant M(\varepsilon_0+\varepsilon)x^{-1}\omega(x).$$

Finally, from Lemma 2 we have that

$$|I_1| = \left|\sum_{n=1}^{\infty}\omega(n^{-1})e^{inx}\right| \geqslant M_0\frac{\omega(|x|)}{|x|}$$

as $x\to 0$. Thus we have proved Theorem 2. □

3. A Final Remark

In this section, by citing a theorem by S. P. Zhou, P. Zhou and Yu [7], we show that the MVBV condition cannot be weakened any further for Theorem 1 to hold.

THEOREM 3. *Let S_n be a given nonnegative increasing sequence tending to infinity. Then for any given $\lambda \geqslant 2$, there exits a positive sequence $\mathbf{C} = \{C_n\}_{n=0}^{\infty}$ satisfying*

$$\lim_{n\to\infty}\frac{\sum_{k=n}^{2n}|\Delta C_k|}{\frac{S_n}{n}\sum_{k=[\lambda^{-1}n]}^{\lambda n}C_k} = 0, \tag{6}$$

however, the trigonometric inequality in Theorem 1 does not hold for some sequence $\{n_m\}$ satisfying (3).

Proof. S. P. Zhou, P. Zhou and Yu [7] constructed the following sequence $\{C_n\}$: Assume that $S_1 \geqslant 10$, therefore $S_j \geqslant 10$ for all $j \geqslant 1$. Set $n_1 = 1, n_2 = 10$, and $n_{j+1} = 2[S_{4n_j}^{1/2}]n_j$ for $j = 2,3,\cdots$. Let

$$C_k = 1, \quad 1 \leqslant k < 40.$$

For $j \geqslant 2$ and $k = 1,2,\cdots,2[S_{4n_j}^{1/2}]-1$, let

$$\begin{aligned} C_m &= \frac{1}{\sqrt{\log S_{4n_j}}}\frac{1}{m}, \quad \text{if} \quad 4kn_j \leqslant m < (4k+2)n_j, \\ C_m &= \frac{1}{8\sqrt{\log S_{4n_j}}}\frac{1}{m}, \quad \text{if} \quad (4k+2)n_j \leqslant m < 4(k+1)n_j. \end{aligned}$$

Among the others, they proved that (6) holds for this sequence $\{C_n\}$, and

$$\left|\sum_{k=n_j}^{n_{j+1}-1} C_k \sin kt_j\right| \geqslant M\sqrt{\log S_{4n_j}}$$

by setting $t_j = \pi/(2n_j)$. Thus

$$\sup_{-\infty<x<\infty} \sum_{j=1}^{\infty}\left|\sum_{k=n_j}^{n_{j+1}-1} C_k \sin kx\right| \geqslant \left|\sum_{k=n_j}^{n_{j+1}-1} C_k \sin kt_j\right| \geqslant M\sqrt{\log S_{4n_j}}$$

for all $j = 1, 2, \cdots$, i. e., the conclusion of the Theorem 1 cannot hold in this case. Futhermore, the construction also implies that inequality (2) or (3) is satisfied for such a sequence $\{C_n\}$ or $\{n_m\}$.

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