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FUNDAMENTAL ITERATED CONVOLUTION INEQUALITIES IN WEIGHTED L_p SPACES AND THEIR APPLICATIONS

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Abstract. In this paper, we obtain the inequalities for the iterated convolution and their applications to physical problems. We also get the inequality

$$\left\| \sum_m \left(\prod_{j=1}^r {}^*(F_{m,j}\rho_{m,j}) \right) \left(\prod_{j=1}^r {}^*|\rho_{m,j}| \right)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}^n)} \leqslant \sum_m \prod_{j=1}^r \|F_{m,j}\|_{L_p(\mathbb{R}^n,|\rho_{m,j}|)}$$

and its applications in $L_p(\mathbb{R}^n,|\rho|)$ space.

1. Introduction

In a series of papers, ([2], [9], [10], [11]) some new type norm inequalities in convolutions in some several weighted L_2 spaces using the theory of reproducing kernels are derived. In particular, S. Saitoh ([9]) gave iterated inequalities in the convolution in L_2 space.

Let $f \in L_p(\mathbb{R})$, $g \in L_q(\mathbb{R})$, and $p^{-1}+q^{-1} > 1$. Then, Young's inequality (see [12]) says that the Fourier convolution

$$f * g := \int_{-\infty}^{\infty} f(y)g(x-y)\,dy$$

belongs to $L_r(\mathbb{R})$, where $r^{-1} = p^{-1}+q^{-1}-1$, and moreover,

$$\|f*g\|_r \leqslant \|f\|_p\|g\|_q.$$

Surprisingly enough, S. Saitoh ([8]) gave convolution norm inequalities in the form

$$\|f*g\|_p \leqslant \|f\|_p\|g\|_p,$$

by considering the L_p - norms in more naturally determined weighted spaces.

This type inequality will be very convenient for various applications for the "same" L_p norms.

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In our recent publications ([6], [7]), we have given several new type of convolution inequalities in weighted $L_p(\mathbb{R}^n, |\rho|)(p > 1)$ spaces in the following form

$$\left\| ((F_1\rho_1) * (F_2\rho_2))(\rho_1 * \rho_2)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}^n)} \leqslant \|F_1\|_{L_p(\mathbb{R}^n,|\rho_1|)} \|F_2\|_{L_p(\mathbb{R}^n,|\rho_2|)}. \tag{1.1}$$

For our specific purpose, we will establish some fundamental iterated convolution inequalities in $L_p(\mathbb{R}^n, |\rho|)$ space and give several their applications to physical problems.

2. Fundamental iterated convolution inequalities

Throughout this paper, for brevity of presentation we shall use the following notations.

By $\mathbb{R}^n$ we denote the $n-$ *dimensional Euclidean space*, $n \in \mathbb{N}$. This is the set of all $n-$ tuples of real numbers, $\mathbf{x} = (x_1, ..., x_n)$, $x_j \in \mathbb{R}$, $j = 1, 2, ..., n$ with the linear operations

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, ..., x_n + y_n), \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \tag{2.2}$$

$$\lambda \mathbf{x} = (\lambda x_1, ..., \lambda x_n), \quad \lambda \in \mathbb{R}, \quad \mathbf{x} \in \mathbb{R}^n, \tag{2.3}$$

the scalar product

$$\mathbf{xy} = x_1 y_1 + \cdots + x_n y_n, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \tag{2.4}$$

and the norm

$$|\mathbf{x}| = (\mathbf{xx})^{\frac{1}{2}} = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}, \quad \mathbf{x} \in \mathbb{R}^n. \tag{2.5}$$

Now let $\mathbf{z}, \boldsymbol{\alpha} \in \mathbb{R}^n$. Then we set

$$\mathbf{z}^{\boldsymbol{\alpha}} = \prod_{j=1}^{n} z_j^{\alpha_j}. \tag{2.6}$$

For brevity we write

$$\int_{\mathbb{R}^n} f(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1, ..., x_n) dx_1 \cdots dx_n. \tag{2.7}$$

Let $f(\mathbf{x}) * g(\mathbf{x})$ is called the convolution (see [1]) of $f(\mathbf{x})$ and $g(\mathbf{x})$ and is defined by the integral

$$(f * g)(\mathbf{x}) := \int_{\mathbb{R}^n} f(\mathbf{z}) g(\mathbf{x} - \mathbf{z}) d\mathbf{z}. \tag{2.8}$$

We shall denote by $\prod_{j=1}^{q} *F_j$ the iterated convolution of F_j, that is

$$(\prod_{j=1}^{r} *F_j)(\mathbf{x}) := (F_1 * F_2 * \cdots * F_r)(\mathbf{x}). \tag{2.9}$$

Then, we obtain the following fundamental results:

THEOREM 1. *For some non-vanishing functions* $\rho_j(\mathbf{x})(j = 1, 2, ..., r)$ *belonging to* $L_1(\mathbb{R}^n, d\mathbf{x})$ *and for* $p > 1$ *we have the* L_p *weighted inequality for iterated convolution*

$$\left\| \left(\prod_{j=1}^{r} *(F_j\rho_j) \right) \left(\prod_{j=1}^{r} *\rho_j \right)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}^n)} \leqslant \prod_{j=1}^{r} \|F_j\|_{L_p(\mathbb{R}^n, |\rho_j|)} \tag{2.10}$$

for functions $F_j \in L_p(\mathbb{R}^n, |\rho_j|)$. *Equality holds for* F_j *if and only if* F_j *are represented in the form*

$$F_j(\mathbf{x}) = C_j e^{\boldsymbol{\alpha}\mathbf{x}}; \quad C_j: \text{ constants,} \tag{2.11}$$

where $\boldsymbol{\alpha} \in \mathbb{R}^n$ *is a constant such that* $F_j \in L_p(\mathbb{R}^n, |\rho_j(\mathbf{x})|d\mathbf{x})(j = 1, 2, ..., r)$.

This Theorem will be proved by application of the following lemma:

LEMMA 1. *If* $\rho_j(\mathbf{x})$ $(j = 1, 2, ..., r)$ *are some non-vanishing functions belonging to* $L_1(\mathbb{R}^n, d\mathbf{x})$ *and for* $p > 1$ *then*

$$\left| \Big(\prod_{j=1}^{r} *(F_j\rho_j) \Big)(\mathbf{x}) \right|^p \leqslant \left\{ \Big(\prod_{j=1}^{r} *|\rho_j| \Big)(\mathbf{x}) \right\}^{p-1} \Big(\prod_{j=1}^{r} *(|F_j|^p|\rho_j|) \Big)(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^n. \tag{2.12}$$

Here, equality holds if and only if

$$F_j(\mathbf{x}) = C_j e^{\boldsymbol{\alpha}\mathbf{x}}; \quad C_j: \text{ constants.} \tag{2.13}$$

Proof. We use induction on r. When $r = 2$, the inequality (2.12) is reduced to the Hölder's inequality. Equality holds if and only if for a function $k(\mathbf{x})$ in $\mathbf{x}$

$$F_1(\mathbf{y})F_2(\mathbf{x} - \mathbf{y}) = k(\mathbf{x}) \quad \text{a.e.} \quad \text{on} \quad \mathbb{R}^n$$

or, so is (2.13) for $r = 2$.

Now suppose (2.12) and (2.13) hold for some integer $r \geqslant 2$. We claim that they also hold for $r + 1$.

For all $\mathbf{x} \in \mathbb{R}^n$, put

$$f(\mathbf{y}) = \left\{ \Big(\prod_{j=1}^{r} *|\rho_j| \Big)(\mathbf{y}) |\rho_{r+1}(\mathbf{x} - \mathbf{y})| \right\}^{\frac{p-1}{p}}, \quad \forall \mathbf{y} \in \mathbb{R}^n$$

and

$$g(\mathbf{y}) = \left\{ \Big(\prod_{j=1}^{r} *(|F_j|^p|\rho_j|) \Big)(\mathbf{y}) |\rho_{r+1}(\mathbf{x} - \mathbf{y})| \right\}^{\frac{1}{p}} |F_{r+1}(\mathbf{x} - \mathbf{y})|, \quad \forall \mathbf{y} \in \mathbb{R}^n.$$

By induction hypothesis, we arrive at

$$\begin{aligned}\left|\Big(\prod_{j=1}^{r+1}*(F_j\rho_j)\Big)(\mathbf{x})\right| &\leqslant \Big(\Big[\prod_{j=1}^{r}*|F_j\rho_j|\Big]*|F_{r+1}\rho_{r+1}|\Big)(\mathbf{x})\\ &= \int_{\mathbb{R}^n}\Big(\prod_{j=1}^{r}*|F_j\rho_j|\Big)(\mathbf{y})|F_{r+1}(\mathbf{x}-\mathbf{y})||\rho_{r+1}(\mathbf{x}-\mathbf{y})|d\mathbf{y}\\ &\leqslant \int_{\mathbb{R}^n} f(\mathbf{y})g(\mathbf{y})d\mathbf{y}.\end{aligned}$$

Application of the Hölder's inequality for $f(\mathbf{y})$ and $g(\mathbf{y})$ gives

$$\begin{aligned}\left(\int_{\mathbb{R}^n} f(\mathbf{y})g(\mathbf{y})d\mathbf{y}\right)^p &\leqslant \left\{\int_{\mathbb{R}^n} f^{\frac{p}{p-1}}(\mathbf{y})d\mathbf{y}\right\}^{p-1}\int_{\mathbb{R}^n} g^p(\mathbf{y})d\mathbf{y}\\ &= \left\{\Big(\prod_{j=1}^{r+1}*|\rho_j|\Big)(\mathbf{x})\right\}^{p-1}\Big(\prod_{j=1}^{r+1}*(|F_j|^p|\rho_j|)\Big)(\mathbf{x}).\end{aligned}$$

Hence, we have

$$\left|\Big(\prod_{j=1}^{r+1}*(F_j\rho_j)\Big)(\mathbf{x})\right|^p \leqslant \left\{\Big(\prod_{j=1}^{r+1}*|\rho_j|\Big)(\mathbf{x})\right\}^{p-1}\Big(\prod_{j=1}^{r+1}*(|F_j|^p|\rho_j|)\Big)(\mathbf{x}). \tag{2.14}$$

Equality holds in (2.14) if and only if

$$\left|\Big(\prod_{j=1}^{r}*(F_j\rho_j)\Big)(\mathbf{y})\right|^p = \left\{\Big(\prod_{j=1}^{r}*|\rho_j|\Big)(\mathbf{y})\right\}^{p-1}\Big(\prod_{j=1}^{r}*(|F_j|^p|\rho_j|)\Big)(\mathbf{y}) \tag{2.15}$$

and

$$[f(\mathbf{y})]^{\frac{p}{p-1}}\int_{\mathbb{R}^n}[g(\mathbf{y})]^p d\mathbf{y} = [g(\mathbf{y})]^p\int_{\mathbb{R}^n}[f(\mathbf{y})]^{\frac{p}{p-1}}d\mathbf{y}. \tag{2.16}$$

By induction hypothesis, (2.15) and (2.16), we obtain

$$F_j(\mathbf{y}) = C_j e^{\boldsymbol{\alpha}\mathbf{y}}, \quad C_j: \text{ constants } (j = 1, 2, ..., r) \tag{2.17}$$

and

$$e^{\boldsymbol{\alpha}\mathbf{y}}F_{r+1}(\mathbf{x}-\mathbf{y}) = h(\mathbf{x}), \quad a.e. \quad \text{on } \mathbb{R}^n \tag{2.18}$$

for a function $h(\mathbf{x})$ in $\mathbf{x}$. From this functional equation, we have

$$F_{r+1}(\mathbf{y}) = C_{r+1}e^{\boldsymbol{\alpha}\mathbf{y}}, \quad C_{r+1}: \text{ constant}$$

and so the assertion follows. □

Proof of Theorem 1. In view of the lemma above, there is

$$\left\| \left(\prod_{j=1}^{r} *(F_j\rho_j) \right) \left(\prod_{j=1}^{r} *\rho_j \right)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}^n)} \leqslant \left\{ \int_{\mathbb{R}^n} \left(\prod_{j=1}^{r} *(|F_j|^p|\rho_j|) \right)(\mathbf{x})d\mathbf{x} \right\}^{\frac{1}{p}}.$$

Using the Fubini's theorem we have immediately (2.10). Equality holds if and only if equality holds in Lemma 1, so we have (2.11). □

In Theorem 1, in many cases the convolution will be given in the form

$$\rho_r \equiv 1, \quad \text{and} \quad F_r(\mathbf{x}-\mathbf{z}) = G(\mathbf{x}-\mathbf{z})$$

for some Green's function $G(\mathbf{x}-\mathbf{z})$. Then we have the inequality

$$\left\| \left(\prod_{j=1}^{r} *(F_j\rho_j) \right) * G \right\|_{L_p(\mathbb{R}^n)} \leqslant \|G\|_{L_p(\mathbb{R}^n)} \prod_{j=1}^{r} \|\rho_j\|_{L_1(\mathbb{R}^n)}^{1-\frac{1}{p}} \prod_{j=1}^{r} \|F_j\|_{L_p(\mathbb{R}^n,|\rho_j|)}. \tag{2.19}$$

By considering the inequality (2.19) in the L_2-weighted space, we also obtain several inequalities for the Fourier transform. We shall denote by $\mathfrak{F}\{f\}$ the Fourier transform of a function f, that is

$$\mathfrak{F}\{f\}(\mathbf{x}) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} e^{-i\mathbf{x}\mathbf{y}} f(\mathbf{y})d\mathbf{y}. \tag{2.20}$$

Application of Parseval's equality and convolution theorem for the Fourier transform gives

$$\left\| \left(\prod_{j=1}^{r} *(F_j\rho_j) \right) * G \right\|_{L_2(\mathbb{R}^n)} = \left(\sqrt{2\pi}\right)^{rn} \left\| \mathfrak{F}\{G\} \prod_{j=1}^{r} \left(\mathfrak{F}\{F_j\rho_j\} \right) \right\|_{L_2(\mathbb{R}^n)}. \tag{2.21}$$

Combining (2.19) and (2.21) yields

$$\left\| \mathfrak{F}\{G\} \prod_{j=1}^{r} \left(\mathfrak{F}\{F_j\rho_j\} \right) \right\|_{L_2(\mathbb{R}^n)} \leqslant \frac{\|G\|_{L_2(\mathbb{R}^n)}}{\left(\sqrt{2\pi}\right)^{rn}} \prod_{j=1}^{r} \|\rho_j\|_{L_1(\mathbb{R}^n)}^{\frac{1}{2}} \prod_{j=1}^{r} \|F_j\|_{L_2(\mathbb{R}^n,|\rho_j|)}. \tag{2.22}$$

In (2.22), for $r=1$, we can obtain L_2-weighted integral estimates for the solutions of partial differential equations whose representations are given by

$$u = \mathfrak{F}^{-1}\left\{ \mathfrak{F}\{F\rho\}\mathfrak{F}\{G\} \right\},$$

namely,

$$\begin{aligned} \|u\|_{L_2(\mathbb{R}^n)}^2 &\leqslant \frac{1}{(2\pi)^n} \|G\|_{L_2(\mathbb{R}^n)}^2 \|\rho\|_{L_1(\mathbb{R}^n)} \|F\|_{L_2(\mathbb{R}^n,|\rho|)}^2 \\ &= \frac{1}{(2\pi)^n} \|\mathfrak{F}\{G\}\|_{L_2(\mathbb{R}^n)}^2 \|\rho\|_{L_1(\mathbb{R}^n)} \|F\|_{L_2(\mathbb{R}^n,|\rho|)}^2. \end{aligned} \tag{2.23}$$

For example, we consider the equation

$$u_{xxxx} - u_{yy} + k^2 u = F(x,y)\rho(x,y), \quad (x,y) \in \mathbb{R}^2, \quad k \in \mathbb{R}_+. \tag{2.24}$$

Using the Fourier transform, we state that

$$\mathfrak{F}\{u\}(x,y) = \frac{\mathfrak{F}\{F\rho\}(x,y)}{x^4 + y^2 + k^2}. \tag{2.25}$$

Then, we have

$$\|u\|^2_{L_2(\mathbb{R}^2)} \leqslant \frac{3}{8\sqrt{2}(\sqrt{k})^5 \pi} B\left(\frac{1}{2}, \frac{5}{4}\right) \|\rho\|_{L_1(\mathbb{R}^2)} \|F\|^2_{L_2(\mathbb{R}^2, |\rho|)} \tag{2.26}$$

for $\rho \in L_1(\mathbb{R}^2, dxdy)$ and for $F \in L_2(\mathbb{R}^2, |\rho(x,y)|dxdy)$.

We next use our results above to derive an important extension of Saitoh's inequality in [11].

COROLLARY 1. *For any functions* $F_j \in L_p\big(\mathbb{R}, (a_j^2\xi^2 + b_j^2)^{-1} d\xi\big) (j = 1, ..., r)$*, and for the iterated convolution* $\prod_{j=1}^{r} *$*, we have the inequality*

$$\int_{-\infty}^{\infty} \left| \left(\prod_{j=1}^{r} * \frac{F_j(\xi)}{a_j^2 \xi^2 + b_j^2} \right)(\xi) \right|^p \left\{ \left(\sum_{j=1}^{r} b_j \prod_{i \neq j} a_i \right)^2 + \xi^2 \left(\prod_{j=1}^{r} a_j \right)^2 \right\}^{p-1} d\xi$$
$$\leqslant \left\{ \pi^{r-1} \sum_{j=1}^{r} \prod_{i \neq j} \frac{a_i}{b_i} \right\}^{p-1} \prod_{j=1}^{r} \int_{-\infty}^{\infty} \frac{|F_j(\xi)|^p}{a_j^2 \xi^2 + b_j^2} d\xi. \tag{2.27}$$

Equality holds for F_j *if and only if* F_j *are represented in the form*

$$F_j(x) = C_j e^{\alpha x}; \quad C_j: \text{ constants}, \tag{2.28}$$

where $\alpha \in \mathbb{R}$ *is a constant such that* $F_j \in L_p\big(\mathbb{R}, (a_j^2\xi^2 + b_j^2)^{-1} d\xi\big) (j = 1, ..., r)$.

In particular, for $r = 2, p = 2$*, we have (see more [11])*

$$\left[\frac{1}{2}\left(\frac{a_1}{b_1} + \frac{a_2}{b_2}\right)\right]^{-1} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \left(\frac{F_1(\xi)}{a_1^2\xi^2 + b_1^2} * \frac{F_2(\xi)}{a_2^2\xi^2 + b_2^2} \right)(\xi) \right|^2 \cdot \left\{ (a_1a_2)^2 \xi^2 + (a_1b_2 + a_2b_1)^2 \right\} d\xi$$
$$\leqslant \int_{-\infty}^{\infty} \frac{|F_1(\xi)|^2}{a_1^2\xi^2 + b_1^2} d\xi \int_{-\infty}^{\infty} \frac{|F_2(\xi)|^2}{a_2^2\xi^2 + b_2^2} d\xi. \tag{2.29}$$

Proof. In Theorem 1, take

$$\rho_j(\xi) = \frac{1}{a_j^2 \xi^2 + b_j^2}, \quad j = 1, ..., r, \quad \xi \in \mathbb{R},$$

we have

$$\mathfrak{F}\{\rho_j\}(\xi) = \sqrt{\frac{\pi}{2}}\frac{1}{a_j b_j}\exp\left\{-\frac{b_j}{a_j}|\xi|\right\}.$$

Hence, by using the convolution theorem, we obtain

$$\begin{aligned}
\Big(\prod_{j=1}^{r} *\rho_j\Big)(\xi) &= (\sqrt{2\pi})^{r-1}\mathfrak{F}^{-1}\left\{\prod_{j=1}^{r}\mathfrak{F}\{\rho_j\}\right\}(\xi) \\
&= (\sqrt{2\pi})^{r-1}\left(\sqrt{\frac{\pi}{2}}\right)^{r}\left(\prod_{j=1}^{r}\frac{1}{a_j b_j}\right)\mathfrak{F}^{-1}\left\{\exp\left(-|\xi|\sum_{j=1}^{r}\frac{b_j}{a_j}\right)\right\}(\xi) \\
&= \pi^{r-1}\left(\prod_{j=1}^{r}\frac{1}{a_j b_j}\right)\left(\sum_{j=1}^{r}\frac{b_j}{a_j}\right)\left(\xi^2+\left(\sum_{j=1}^{r}\frac{b_j}{a_j}\right)^2\right)^{-1} \\
&= \pi^{r-1}\left(\sum_{j=1}^{r}\prod_{i\neq j}\frac{a_i}{b_i}\right)\left(\xi^2+\left(\sum_{j=1}^{r}b_j\prod_{i\neq j}a_i\right)^2\right)^{-1}.
\end{aligned}$$

We then have the desired inequalities. □

We next generalize inequality (1.1) to a sum of iterated convolution:

THEOREM 2. *For some non-vanishing functions $\rho_{m,j}(\mathbf{x})(j=1,2,...,r)$ belonging to $L_1(\mathbb{R}^n, d\mathbf{x})$ and for $p>1$ we have the L_p weighted inequality for the sum of iterated convolution*

$$\left\|\sum_m\left(\prod_{j=1}^{r}*(F_{m,j}\rho_{m,j})\right)\left(\prod_{j=1}^{r}*|\rho_{m,j}|\right)^{\frac{1}{p}-1}\right\|_{L_p(\mathbb{R}^n)} \leqslant \sum_m\prod_{j=1}^{r}\|F_{m,j}\|_{L_p(\mathbb{R}^n,|\rho_{m,j}|)} \tag{2.30}$$

for $F_{m,j}(\mathbf{x}) \in L_p(\mathbb{R}^n, |\rho_{m,j}(\mathbf{x})|d\mathbf{x})(j=1,2,...,r)$ and we assume that the right hand side of (2.30) *is finite. Equality holds if and only if*

$$F_{m,j}(\mathbf{x}) = C_{m,j}e^{\boldsymbol{\alpha}\mathbf{x}} \text{ and } \Big(\prod_{j=1}^{r}*|\rho_{m,j}|\Big)(\mathbf{x}) = C_m\varphi(\mathbf{x}), \quad C_{m,j}, C_m: \textit{ constants}, \tag{2.31}$$

where $\varphi(\mathbf{x})$ some integrable function and $\boldsymbol{\alpha}\in\mathbb{R}^n$ is a constant such that $F_{m,j}\in L_p(\mathbb{R}^n,|\rho_{m,j}|d\mathbf{x})$ and the right hand side of (2.30) *is finite.*

We emphasize that the inequality (1.1) is a special case of inequality (2.30) when $m=1$ and $r=2$.

In Theorem 2, in many cases the convolution will be given in the form

$$\rho_{m,r}\equiv 1, \quad \text{and} \quad F_{m,r}(\mathbf{x}-\mathbf{z}) = G_m(\mathbf{x}-\mathbf{z})$$

for some Green's functions $G_m(\mathbf{x}-\mathbf{z})$. Then we have the inequality

$$\left\| \sum_m \left(\{ \prod_{j=1}^{r} *(F_{m,j}\rho_{m,j}) \} * G_m \right) \right\|_{L_p(\mathbb{R}^n)}$$

$$\leqslant \sum_m \left\{ \|G_m\|_{L_p(\mathbb{R}^n)} \prod_{j=1}^{r} \|\rho_{m,j}\|_{L_1(\mathbb{R}^n)}^{1-\frac{1}{p}} \prod_{j=1}^{r} \|F_{m,j}\|_{L_p(\mathbb{R}^n,|\rho_{m,j}|)} \right\}. \tag{2.32}$$

For $r=2$, from (2.32), we have

$$\left[\int_{\mathbb{R}^n} \left| \sum_m ((F_m\rho_m) * G_m)(\mathbf{x}) \right|^p d\mathbf{x} \right]^{\frac{1}{p}}$$

$$\leqslant \sum_m \left\{ \|G_m\|_{L_p(\mathbb{R}^n)} \|\rho_m\|_{L_1(\mathbb{R}^n)}^{\frac{p-1}{p}} \|F_m\|_{L_p(\mathbb{R}^n,|\rho_m|)} \right\}. \tag{2.33}$$

In general, in (2.33) we have a generalization

$$\left[\int_{\mathbf{c}}^{\mathbf{d}} \left| \sum_m ((F_m\rho_m) * G_m)(\mathbf{x}) \right|^p d\mathbf{x} \right]^{\frac{1}{p}}$$

$$\leqslant \sum_m \left\{ \left(\int_{\mathbb{R}^n} |\rho_m(\mathbf{z})|\, d\mathbf{z} \right)^{p-1} \int_{\mathbb{R}^n} |F_m(\mathbf{z})|^p |\rho_m(\mathbf{z}) d\mathbf{z} \int_{\mathbf{c}-\mathbf{z}}^{\mathbf{d}-\mathbf{z}} |G_m(\mathbf{x})|^p d\mathbf{x} \right\}^{\frac{1}{p}}. \tag{2.34}$$

Proof of Theorem 2.. Take

$$f_m(\mathbf{x}) = \big(\prod_{j=1}^{r} *(F_{m,j}\rho_{m,j}) \big)(\mathbf{x}) \left\{ \big(\prod_{j=1}^{r} *|\rho_{m,j}| \big)(\mathbf{x}) \right\}^{\frac{1}{p}-1}, \quad \mathbf{x} \in \mathbb{R}^n.$$

Application of the inequality of Hardy, Littlewood and Pólya ([5], pp. 148–150) for $f_m(\mathbf{x})$ gives

$$\left\| \sum_m \left(\prod_{j=1}^{r} *(F_{m,j}\rho_{m,j}) \right) \left(\prod_{j=1}^{r} *|\rho_{m,j}| \right)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}^n)}$$

$$\leqslant \sum_m \left\{ \left\| \left(\prod_{j=1}^{r} *(F_{m,j}\rho_{m,j}) \right) \left(\prod_{j=1}^{r} *\rho_{m,j} \right)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}^n)} \right\}. \tag{2.35}$$

Moreover, by using the inequality (2.10), we state that

$$\left\| \left(\prod_{j=1}^{r} *(F_{m,j}\rho_{m,j}) \right) \left(\prod_{j=1}^{r} *\rho_{m,j} \right)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}^n)} \leqslant \prod_{j=1}^{r} \|F_{m,j}\|_{L_p(\mathbb{R}^n,|\rho_{m,j}|)}. \tag{2.36}$$

Combining (2.35) and (2.36) yields (2.30).

Equality holds if and only if equality holds in (2.35) and equality in (2.36). So, we have (2.30). □

We next introduce several applications of the above convolution inequalities to physical problems. We obtain some estimates in the $L_p(p > 1)$-weighted space and especially in the case of $p = 2$ (see [1], [3]).

3. Applications

3.1. The Bernoulli-Euler Beam Equation

We consider the vertical deflection $u(x)$ of an infinite beam on an elastic foundation under the action of a prescribed vertical load $W(x)$. The deflection $u(x)$ satisfies the ordinary differential equation

$$EI\frac{d^4u}{dx^4} + \kappa u = W(x), \quad -\infty < x < \infty, \tag{3.37}$$

where EI is the flexural rigidity and κ is the foundation modulus of the beam. We find the solution assuming that $W(x)$ has a compact support and u, u', u'', u''' all tend to zero as $|x| \to \infty$. Put

$$a^4 = \frac{\kappa}{EI}, \quad F(x)\rho(x) = \frac{W(x)}{EI}.$$

By using the Fourier transform, we (see [3], pp. 63-64) obtain

$$\mathfrak{F}\{u\}(x) = \frac{\mathfrak{F}\{F\rho\}(x)}{x^4 + a^4}. \tag{3.38}$$

Then, we have the inequality

$$\|u\|^2_{L_2(\mathbb{R})} \leqslant \frac{3}{8\sqrt{2}a^7}\|\rho\|_{L_1(\mathbb{R})}\|F\|^2_{L_2(\mathbb{R},|\rho|)}, \tag{3.39}$$

where ρ is an $L_1(\mathbb{R}, dx)$ function and for functions $F \in L_2(\mathbb{R}, |\rho(x)|dx)$.

In the Bernoulli-Euler equation on an elastic foundation

$$EI\frac{\partial^4 u}{\partial x^4} + \kappa u + m\frac{\partial^2 u}{\partial t^2} = F(x)\rho(x)\delta(t), \quad -\infty < x < \infty, \quad t > 0, \tag{3.40}$$

with the initial data

$$u(x,0) = 0 \quad \text{and} \quad u_t(x,0) = 0, \tag{3.41}$$

we (see [3], pp. 245) have

$$\mathfrak{F}\{u(.,t)\}(x,t) = \frac{\mathfrak{F}\{F\rho\}(x)}{m}\left(\frac{\sin \alpha t}{\alpha}\right), \quad \alpha = (a^2x^4 + \omega^2)^{\frac{1}{2}}, \tag{3.42}$$

where

$$a^2 = \frac{EI}{m} \quad \text{and} \quad \omega^2 = \frac{\kappa}{m}.$$

From the inequality (2.23), we obtain

$$\|u(.,t)\|^2_{L_2(\mathbb{R})} \leqslant \frac{1}{4m\kappa}\sqrt{\frac{2\omega}{a}}\|\rho\|_{L_1(\mathbb{R})}\|F\|^2_{L_2(\mathbb{R},|\rho|)} \tag{3.43}$$

for $\rho \in L_1(\mathbb{R}, dx)$ and for $F \in L_2(\mathbb{R}, |\rho(x)|dx)$.

3.2. Diffusion of Vorticity from a Vortex Sheet

We consider the two-dimensional vorticity equation in the x, y plane given by

$$\zeta_t = \nu\Delta\zeta \tag{3.44}$$

with the initial condition

$$\zeta(x,y,0) = F(x,y)\rho(x,y), \tag{3.45}$$

where $\zeta = v_x - u_y$.

Application of the Fourier transform gives (see [3], pp. 117–118)

$$\mathfrak{F}\{\zeta(.,.,t)\}(x,y,t) = \mathfrak{F}\{F\rho\}(x,y)\exp\{-\nu(x^2+y^2)t\}. \tag{3.46}$$

Then, we have the inequality

$$\int_{\mathbb{R}^2}|\zeta(x,y,t)|^2 dxdy \leqslant \frac{1}{4\pi\nu t}\int_{\mathbb{R}^2}|\rho(x,y)|dxdy\int_{\mathbb{R}^2}|F(x,y)|^2|\rho(x,y)|dxdy, \tag{3.47}$$

where ρ is an $L_1(\mathbb{R}^2, dxdy)$ function and for $F \in L_2(\mathbb{R}^2, |\rho(x,y)|dxdy)$.

3.3. Helmholtz Equation

We next consider the Dirichlet problem for the Helmholtz equation in a half space of $\mathbb{R}^{n+1}$ (see [1], pp. 75-76), i.e. the determination of the bounded solution of

$$\Delta_{n+1}u(\mathbf{x}') + ku(\mathbf{x}') = 0, \quad \mathbf{x}' \in \mathbb{R}_+ \times \mathbb{R}^n, \quad k \in \mathbb{R}_+ \tag{3.48}$$

under the boundary value condition

$$u(0,\mathbf{x}) = F(\mathbf{x})\rho(\mathbf{x}), \quad F\rho \in L_1(\mathbb{R}^n). \tag{3.49}$$

Here, $\mathbf{x}' = (t, x_1, ..., x_n) = (t, \mathbf{x})$. Setting $U(t,\mathbf{x}) = \mathfrak{F}\{u(t,.)\}(t,\mathbf{x})$, we obtain

$$U(t,\mathbf{x}) = \mathfrak{F}\{F\rho\}(\mathbf{x})\exp\big\{-t(k^2+|\mathbf{x}|^2)^{1/2}\big\}. \tag{3.50}$$

So, we have the inequality

$$\begin{aligned}\int_{\mathbb{R}^n}|u(t,\mathbf{x})|^2 d\mathbf{x} \leqslant \left(\frac{\sqrt{n+1}}{t2\pi}\right)^n \exp\left\{-\frac{2tk}{\sqrt{n+1}}\right\}\\ \cdot\int_{\mathbb{R}^n}|\rho(\mathbf{x})|d\mathbf{x}\int_{\mathbb{R}^n}|F(\mathbf{x})|^2|\rho(\mathbf{x})|d\mathbf{x},\end{aligned} \tag{3.51}$$

where $\rho \in L_1(\mathbb{R}^n, d\mathbf{x})$, $F \in L_2(\mathbb{R}^n, |\rho(\mathbf{x})|d\mathbf{x})$.

3.4. Axisymmetric Heat Conduction Equation

We consider the bounded solution of the axisymmetric heat conduction equation

$$u_t = \kappa \left(u_{rr} + \frac{1}{r} u_r \right), \quad 0 \leqslant r < a, \quad t > 0, \tag{3.52}$$

with the initial and boundary data

$$u(r,0) = 0 \quad \text{for} \quad 0 < r < a, \tag{3.53}$$

$$u(r,t) = F(t)\rho(t) \quad \text{at } r = a \quad \text{for } t > 0, \tag{3.54}$$

where κ and a are constants.

Application of Laplace transform gives (see [3], pp. 217-219)

$$u(r,t) = \frac{2\kappa}{a} \sum_{n=1}^{\infty} \frac{\alpha_n J_0(r\alpha_n)}{J_1(a\alpha_n)} \int_0^t F(\tau)\rho(\tau) \exp\big(-\kappa(t-\tau)\alpha_n^2 \big) d\tau, \tag{3.55}$$

where the summation is taken over the positive roots α_n of $J_0(a\alpha) = 0$.

Then, we obtain the inequality

$$\|u(r,.)\|_{L_p(\mathbb{R}_+)} \leqslant \|\rho\|_{L_1(\mathbb{R}_+)}^{\frac{p-1}{p}} \|F\|_{L_p(\mathbb{R}_+,|\rho|)} \left(\frac{2\kappa}{a} \right) \left(\frac{1}{p\kappa} \right)^{\frac{1}{p}} \sum_{n=1}^{\infty} \left| \frac{J_0(r\alpha_n)}{J_1(a\alpha_n)} \right| \alpha_n^{1-\frac{2}{p}} \tag{3.56}$$

for $\rho \in L_1(\mathbb{R}_+, dt)$ and for $F \in L_p(\mathbb{R}_+, |\rho(t)|dt)$.

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