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CONCERNING THE INTERMEDIATE POINT IN THE MEAN VALUE THEOREM

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Abstract. If the function $f : I \to \mathbb{R}$ is differentiable on the interval $I \subseteq \mathbb{R}$, then for each $x, a \in I$, according to the mean value theorem, there exists a number $c(x)$ belonging to the open interval determined by x and a, and there exists a real number $\theta(x) \in]0,1[$ such that

$$f(x) - f(a) = (x-a) f^{(1)}(c(x))$$

and

$$f(x) - f(a) = (x-a) f^{(1)}(a + (x-a)\theta(x)).$$

In this paper we shall study the differentiability of the functions c and θ in a neighbourhood of a.

1. Introduction

The mean-value theorem of differential calculus, the theorem of finite variation, or I. J. Lagrange's theorem has one of the following forms:

THEOREM 1. (I. J. Lagrange-A. M. Ampere) *Let a and b be real numbers with $a < b$ and $f : [a,b] \to \mathbb{R}$. If*

(i) the function f is continuous on $[a,b]$,

(ii) the function f is differentiable on $]a,b[$,

then there exists at least one real number $c \in]a,b[$ such that

$$f(b) - f(a) = (b-a) f^{(1)}(c).$$

If we note $\theta = \dfrac{c-a}{b-a}$, then Theorem 1 becomes:

THEOREM 2. (I. J. Lagrange-A.M. Ampere) *Let a and b be two real numbers with $a < b$ and $f : [a,b] \to \mathbb{R}$ be a function. If*

(i) the function f is continuous on $[a,b]$,

(ii) the function f is differentiable on $]a,b[$,

then there exists at least one real number $\theta \in]0,1[$ such that

$$f(b) - f(a) = (b-a) f^{(1)}(a + (b-a)\theta).$$

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For some classes of functions $f:[a,b]\to\mathbb{R}$, continuous on $[a,b]$ and differentiable on $]a,b[$, the position of the points $c\in]a,b[$ and $\theta\in]0,1[$ from the mean value theorem may be better determined. For example:

$a)$ If $0<a<b<+\infty$, then for the function $f:[a,b]\to\mathbb{R}$, defined by

$$f(x)=\ln x,\ \text{for all } x\in[a,b],$$

we have [15]:

$$\frac{a\sqrt[3]{b}+b\sqrt[3]{a}}{\sqrt[3]{b}+\sqrt[3]{a}}<c<\frac{a+b}{2}\quad\text{and}\quad\sqrt[3]{a}<\theta<\frac{1}{2}$$

$b)$ If $0\leqslant a<b\leqslant\frac{\sqrt{3}}{3}$, then for the function $f:[a,b]\to\mathbb{R}$, defined by

$$f(x)=\ \operatorname{arctg} x,\ \text{for all } x\in[a,b],$$

we have:

$$\frac{a+b}{2}<c<\sqrt{\frac{a^2+b^2}{2}},\ \text{and}\ \frac{1}{2}<\theta<\frac{(a+b)\sqrt{2}}{2\left(a\sqrt{2}+\sqrt{a^2+b^2}\right)}.$$

$c)$ For the function $f:\mathbb{R}\to\mathbb{R}$, defined by

$$f(x)=a_3x^3+a_2x^2+a_1x+a_0,\ \text{for all } x\in\mathbb{R},$$

$(a_3,a_2,a_1,a_0\in\mathbb{R},\ a_3\neq 0)$, D. Pompeiu [18] proved that there exists an interval $[a^*,b^*]\subseteq\,]a,b[$, such that $a^*\leqslant c\leqslant b^*$. Moreover, the subinterval of the smallest length has

$$a^*=\frac{a+b}{2}-\frac{b-a}{2}w,\ \text{and}\ b^*=\frac{a+b}{2}+\frac{b-a}{2}w,\ \text{where}\ w=\frac{\sqrt{3}}{3}.$$

In this case

$$\frac{1}{2}-\frac{w}{2}<\theta<\frac{1}{2}+\frac{w}{2}.$$

$d)$ I. Tchakaloff [22] proved that, if $f:\mathbb{R}\to\mathbb{R}$ is a polynomial function of the degree n, i.e.

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0,\ \text{for all}\ x\in\mathbb{R},$$

$(a_n,a_{n-1},\ldots,a_1,a_0\in\mathbb{R},\ a_n\neq 0)$, then

$$\frac{a+b}{2}-\frac{b-a}{2}w\leqslant c\leqslant\frac{a+b}{2}+\frac{b-a}{2}w\ \text{and}\ \frac{1}{2}-\frac{w}{2}<\theta<\frac{1}{2}-\frac{w}{2},$$

where w is the biggest solution of the Legendre polynom P_m of the degree $m=\left[\frac{n+1}{2}\right]$.

But this paper has a different purpose.

REMARK 3. In the conditions of Theorems 1 and 2, if $f^{(1)}$ is injective on $]a,b[$, then the numbers c and θ are uniquely determined (see [6]).

Let now $I \subseteq \mathbb{R}$ be an interval, $a \in I$ and $f : I \to \mathbb{R}$ be a differentiable function on I. Then, according to Theorems 1 and 2, for each $x \in I \setminus \{a\}$, there exists at least one real number c_x belonging to the open interval determined by a and x and at least one $\theta_x \in]0,1[$ such that

$$f(x) - f(a) = (x-a)f^{(1)}(c_x) \tag{1}$$

and

$$f(x) - f(a) = (x-a)f^{(1)}(a + (x-a)\theta_x). \tag{2}$$

If the function $f^{(1)}$ is injective on I, then for each $x \in I \setminus \{a\}$, the numbers c_x and θ_x are uniquely determined, such that (1) and (2) hold. In this case, we can define the function $c : I \setminus \{a\} \to I \setminus \{a\}$ by

$$c(x) = c_x, \text{ for all } x \in I \setminus \{a\},$$

that verifies

$$f(x) - f(a) = (x-a)f^{(1)}(c(x)), \text{ for all } x \in I \setminus \{a\}$$

and the function $\theta : I \setminus \{a\} \to]0,1[$ by

$$\theta(x) = \theta_x \text{ for all } x \in I \setminus \{a\},$$

that verifies

$$f(x) - f(a) = (x-a)f^{(1)}(a + (x-a)\theta(x)), \text{ for all } x \in I \setminus \{a\}.$$

The two functions are linked by the relation

$$c(x) = a + (x-a)\theta(x), \quad \text{for all} \quad x \in I \setminus \{a\}.$$

The following results can be found in [6].

THEOREM 4. *Let $I \subseteq \mathbb{R}$ be an interval and a be an interior point of I. Let $f : I \to \mathbb{R}$ be a function which satisfies the following conditions:*

(i) *the function f is two times differentiable on I,*
(ii) *the function $f^{(2)}$ is continuous on I,*
(iii) $f^{(2)}(a) \neq 0$.
Then the following hold:

1^0 *There exists a real number $\delta > 0$ such that $]a-\delta, a+\delta[\subseteq I$,*

$$f^{(2)}(x) \neq 0, \textit{ for all } x \in]a-\delta, a+\delta[$$

and $f^{(1)}$ is injective on $]a-\delta, a+\delta[$.

2^0 *There exists a unique function $c :]a-\delta, a+\delta[\setminus \{a\} \to]a-\delta, a+\delta[\setminus \{a\}$ that verifies the relation*

$$f(x) - f(a) = (x-a)f^{(1)}(c(x)),$$

for all $x \in]a-\delta, a+\delta[\setminus \{a\}$.

3^0 *The function* c *has limit at the point* $x = a$ *and*

$$\lim_{x \to a} c(x) = a.$$

4^0 *There exists a unique function* $\theta :]a-\delta, a+\delta[\setminus \{a\} \to]0,1[$ *that verifies the relation*

$$f(x) - f(a) = (x-a) f^{(1)}(a + (x-a)\theta(x)),$$

for all $x \in]a-\delta, a+\delta[\setminus \{a\}$.

5^0 *The function* θ *has limit at the point* $x = a$ *and*

$$\lim_{x \to a} \theta(x) = \frac{1}{2}.$$

Theorem 4 remains true if the point $a \in I$ is an extremity of the interval I.

THEOREM 5. *Let* $I \subseteq \mathbb{R}$ *be an interval,* $a \in I$ *the left (respectively right) extremity of* I *and* $f : I \to \mathbb{R}$ *a function that satisfies the conditions:*

(i) f *is two times differentiable on* I,

(ii) $f^{(2)}$ *is continuous on* I,

(iii) $f^{(2)}(a) \neq 0$.

Then the following statements hold:

1^0 *There exists a real number* $\delta > 0$ *such that* $]a, a+\delta[\subseteq I$ *(respectively* $]a-\delta, a[\subseteq I$*),*

$$f^{(2)}(x) \neq 0, \text{ for all } x \in]a, a+\delta[\text{ (respectively } x \in]a-\delta, a[)$$

and $f^{(1)}$ *is injective on* $]a, a+\delta[$ *(respectively* $]a-\delta, a[$*).*

2^0 *There exists a unique function* $c :]a, a+\delta[\to]a, a+\delta[$ *(respectively* $c :]a-\delta, a[\to]a-\delta, a[$*) such that*

$$f(x) - f(a) = (x-a) f^{(1)}(c(x)),$$

for all $x \in]a, a+\delta[$ *(respectively* $x \in]a-\delta, a[$*).*

3^0 *The function* c *has a right-hand side limit (respectively a left-hand side limit) at the point* $x = a$ *and*

$$\lim_{\substack{x \to a \\ x > a}} c(x) = a \;\; \text{(respectively } \lim_{\substack{x \to a \\ x < a}} c(x) = a\text{)}.$$

4^0 *There exists a unique function* $\theta :]a, a+\delta[\to]0,1[$ *(respectively* $\theta :]a-\delta, a[\to]0,1[$*) such that*

$$f(x) - f(a) = (x-a) f^{(1)}(a + (x-a)\theta(x))$$

for all $x \in]a, a+\delta[$ *(respectively* $x \in]a-\delta, a[$*).*

5^0 *The function* θ *has a right-hand side limit (respectively a left-hand side limit) at the point* $x = a$ *and*

$$\lim_{\substack{x \to a \\ x > a}} \theta(x) = \frac{1}{2} \;\; \text{(respectively } \lim_{\substack{x \to a \\ x < a}} \theta(x) = \frac{1}{2}\text{)}.$$

Other properties of the functions c and θ can be found in the papers [6], [8], [9], [13], [15].

In this paper we shall study the differentiability, of the first and higher orders, of the functions c and θ in a neighbourhood of the point a.

2. Preliminaries

LEMMA 6. *Let $J \subseteq I \subseteq \mathbb{R}$ be two intervals and $f : I \to \mathbb{R}$ be a function differentiable on I. Denote by $f^{(1)}(J)$ the set of values for the function $f^{(1)}$ on J. If $a \in J$, then*

$$\frac{f(x)-f(a)}{x-a} \in f^{(1)}(J), \text{ for all } x \in J, x \neq a.$$

Proof. Let $x \in J$, $x \neq a$. According to Theorem 1, applied to the function f on the interval with extremities a and x, there exists c_x in the open interval determined by a and x such that

$$\frac{f(x)-f(a)}{x-a} = f^{(1)}(c_x).$$

But $f^{(1)}(c_x) \in f^{(1)}(J)$ and therefore Lemma 6 is proved.

THEOREM 7. *Let $I \subseteq \mathbb{R}$ be an interval, a an interior point of I and $f : I \to \mathbb{R}$ be a function that satisfies the conditions:*

(i) *there exists a neighbourhood V of a so that f is $n+1$ times differentiable on $V \cap I$,*

(ii) *the function $f^{(n+1)} : V \cap I \to \mathbb{R}$ is continuous at a.*

Then exists

$$\lim_{x \to a} \left(\frac{f(x)-f(a)}{x-a} \right)^{(n)}$$

and

$$\lim_{x \to a} \left(\frac{f(x)-f(a)}{x-a} \right)^{(n)} = \frac{1}{n+1} f^{(n+1)}(a). \tag{3}$$

Proof. For $n \in \{0,1,2\}$ this resumes to an easy check. Let n be an integer, n greater or equal to 3. Taking Leibniz's rule into account, for each $x \in V \cap I$ we have that

$$\left(\frac{f(x)-f(a)}{x-a} \right)^{(n)} = \left[(f(x)-f(a)) \frac{1}{x-a} \right]^{(n)}$$

$$= \binom{n}{0} f^{(n)}(x) \frac{1}{x-a} + \sum_{k=1}^{n-1} \binom{n}{k} [f(x)-f(a)]^{(n-k)} \left(\frac{1}{x-a} \right)^{(k)}$$

$$+ \binom{n}{n} [f(x)-f(a)] \left(\frac{1}{x-a} \right)^{(n)} = \binom{n}{0} f^{(n)}(x) \frac{1}{x-a}$$

$$+\sum_{k=1}^{n-1}\binom{n}{k}\left[f(x)-f(a)\right]^{(n-k)}\frac{(-1)^k k!}{(x-a)^{k+1}}+\binom{n}{n}\left[f(x)-f(a)\right]\frac{(-1)^n n!}{(x-a)^{n+1}}$$

$$=\frac{\binom{n}{0}f^{(n)}(x)(x-a)^n+\sum_{k=1}^{n-1}(-1)^k k!\binom{n}{k}\left[f(x)-f(a)\right]^{(n-k)}(x-a)^{n-k}}{(x-a)^{n+1}}$$

$$+\frac{(-1)^n n!\binom{n}{n}\left[f(x)-f(a)\right]}{(x-a)^{n+1}}.$$

Taking this to the limit and applying l'Hôpital's rule yields

$$\lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)^{(n)}$$

$$=\lim_{x\to a}\frac{f^{(n+1)}(x)(x-a)^n+\sum_{k=2}^{n-1}(-1)^k k!\binom{n}{k}f^{(n-k+1)}(x)(x-a)^{n-k}}{(n+1)(x-a)^n}$$

$$+\frac{\sum_{k=1}^{n-2}(-1)^k k!\binom{n}{k}f^{(n-k)}(x)(n-k)(x-a)^{n-k-1}}{(n+1)(x-a)^n}.$$

But

$$\sum_{k=2}^{n-1}(-1)^k k!\binom{n}{k}f^{(n-k+1)}(x)(x-a)^{n-k}$$

$$+\sum_{k=1}^{n-2}(-1)^k k!\binom{n}{k}(n-k)f^{(n-k)}(x)(x-a)^{n-k+1}$$

$$=\sum_{k=1}^{n-2}(-1)^{k+1}(k+1)!\binom{n}{k+1}f^{(n-k)}(x)(x-a)^{n-k-1}$$

$$+\sum_{k=1}^{n-2}(-1)^k k!\binom{n}{k}(n-k)f^{(n-k)}(x)(x-a)^{n-k-1}$$

$$=\sum_{k=1}^{n-2}(-1)^k k! f^{(n-k)}(x)(x-a)^{n-k-1}\left[-(k+1)\binom{n}{k+1}+(n-k)\binom{n}{k}\right]=0,$$

hence

$$\lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)^{(n)}=\lim_{x\to a}\frac{f^{(n+1)}(x)}{n+1}$$

and now (3) follows immediately.

Theorem 7 remains true if the point a is an extremity of the interval I.

THEOREM 8. *Let $I \subseteq \mathbb{R}$ be an interval, $a \in I$ the left (respectively right) extremity of I and $f : I \to \mathbb{R}$ be a function that satisfies the conditions:*

(i) there exists a neighbourhood V of a such that f is $n+1$ times differentiable on $V \cap I$,

(ii) the function $f^{(n+1)} : V \cap I \to \mathbb{R}$ is continuous at the right (respectively at the left) at a.

Then

$$\lim_{\substack{x \to a \\ x > a}} \left(\frac{f(x) - f(a)}{x - a} \right)^{(n)} \quad \left(\text{respectively } \lim_{\substack{x \to a \\ x < a}} \left(\frac{f(x) - f(a)}{x - a} \right)^{(n)} \right)$$

exists and

$$\lim_{\substack{x \to a \\ x > a}} \left(\frac{f(x) - f(a)}{x - a} \right)^{(n)} = \frac{1}{n+1} f^{(n+1)}(a)$$

$$\left(\text{respectively } \lim_{\substack{x \to a \\ x > a}} \left(\frac{f(x) - f(a)}{x - a} \right)^{(n)} = \frac{1}{n+1} f^{(n+1)}(a) \right).$$

Proof. The proof is similar to the one given for Theorem 7.

Next, we state a consequence of Theorem 7.

COROLLARY 9. *Let $I \subseteq \mathbb{R}$ be an interval, a an interior point of I and $f : I \to \mathbb{R}$ be a function that verifies:*

(i) there exists a neighbourhood V of a such that f is $n+1$ times differentiable on $V \cap I$,

(ii) the function $f^{(n+1)} : V \cap I \to \mathbb{R}$ is continuous at a.

Then the function $g : V \cap I \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} \frac{f(x) - f(a)}{x - a}, \text{ if } x \in (V \cap I) \setminus \{a\} \\ f^{(1)}(a), \quad \text{if } x = a, \end{cases}$$

is n times differentiable on $V \cap I$ and, for each $m \in \mathbb{N}$, $m \leqslant n$, we have

$$g^{(m)}(x) = \begin{cases} \left(\frac{f(x) - f(a)}{x - a} \right)^{(m)}, \text{ if } x \in (V \cap I) \setminus \{a\} \\ \frac{1}{m+1} f^{(m+1)}(a), \text{ if } x = a. \end{cases} \tag{4}$$

REMARK 10. Corollary 9 remains true if the point $a \in I$ is an extremity of the interval I.

We shall recall two known results, which can be found in [12], [16] or [23].

THEOREM 11. *Let $I,J \subseteq \mathbb{R}$ be two intervals and $f : I \to \mathbb{R}$, $g : J \to \mathbb{R}$ two functions such that $f(I) \subseteq J$. If f is n times differentiable on I, and g is n times differentiable on J, then the function $g \circ f : I \to \mathbb{R}$ is also n times differentiable on I and the following holds for every $x \in I$*

$$(g\circ f)^{(n)}(x) = \sum_{m=1}^{n} \left(g^{(m)} \circ f\right)(x) \times \tag{5}$$

$$\times \sum_{\substack{i_1+2i_2+\ldots+ni_n=n \\ i_1+i_2+\ldots+i_n=m}} \frac{n!}{i_1!i_2!\cdot\ldots\cdot i_n!} \left(\frac{f^{(1)}(x)}{1!}\right)^{i_1} \left(\frac{f^{(2)}(x)}{2!}\right)^{i_2} \cdot\ldots\cdot \left(\frac{f^{(n)}(x)}{n!}\right)^{i_n}.$$

THEOREM 12. *Let $I,J \subseteq \mathbb{R}$ be two intervals and $f : I \to J$ a bijective function. If f is n times differentiable on I and $f'(x) \neq 0$, for all $x \in I$, then the function f^{-1} is n times differentiable on J and, for each $y \in J$*

$$(f^{-1})^{(n)}(y) = \sum_{\substack{i_2+2i_3+\ldots+(n-1)i_n=n-1 \\ i_1+i_2+\ldots+i_n=n-1}} \frac{(-1)^{n-1+i_1}(2n-2-i_1)!}{i_2!i_3!\cdot\ldots\cdot i_n!} \times \tag{6}$$

$$\times \frac{1}{\left(f^{(1)}(x)\right)^{2n-1}} \left(\frac{f^{(1)}(x)}{1!}\right)^{i_1} \left(\frac{f^{(2)}(x)}{2!}\right)^{i_2} \cdot\ldots\cdot \left(\frac{f^{(n)}(x)}{n!}\right)^{i_n}$$

where $x = f^{-1}(y)$.

3. Main Results

THEOREM 13. *Let $I \subseteq \mathbb{R}$ be an interval, a an interior point of I and $f : I \to \mathbb{R}$ be a function satisfying the conditions:*

(i) the function f is $n+1$ times differentiable on I,

(ii) the function $f^{(n+1)}$ is continuous on I,

(iii) $f^{(2)}(a) \neq 0$.

Then, there exists a real number $\delta > 0$, such that $]a-\delta, a+\delta[\subseteq I$ and

1^0 $f^{(2)}(x) \neq 0$, for all $x \in]a-\delta, a+\delta[$.

2^0 The function $\varphi :]a-\delta, a+\delta[\to J$, where $J = f^{(1)}\left(]a-\delta, a+\delta[\right)$, defined by

$$\varphi(x) = f^{(1)}(x), \text{ for all } x \in]a-\delta, a+\delta[$$

is bijective.

3^0 There exists a uniquely determined function $c :]a-\delta, a+\delta[\setminus\{a\} \to]a-\delta, a+\delta[\setminus\{a\}$ such that

$$f(x) - f(a) = (x-a)f^{(1)}(c(x)), \tag{7}$$

for all $x \in]a-\delta, a+\delta[\setminus\{a\}$.

4^0 There exists a uniquely determined function $\theta :]a-\delta, a+\delta[\setminus\{a\} \to]0,1[$ such that

$$f(x) - f(a) = (x-a)f^{(1)}(a + (x-a)\theta(x)), \tag{8}$$

for all $x \in]a-\delta, a+\delta[\setminus \{a\}$.

5^0 *The function* $\overline{c} :]a-\delta, a+\delta[\to]a-\delta, a+\delta[$ *defined by*

$$\overline{c}(x) = \begin{cases} c(x), & \text{if } x \in]a-\delta, a+\delta[\setminus \{a\} \\ a, & \text{if } x = a, \end{cases} \tag{9}$$

is n *times differentiable on* $]a-\delta, a+\delta[$ *and the relations*

$$\overline{c}^{(n)}(a) = \sum_{m=1}^{n} \left(\varphi^{-1}\right)^{(m)} \left(f^{(1)}(a)\right) \times \tag{10}$$

$$\times \sum_{\substack{i_1+2i_2+\ldots+ni_n=n \\ i_1+i_2+\ldots+i_n=m}} \frac{n!}{i_1! i_2! \cdot \ldots \cdot i_n!} \left(\frac{f^{(2)}(a)}{2!}\right)^{i_1} \left(\frac{f^{(3)}(a)}{3!}\right)^{i_2} \cdot \ldots \cdot \left(\frac{f^{(n+1)}(a)}{(n+1)!}\right)^{i_n}$$

and

$$\overline{c}^{(n)}(a) = \sum_{m=1}^{n} \sum_{\substack{j_2+2j_3+\ldots+(m-1)j_m=m-1 \\ j_1+j_2+\ldots+j_m=m-1}} \frac{(-1)^{m-1+j_1}(2m-2-j_1)!}{j_2! j_3! \cdot \ldots \cdot j_m! \left[f^{(2)}(a)\right]^{2m-1}} \times \tag{11}$$

$$\times \left(\frac{f^{(2)}(a)}{1!}\right)^{j_1} \left(\frac{f^{(3)}(a)}{2!}\right)^{j_2} \cdot \ldots \cdot \left(\frac{f^{(m+1)}(a)}{m!}\right)^{j_m} \times$$

$$\times \sum_{\substack{i_1+2i_2+\ldots+ni_n=n \\ i_1+i_2+\ldots+i_n=m}} \frac{n!}{i_1! i_2! \cdot \ldots \cdot i_n!} \left(\frac{f^{(2)}(a)}{2!}\right)^{i_1} \left(\frac{f^{(3)}(a)}{3!}\right)^{i_2} \cdot \ldots \cdot \left(\frac{f^{(n+1)}(a)}{(n+1)!}\right)^{i_n}$$

hold.

6^0 *The function* $\overline{\theta} :]a-\delta, a+\delta[\to]0,1[$ *defined by*

$$\overline{\theta}(x) = \begin{cases} \theta(x), & x \in]a-\delta, a+\delta[\setminus \{a\} \\ \frac{1}{2}, & x = a \end{cases}$$

is $n-1$ *times differentiable on* $]a-\delta, a+\delta[$ *and*

$$\overline{\theta}^{(n-1)}(a) = \frac{1}{n} \overline{c}^{(n)}(a). \tag{12}$$

Proof. 1^0 Assume that $f^{(2)}(a) > 0$. Since a is an interior point of I, then there exists a real number $\delta > 0$ such that $]a-\delta, a+\delta[\subseteq I$ and $f^{(2)}(x) > 0$, for all $x \in]a-\delta, a+\delta[$.

We infer that the function $f^{(1)}$ is increasing on $]a-\delta, a+\delta[$ and therefore injective on $]a-\delta, a+\delta[$. If $f^{(2)}(a) < 0$, the proof is similar.

2^0 We have proved that $f^{(1)}$ is injective on $]a-\delta,a+\delta[$ and, due to the definition of $J,$ we have that $f^{(1)}$ is bijective.

3^0-4^0 It follows from statement 1^0 above and Theorem 4.

5^0 Taking Lemma 6 and 3^0 into account, (7) yields that c has the following expression

$$c(x)=\varphi^{-1}\left(\frac{f(x)-f(a)}{x-a}\right),\text{ for all }x\in]a-\delta,a+\delta[\setminus\{a\}. \tag{13}$$

Now, from (8) and due to $3^0,$ we obtain

$$c(x)=a+(x-a)\,\theta(x),\text{ for all }x\in]a-\delta,a+\delta[\setminus\{a\}. \tag{14}$$

Taking Theorem 4 it results that $\lim\limits_{x\to a}c(x)$ exists and

$$\lim_{x\to a}c(x)=a.$$

Then, the function $\overline{c}$ defined by (9) is continuous at $x=a.$

Let $g:]a-\delta,a+\delta[\to\mathbb{R}$ be the function defined by

$$g(x)=\begin{cases}\frac{f(x)-f(a)}{x-a}, & \text{if } x\in]a-\delta,a+\delta[\setminus\{a\}\\[2ex] f^{(1)}(a), & \text{if } x=a.\end{cases} \tag{15}$$

According to (13) and (15), relation (9) becomes

$$\overline{c}(x)=\begin{cases}\left(\varphi^{-1}\circ g\right)(x), & \text{if } x\in]a-\delta,a+\delta[\setminus\{a\}\\[2ex] a, & \text{if } x=a.\end{cases}$$

From (i) and the definition of g we have that the function $\overline{c}$ is n times differentiable on $]a-\delta,a+\delta[$ and heeding (5), for all $x\in]a-\delta,a+\delta[\setminus\{a\}$ we have that

$$\overline{c}^{(n)}(x)=\sum_{m=1}^{n}\left(\left(\varphi^{-1}\right)^{(m)}\circ g\right)(x)\times$$

$$\times\sum_{\substack{i_1+2i_2+\ldots+ni_n=n\\ i_1+i_2+\ldots+i_n=m}}\frac{n!}{i_1!i_2!\cdot\ldots\cdot i_n!}\left(\frac{g^{(1)}(x)}{1!}\right)^{i_1}\left(\frac{g^{(2)}(x)}{2!}\right)^{i_2}\cdot\ldots\cdot\left(\frac{g^{(n)}(x)}{n!}\right)^{i_n}.$$

Taking it to the limit and recalling (4), one obtains (10). Using formula (6) in (10) one obtains (11).

6^0 From (8) and (9) it follows that the function $\overline{\theta}$ is $n-1$ times differentiable on $]a-\delta,a+\delta[$ and

$$\overline{\theta}^{(n-1)}(x)=\left(\frac{\overline{c}(x)-\overline{c}(a)}{x-a}\right)^{(n-1)},$$

for all $x \in]a-\delta, a+\delta[\setminus \{a\}$ and hence

$$\overline{\theta}^{(n-1)}(a) = \lim_{x \to a} \left(\frac{\overline{c}(x) - \overline{c}(a)}{x-a} \right)^{(n-1)}.$$

Considering (13) one obtains (12).

REMARK 14. Theorem 13 remains true if the point $a \in I$ is an extremity of the interval I.

THEOREM 15. *Let $I \subseteq \mathbb{R}$ be an interval, $a \in I$ the left (respectively right) extremity of I and $f : I \to \mathbb{R}$ be a function satisfies the following conditions:*

(i) f is $n+1$ times differentiable on I,

(ii) the function $f^{(n+1)}$ is continuous on I,

(iii) $f^{(2)}(a) \neq 0$.

Then, there exists a real number $\delta > 0$, such that $]a, a+\delta[\subseteq I$ (respectively $]a-\delta, a[\subseteq I$) and

1^0 $f^{(2)}(x) \neq 0$ for all $x \in]a, a+\delta[$ (respectively $x \in]a-\delta, a[$).

2^0 The function $\varphi : [a, a+\delta[\to J$, where $J = f^{(1)}([a, a+\delta[)$ (respectively $\varphi :]a-\delta, a] \to J$, where $J = f^{(1)}(]a-\delta, a])$ defined by

$$\varphi(x) = f^{(1)}(x), \text{ for all } x \in [a, a+\delta[\text{ (respectively } x \in]a-\delta, a])$$

is bijective.

3^0 There exists a uniquely determined function $c :]a, a+\delta[\to]a, a+\delta[$ (respectively $c :]a-\delta, a[\to]a-\delta, a[$) such that

$$f(x) - f(a) = (x-a) f^{(1)}(c(x)),$$

for all $x \in]a, a+\delta[$ (respectivrly $x \in]a-\delta, a[$).

4^0 There exists a uniquely determined function $\theta :]a, a+\delta[\to]0,1[$ (respectively $\theta :]a-\delta, a[\to]0,1[$) such that

$$f(x) - f(a) = (x-a) f^{(1)}(a + (x-a)\theta(x)),$$

for all $x \in]a, a+\delta[$ (respectively $x \in]a-\delta, a[$).

5^0 The function $\overline{c} : [a, a+\delta[\to [a, a+\delta[$ (respectively $\overline{c} :]a-\delta, a] \to]a-\delta, a]$) defined by

$$\overline{c}(x) = \begin{cases} c(x), & \text{if } x \in]a, a+\delta[\text{ (respectively } x \in]a-\delta, a[\\ \\ a, & \text{if } x = a \end{cases}$$

is n times differentiable on $[a, a+\delta[$ (respectively $]a-\delta, a]$) and the relations

$$\overline{c}^{(n)}(a) = \sum_{m=1}^{n} \left(\varphi^{-1}\right)^{(m)} \left(f^{(1)}(a)\right) \times$$

$$\times \sum_{\substack{i_1+2i_2+\ldots+ni_n=n \\ i_1+i_2+\ldots+i_n=m}} \frac{n!}{i_1!i_2!\cdot\ldots\cdot i_n!} \left(\frac{f^{(2)}(a)}{2!}\right)^{i_1} \left(\frac{f^{(3)}(a)}{3!}\right)^{i_2} \cdot\ldots\cdot \left(\frac{f^{(n+1)}(a)}{(n+1)!}\right)^{i_n}$$

and

$$\overline{c}^{(n)}(a) = \sum_{m=1}^{n} \sum_{\substack{j_2+2j_3+\ldots+(m-1)j_m=m-1 \\ j_1+j_2+\ldots+j_m=m-1}} \frac{(-1)^{m-1+j_1}(2m-2-j_1)!}{j_2!j_3!\cdot\ldots\cdot j_m!\left[f^{(2)}(a)\right]^{2m-1}} \times$$

$$\times \left(\frac{f^{(2)}(a)}{1!}\right)^{j_1} \left(\frac{f^{(3)}(a)}{2!}\right)^{j_2} \cdot\ldots\cdot \left(\frac{f^{(m+1)}(a)}{m!}\right)^{j_m} \times$$

$$\times \sum_{\substack{i_1+2i_2+\ldots+ni_n=n \\ i_1+i_2+\ldots+i_n=m}} \frac{n!}{i_1!i_2!\cdot\ldots\cdot i_n!} \left(\frac{f^{(2)}(a)}{2!}\right)^{i_1} \left(\frac{f^{(3)}(a)}{3!}\right)^{i_2} \cdot\ldots\cdot \left(\frac{f^{(n+1)}(a)}{(n+1)!}\right)^{i_n}$$

hold.

6^0 *The function* $\overline{\theta} : [a, a+\delta[\to]0,1[$ (*respectively* $\overline{\theta} :]a-\delta, a] \to]0,1[$) *defined by*

$$\overline{\theta}(x) = \begin{cases} \theta(x), & \textit{if } x \in]a, a+\delta[\ (\textit{respectively } x \in]a-\delta, a[) \\ \frac{1}{2}, & \textit{if } x = a \end{cases}$$

is $n-1$ *times differentiable on* $[a, a+\delta[$ (*respectively* $]a-\delta, a]$) *and*

$$\overline{\theta}^{(n-1)}(a) = \frac{1}{n}\overline{c}^{(n)}(a).$$

Proof. The proof is similar to the one given for Theorem 13.

Now, we give three applications in the conditions of Theorem 13.

EXAMPLE 16. If $n = 1$, then from (10) we have that

$$\overline{c}^{(1)}(a) = \frac{1}{2}$$

and then, according to (12), it results that

$$\overline{\theta}(a) = \overline{c}^{(1)}(a) = \frac{1}{2}.$$

(see [6]).

EXAMPLE 17. If $n = 2$, then from (10) we have that

$$\overline{c}^{(2)}(a) = \frac{f^{(3)}(a)}{12 f^{(2)}(a)},$$

hence

$$\overline{\theta}^{(1)}(a) = \frac{f^{(3)}(a)}{24 f^{(2)}(a)}.$$

(see [9]).

EXAMPLE 18. If $n = 3$, then (10) and (12) yield after computations

$$\frac{1}{3}\overline{c}^{(3)}(a) = \overline{\theta}^{(2)}(a) = \frac{f^{(2)}(a) f^{(4)}(a) - \left(f^{(3)}(a)\right)^2}{24\left(f^{(2)}(a)\right)^2}.$$

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