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A SHARP CONVERSE INEQUALITY OF THREE WEIGHTED ARITHMETIC AND GEOMETRIC MEANS OF POSITIVE DEFINITE OPERATORS

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Abstract. In this paper we consider three-weighted arithmetic and geometric means of positive definite operators constructed by the symmetrization method that appeared in the definition of Ando-Li-Mathias's geometric mean of several positive definite operators and we establish a converse inequality of three-weighed arithmetic-geometric means via Specht ratio.

1. Introduction

In [5], Lawson and Lim constructed three-weighted arithmetic and geometric means of positive definite operators based on the symmetrization method that appeared in the definition of Ando-Li-Mathias's geometric mean of several positive definite operators [2]: Let A, B and C be positive definite operators in a Hilbert space and $0 < s, t, u < 1$. Then the barycentric operators of (s, t, u)-weighted arithmetic and geometric means

$$\alpha(A,B,C) = ((1-s)B+sC, (1-t)A+tC, (1-u)A+uB) \tag{1.1}$$
$$\beta(A,B,C) = (B\#_s C, A\#_t C, A\#_u B), \tag{1.2}$$

where $A\#_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$ denotes the t-weighted geometric mean of A and B, are power convergent under the Thompson metric, existing an invariant metric $d(A,B) = ||\log A^{-1/2}BA^{-1/2}||$ on the convex cone of positive definite operators ([3], [7]). That is, there exist positive definite operators denoted by $A(s,t,u : A,B,C)$ and $G(s,t,u : A,B,C)$ such that

$$\lim_{n\to\infty} \alpha^n(A,B,C) = (A(s,t,u:A,B,C), A(s,t,u:A,B,C), A(s,t,u:A,B,C))$$
$$\lim_{n\to\infty} \beta^n(A,B,C) = (G(s,t,u:A,B,C), G(s,t,u:A,B,C), G(s,t,u:A,B,C))$$

respectively. An explicit formula for the arithmetic mean $A(s,t,u : A,B,C)$ is given by

$$A(s,t,u:A,B,C) = \frac{1-tu}{C(s,t,u)}A + \frac{1-s+su}{C(s,t,u)}B + \frac{s+t-st}{C(s,t,u)}C$$

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where $C(s,t,u) = 2+t-(s+u)t+su$ (see [5]). It is shown that the geometric mean $G(s,t,u : A,B,C)$ satisfies the properties suggesting criteria for weighted geometric means of higher orders to satisfy. The geometric mean $G(s,t,u : A,B,C)$ is invariant under the congruence transformations and inversion: For any invertible operator M,

$$G(s,t,u : MAM^*, MBM^*, MCM^*) = MG(s,t,u : A,B,C)M^*,$$
$$G(s,t,u : A^{-1}, B^{-1}, C^{-1}) = G(s,t,u : A,B,C)^{-1}.$$

Furthermore, the arithmetic-geometric mean inequality holds:

$$A(s,t,u : A,B,C) \geqslant G(s,t,u : A,B,C).$$

In this paper, we prove a converse inequality via Specht ratio

$$A(s,t,u : A,B,C) \leqslant (S_h)^{\frac{1}{1-\rho}} G(s,t,u : A,B,C),$$

where $\rho := \max\{s,t,u,1-s,1-t,1-u\}$, $S_h = \frac{(h-1)h^{(h-1)^{-1}}}{e\log h}$ denotes the Specht ratio, $h = e^{\Delta(A,B,C)}$ and $\Delta(A,B,C)$ is the diameter of $\{A,B,C\}$ for the Thompson metric d. This improves the previous result for the case $s=t=u$ ([4]).

Throughout this paper, we assume that $\mathscr{H}$ is a Hilbert space and Ω is the convex cone of positive definite operators of $\mathscr{H}$. For Hermitian operators X and Y, we write that $X \leqslant Y$ if $Y-X$ is positive semidefinite, and $X < Y$ if $Y-X$ is positive definite (positive semidefinite and invertible).

2. A converse inequality

For $h,s \geqslant 1$, we define the Specht ratio by

$$S_h(s) = \frac{(h^s-1)h^{s(h^s-1)^{-1}}}{e\log h^s}, \; S_1(s) = 1. \tag{2.3}$$

For $s=1$, we denote $S_h := S_h(1)$.

LEMMA 2.1. *The Specht ratio has the following properties.*

(1) $s \mapsto S_h(s)^{\frac{1}{s}}$ *and* $h \mapsto S_h$ *are increasing functions for* $s \geqslant 1$ *and* $h \geqslant 1$, *respectively.*

(2) $S_{h^\rho} \leqslant S_h^\rho$ *for* $0 < \rho \leqslant 1$.

Proof. See Lemma 3.2 in [4]. □

A converse inequality of weighted arithmetic and geometric means of two positive definite operators by the Specht ratio appears in ([6], [1]).

THEOREM 2.2. *Let A and B be two positive definite operators and let $0 < t < 1$. Then*

$$A(t : A,B) \leqslant S_h \cdot G(t : A,B),$$

where $h = e^{d(A,B)}$ for the Thompson metric d.

Throughout this section we fix positive real numbers $0 < s,t,u < 1$. The barycentric operator for (s,t,u)-weighted geometric mean is denoted by

$$\beta(A,B,C) = (B\#_s C, A\#_t C, A\#_u B).$$

The following property will paly a crucial role for our purpose.

PROPOSITION 2.3. (Proposition 6.2, [5]) *For* $\mathbb{A} = (A,B,C) \in \Omega^3$,

$$\Delta(\beta(\mathbb{A})) \leqslant \rho\Delta(\mathbb{A}), \quad \rho := \max\{s,t,u,1-s,1-t,1-u\}$$

where $\Delta(\mathbb{A}) = \max\{d(A,B), d(A,C), d(B,C)\}$ *is the diameter of* $\{A,B,C\}$ *for the Thompson metric* d.

For the readers convenience, we provide a proof of the power convergence of β. By the preceding result, $\Delta(\beta(A,B,C)) \leqslant \rho\Delta(A,B,C)$ for any $(A,B,C) \in \Omega^3$. Applying this inequality inductively,

$$\Delta(\beta^n(A,B,C)) \leqslant \rho^n\Delta(A,B,C). \tag{2.4}$$

Now, the metric convexity ([3]) implies that

$$d(B\#_s C, A) = d(B\#_s C, A\#_s A) \leqslant (1-s)d(A,B) + sd(A,C) \leqslant \Delta(A,B,C)$$

and similarly $d(A\#_t C, B)$, $d(A\#_u B, C) \leqslant \Delta(A,B,C)$. Therefore equipped Ω^3 with sup metric we have

$$d(\beta(A,B,C),(A,B,C)) \leqslant \Delta(A,B,C)$$

for all $(X,Y,Z) \in \Omega^3$. Combining this inequality with (2.4), we conclude that

$$d\big(\beta^{n+1}(A,B,C), \beta^n(A,B,C)\big) \leqslant \Delta(\beta^n(A,B,C)) \leqslant \rho^n\Delta(A,B,C).$$

From $0 < \rho < 1$, the sequence $\{\beta^n(A,B,C)\}$ is a Cauchy sequence in Ω^3 and thus converges in this complete metric space Ω^3. The fact (again from (2.4)) that the diameter of the sets consisting of the coordinates of each $\beta^n(A,B,C)$ approach 0 implies that the limits in each coordinates are all equal.

PROPOSITION 2.4. *For* $\mathbb{A} \in \Omega^3$,

$$\limsup_k S_{h_0(\mathbb{A})} S_{h_1(\mathbb{A})} \cdots S_{h_k(\mathbb{A})} \leqslant S_{h_0}^{\frac{1}{1-\rho}}, \tag{2.5}$$

where $h_k(\mathbb{A}) = e^{\Delta(\beta^k(\mathbb{A}))}$, $(k = 0,1,2,\dots)$.

Proof. Let $\mathbb{A} \in \Omega^3$. Set $h_i = h_i(\mathbb{A})$. By Proposition 2.3, $\Delta(\beta^k(\mathbb{A})) \leqslant \rho^k\Delta(\mathbb{A})$ and hence $1 \leqslant h_k = e^{\Delta(\beta^k(\mathbb{A}))} \leqslant e^{\rho^k\Delta(\mathbb{A})} = h_0^{\rho^k}$. By Lemma 2.1, $S_{h_k} \leqslant S_{h_0^{\rho^k}} \leqslant (S_{h_0})^{\rho^k}$ and therefore $S_{h_0}S_{h_1}\cdots S_{h_k} \leqslant S_{h_0}^{\{1+\rho+\rho^2+\cdots+\rho^k\}} \to S_{h_0}^{\frac{1}{1-\rho}}$. $\square$

THEOREM 2.5. *Let* $\mathbb{A} = (A,B,C) \in \Omega^3$ *and let* $h_0 = e^{\Delta(\mathbb{A})}$. *Then*

$$A(s,t,u : A,B,C) \leqslant (S_{h_0})^{\frac{1}{1-\rho}}\, G(s,t,u : A,B,C). \tag{2.6}$$

Proof. Let α denote the barycentric operator for the (s,t,u)-weighted arithmetic mean on Ω^3, and let $\mathbb{A} = (A,B,C) \in \Omega^3$. Denote $\pi_{\neq 1}(\mathbb{A}) = (B,C), \pi_{\neq 2}(\mathbb{A}) = (A,C)$ and $\pi_{\neq 3}(\mathbb{A}) = (A,B)$. From $\Delta(\pi_{\neq i}\mathbb{A}) \leqslant \Delta(\mathbb{A})$ and Lemma 2.1,

$$S_{h'} \leqslant S_{h_0},\ h' = e^{\Delta(\pi_{\neq i}\mathbb{A})} \leqslant e^{\Delta(\mathbb{A})} = h_0. \tag{2.7}$$

By Proposition 2.2

$$\begin{aligned}\alpha(\mathbb{A}) &= (A(s:\pi_{\neq 1}\mathbb{A}), A(t:\pi_{\neq 2}\mathbb{A}), A(u:\pi_{\neq 3}\mathbb{A}))\\ &\leqslant S_{h_0}(G(s:\pi_{\neq 1}\mathbb{A}), G(t:\pi_{\neq 2}\mathbb{A}), G(u:\pi_{\neq 3}\mathbb{A}))\\ &= S_{h_0}\beta(\mathbb{A}),\end{aligned}$$

and by replacing $\mathbb{A}$ to $\beta(\mathbb{A})$,

$$\alpha(\beta(\mathbb{A})) \leqslant S_{h_1}\beta^2(\mathbb{A}),\ h_1 = e^{\Delta(\beta(\mathbb{A}))}.$$

By monotonicity of α, we have

$$\alpha^2(\mathbb{A}) \leqslant \alpha(S_{h_0}\beta(\mathbb{A})) = S_{h_0}\alpha(\beta(\mathbb{A})) \leqslant (S_{h_0}S_{h_1})\beta^2(\mathbb{A}).$$

Inductively, we have

$$\alpha^r(\mathbb{A}) \leqslant \left(\prod_{k=0}^{r-1} S_{h_k}\right)\beta^r(\mathbb{A}) \overset{(2.5)}{\leqslant} S_{h_0}^{\frac{1}{1-\rho}}\beta^r(\mathbb{A}), \quad h_k = e^{\Delta(\beta^k(\mathbb{A}))}.$$

Taking the limit (for the Thompson metric) of both sides as $r \to \infty$ and projecting into the first coordinate yield

$$A(s,t,u:\mathbb{A}) \ = \pi_1(\lim_{r\to\infty}\alpha^r(\mathbb{A})) \leqslant S_{h_0}^{\frac{1}{1-\rho}}\pi_1(\lim_{r\to\infty}\beta^r(\mathbb{A})) = S_{h_0}^{\frac{1}{1-\rho}}G(s,t,u:\mathbb{A}).$$

□

REMARK 2.6. In [4], we have proved a converse inequality of higher order t-weighted arithmetic and geometric means of positive definite operators : for $\mathbb{A} = (A_1, A_2, \ldots, A_n) \in \Omega^n$,

$$A(t:\mathbb{A}) \leqslant \left(S_{h_0^{(n-1)!}}\right)^{\left(\frac{1}{1-\rho}\right)^{n-2}} G(t:\mathbb{A}), \quad \rho = \max\{t, 1-t\},$$

where $h_0 = e^{\Delta(\mathbb{A})}$ and $0 < t < 1$. For $n = 3$, our result is sharper than the previous one even if $s = t = u$. See [8] for the case $t = 1/2$ via Kantorovich constant.

REMARK 2.7. By Theorem 6.3 and Theorem 6.7 of [5], staring at the weighted arithmetic and geometric mean $A(s,t,u:A,B,C)$ and $G(s,t,u:A,B,C)$ for fixed three weights (s,t,u), the (s,t,u)-weighted arithmetic and geometric means of n-positive definite operators $\mathbb{A} = (A_1, A_2, \ldots, A_n) \in \Omega^n (n \geqslant 3)$ can be defined inductively via the symmetrization procedure with the arithmetic-geometric mean inequality

$$G(s,t,u:\mathbb{A}) \leqslant A(s,t,u:\mathbb{A}).$$

We have checked that the converse inequality holds for $n \leqslant 5$:

$$A(s,t,u:\mathbb{A}) \leqslant (S_{h_0})^{\prod_{k=3}^{n}\frac{1}{1-\rho_k}} G(s,t,u:\mathbb{A}), \quad h_0 = \Delta(\mathbb{A})$$

where

$$\rho = \rho_3, \quad \rho_k = \max\{(1-\alpha)+\alpha\rho_{k-1}, \alpha+(1-\alpha)\rho_{k-1} : \alpha = s,t,u\}.$$

We left open the problem of associated converse inequalities for $n > 5$. A variant of Proposition 2.3 for higher orders will be necessary.

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