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BOUNDEDNESS OF HARDY OPERATORS ON GENERALIZED AMALGAMS

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Abstract. Boundedness of the Hardy operator $(Tf)(x) = \int_{-\infty}^{x} f(t)dt$ between amalgam spaces $\ell^q(X_u)$ and $\ell^{\bar{q}}(L_v^{\bar{p}})$ is obtained, where X_u is a weighted Banach function space. The adjoint opertor $(T^*f)(x) = \int_x^{\infty} f(t)dt$ has also been treated.

1. Introduction

In [3], Lebrun, Heinig and Hofmann have studied the boundedness of the Hardy operator $(Tf)(x) = \int_{-\infty}^{x} f(t)dt$ in the context of weighted amalgams $\ell^q(L_w^p)$. In fact their idea was to study the corresponding inequality in terms of two standard Hardy inequalities one in the continuous form and other in the discrete form.

In this paper, we follow their idea and consider a more general amalgam where the weighted Lebesgue space L_w^p is replaced by a weighted Banach function space X_w, i.e., we consider the space $\ell^q(X_w)$ and characterize the boundedness of the operator T in the framework of such spaces. We derive as a special case, the results for the $\ell^q(X_w^p)$, i.e., when the Banach function space X is taken as X^p. We also study the corresponding boundedness for the adjoint operator $(T^*f)(x) = \int_x^{\infty} f(t)dt$. Unlike the L^p - L^q boundedness or $\ell^q(L^{\bar{q}})$ - $\ell^p(L^{\bar{p}})$ boundedness of T^* which can be obtained directly from the corresponding boundedness of T using duality arguments, the situation here is different. Since the duals of the spaces involved here are not known, the duality argument can not be applied. So, we treat this case directly.

The L^p - L^q Hardy inequality is well known in the literature. For a complete description of such inequalities with several variants, generalizations and applications, one may refer to [7], [8], [12], [13], and the references therein. The space X^p has been considered and used in [11], [14] and Hardy inequalities on these spaces have recently been studied in [5], [6]. The paper is organized as follows:

In order not to distrub our discussions later on, we give some preliminaries in Section 2. In Section 3, we characterize the boundedness of T between the weighted

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amalgams $\ell^q(X_u)$ and $\ell^{\bar{q}}(L_v^{\bar{p}})$ in terms of two inequalities. Also, this characterization is given in terms of the condition on weights u and v. As an example, we consider the case when X is replaced by the space X^p. Finally, a similar study regarding the adjoint operator T^* is made in Section 4.

2. Preliminaries

By a weight function, we mean a function which is measurable, finite and positive a.e. on the appropriate domain. For a weight function defined on $\mathbb{R}$, we shall denote by L_w^p, the weighted Lebesgue space which is the space of all measurable functions f for which

$$\|f\|_{p,w} := \left(\int_{\mathbb{R}} |f(x)|^p w(x)dx\right)^{\frac{1}{p}} < \infty.$$

We denote by ℓ^p, the sequence space which consists of all sequences $\xi = \{\xi_n\}_{n\in\mathbb{Z}}$ for which

$$\|\xi\|_{\ell^p} := \left(\sum_{n\in\mathbb{Z}} |\xi_n|^p\right)^{\frac{1}{p}} < \infty$$

and for a sequence weight $\{w_n\}$, we denote by $\ell^p_{\{w_n\}}$, the weighted sequence space which consists of all sequences $\xi = \{\xi_n\}_{n\in\mathbb{Z}}$ for which

$$\|\xi\|_{\ell^p_{\{w_n\}}} := \left(\sum_{n\in\mathbb{Z}} w_n|\xi_n|^p\right)^{\frac{1}{p}} < \infty .$$

For $1 \leqslant p < \infty$, both the spaces L_w^p and ℓ^p are Banach spaces.

For a weight w, we denote by X_w, the weighted normed linear space of measurable functions with the norm defined by

$$\|f\|_{X_w} := \|fw\|_X ,$$

where X is the underlying non-weighted normed linear space of measurable functions. The weighted normed linear space X_w is called a weighted Banach function space (BFS) if in addition to the usual norm axioms, $\|f\|_{X_w}$ satisfies the following:

(1) $\|f\|_{X_w} = \||f|\|_{X_w}$ for all $f \in X_w$;
(2) $0 \leqslant f \leqslant g$ a.e. $\Rightarrow$ $\|f\|_{X_w} \leqslant \|g\|_{X_w}$;
(3) $0 \leqslant f_n \uparrow f$ a.e. $\Rightarrow$ $\|f_n\|_{X_w} \uparrow \|f\|_{X_w}$.
(4) If E is a measurable subset of $(0,\infty)$ such that $w(E) := \int_E w < \infty$, then $\|\chi_E\|_{X_w} < \infty$.
(5) For all measurable $E \subset (0,\infty)$ with $w(E) < \infty$, there exists a constant $c_E > 0$ such that $\int_E fw \leqslant c_E\|f\|_{X_w}$ for all $f \in X_w$.

The notion of BFS was introduced by Luxemburg [10]. These spaces enjoy the properties as possessed by L^p-spaces and are yet far general than L^p-spaces. A BFS satisfying properties (1)-(5) above with respect to the count measure is called a Banach sequence space (BSS). A good treatment of BFS can be found in [1]. The weighted BFS have been considered in [4] where the compactness of Hardy-type operators in such spaces has been studied.

For a BFS X and $-\infty < p < \infty$, $p \neq 0$, we denote by X^p, the space of all measurable functions f for which

$$\|f\|_{X^p} := \||f|^p\|_X^{\frac{1}{p}} < \infty .$$

For $1 < p < \infty$, X^p is a BFS and for $X = L^1$, X^p coincides with the usual L^p-space. Such spaces have been studied in [11], [14]. Recently, in [5], [6], the authors have obtained weight characterization of the Hardy inequalities in the framework of X^p-spaces.

For $1 < p,\ q < \infty$ and a weight function w the weighted amalgams $\ell^q(L^p_w)$ consists of functions which are locally integrable in the weighted Lebesgue space L^p_w, where the integral over intervals $[n, n+1]$ form a sequence of the sequence space ℓ^q, i.e.,

$$\|f\|_{p,w,q} := \left\{\sum_{n\in\mathbb{N}}\left(\int_n^{n+1} |f(x)|^p w(x)dx\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}} < \infty.$$

We consider in this paper, the weighted amalgams $\ell^q(X_w)$ in which the weighted Lebesgue space L^p_w is replaced by the weighted BFS X_w. The natural norm in such a space is defined by

$$\|f\|_{\ell^q(X_w)} := \left(\sum_{n\in\mathbb{Z}} \|f\chi_n w\|_X^q\right)^{\frac{1}{q}} .$$

Clearly, when $X = L^p$, then $\ell^q(X_w) = \ell^q\left(L^p_{w^{1/p}}\right)$.

The following is the well known result giving the boundedness of the discrete Hardy operator $H(a_n) = \sum_{k=-\infty}^n a_k$ between appropriate sequence spaces.

THEOREM A. *Let* $1 < p,q < \infty$, $u_n \geqslant 0$, $v_n > 0$, $n \in \mathbb{Z}$ *and* $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. *Then there exists a constant* $C > 0$ *such that*

$$\left(\sum_{n\in\mathbb{Z}} u_n \left(\sum_{k=-\infty}^{n} a_k\right)^q\right)^{\frac{1}{q}} \leqslant C\left(\sum_{n\in\mathbb{Z}} a_n^p v_n\right)^{\frac{1}{p}},$$

holds for all non-negative sequence $\{a_k\} \in \ell^q_{\{v_n\}}$, *if and only if*

(a) *for* $1 < p \leqslant q < \infty$

$$\sup_{m\in\mathbb{Z}} \left(\sum_{n=m}^{\infty} u_n\right)^{\frac{1}{q}} \left(\sum_{n=-\infty}^{m} v_n^{1-p'}\right)^{\frac{1}{p'}} < \infty$$

(b) *for* $1 < q < p < \infty$

$$\left\{ \sum_{m\in\mathbb{Z}} \left(\sum_{n=m}^{\infty} u_n \right)^{\frac{r}{q}} \left(\sum_{n=-\infty}^{m} v_n^{1-p'} \right)^{\frac{r}{q'}} v_m^{1-p'} \right\}^{\frac{1}{r}} < \infty.$$

The corresponding result for the adjoint operator $H^*(a_n) = \sum_{k=n}^{\infty} a_k$ is the following:

THEOREM B. *Let* $1 < p, q < \infty$, $u_n \geqslant 0$, $v_n > 0$, $n \in \mathbb{Z}$ *and* $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. *Then there exists a constant* $C > 0$ *such that*

$$\left(\sum_{n\in\mathbb{Z}} u_n \left(\sum_{k=n}^{\infty} a_k \right)^q \right)^{\frac{1}{q}} \leqslant C \left(\sum_{n\in\mathbb{Z}} a_n^p v_n \right)^{\frac{1}{p}}$$

holds for all non-negative sequence $\{a_k\} \in \ell^q_{\{v_n\}}$, *if and only if*

(a) *for* $1 < p \leqslant q < \infty$

$$\sup_{m\in\mathbb{Z}} \left(\sum_{n=-\infty}^{m} u_n \right)^{\frac{1}{q}} \left(\sum_{n=m}^{\infty} v_n^{1-p'} \right) < \infty$$

(b) *for* $1 < q < p < \infty$

$$\left\{ \sum_{m\in\mathbb{Z}} \left(\sum_{n=-\infty}^{m} u_n \right)^{\frac{r}{q}} \left(\sum_{n=m}^{\infty} v_n^{1-p'} \right)^{\frac{r}{q'}} u_m \right\}^{\frac{1}{r}} < \infty .$$

In Theorem A and throughout primes will denote the conjugate indices, e.g., $p' = \frac{p}{p-1}$. As usual, the symbol $\chi_{[a,b]}$ will denote the characteristic function over the interval $[a,b]$. However, for any integer, say n, $\chi_{[n,n+1]}$ will be denoted by χ_n. X' will denote the associate space of a BFS X.

3. Boundedness of the operator T

We begin with our first main result which characterizes the boundedness of the operator T between the amalgams $\ell^q(X_u)$ and $\ell^{\bar{q}}(L^{\bar{p}}_v)$.

THEOREM 1. *Let* u, v *be weight functions,* X_u *be a weighted BFS and* $1 < \bar{p}, \bar{q}, q < \infty$. *Then the inequality*

$$\|Tf\|_{\ell^q(X_u)} \leqslant C \|f\|_{\bar{p},v,\bar{q}} \tag{3.1}$$

holds for all $f \in \ell^{\bar{q}}(L^{\bar{p}}_v)$ *if and only if the following inequalities hold:*

(a) *for all sequences* $\{A_k\} \in \ell^{\bar{q}}_{\{V_n\}}$

$$\left(\sum_{n\in\mathbb{Z}} \left(\sum_{k=-\infty}^{n} A_k \right)^q U_n^q \right)^{\frac{1}{q}} \leqslant C \left(\sum_{n\in\mathbb{Z}} A_n^{\bar{q}} V_n \right)^{\frac{1}{\bar{q}}}, \tag{3.2}$$

where

$$A_k = \int_{k-1}^{k} f, \quad U_n = \|\chi_n\|_{X_u} \quad and \quad V_n = \left(\int_{n-1}^{n} v^{1-\bar{p}'}\right)^{\frac{-\bar{q}}{\bar{p}'}}$$

(b) *for each* $n \in \mathbb{Z}$

$$\left\| \left(\int_n^x f\right) \chi_n \right\|_{X_u} \leqslant C \left(\int_n^{n+1} f^{\bar{p}} v\right)^{\frac{1}{\bar{p}}}. \tag{3.3}$$

Proof. We prove sufficiency first. Without any loss of generality, we may assume that $f \geqslant 0$ since $|Tf| \leqslant T|f|$. We have

$$\begin{aligned}
\|(Tf)\chi_n\|_{X_u} &= \left\| \left(\int_{-\infty}^{n} f + \int_n^x f\right) \chi_n \right\|_{X_u} \\
&\leqslant \left\| \left(\int_{-\infty}^{n} f\right) \chi_n \right\|_{X_u} + \left\| \left(\int_n^x f\right) \chi_n \right\|_{X_u} \\
&= \left\| \left(\sum_{k=-\infty}^{n} \int_{k-1}^{k} f\right) \chi_n \right\|_{X_u} + \left\| \left(\int_n^x f\right) \chi_n \right\|_{X_u}
\end{aligned}$$

so that

$$\begin{aligned}
\|Tf\|_{\ell^q(X_u)} &\leqslant \left(\sum_{n\in\mathbb{Z}} \left(\sum_{k=-\infty}^{n} \int_{k-1}^{k} f\right)^q \|\chi_n\|_{X_u}^q\right)^{\frac{1}{q}} + \left(\sum_{n\in\mathbb{Z}} \left\| \left(\int_n^x f\right) \chi_n \right\|_{X_u}^q\right)^{\frac{1}{q}} \\
&= J_1 + J_2\,.
\end{aligned} \tag{3.4}$$

Taking $U_n = \|\chi_n\|_{X_u}$, $A_k = \int_{k-1}^{k} f$, using (3.2) and applying Hölder's inequality, we get

$$\begin{aligned}
J_1 &= \left(\sum_{n\in\mathbb{Z}} \left(\sum_{k=-\infty}^{n} A_k\right)^q U_n\right)^{\frac{1}{q}} \\
&\leqslant C \left(\sum_{n\in\mathbb{Z}} A_n^{\bar{q}} \left(\int_{n-1}^{n} v^{1-\bar{p}'}\right)^{\frac{-\bar{q}}{\bar{p}'}}\right)^{\frac{1}{\bar{q}}} \\
&= C \left(\sum_{n\in\mathbb{Z}} \left(\int_{n-1}^{n} f\right)^{\bar{q}} \left(\int_{n-1}^{n} v^{1-\bar{p}'}\right)^{\frac{-\bar{q}}{\bar{p}'}}\right)^{\frac{1}{\bar{q}}} \\
&\leqslant C \left(\sum_{n\in\mathbb{Z}} \left(\int_{n-1}^{n} f^{\bar{p}} v\right)^{\frac{\bar{q}}{\bar{p}}}\right)^{\frac{1}{\bar{q}}} \\
&= C\|f\|_{\bar{p},v,\bar{q}}\,.
\end{aligned} \tag{3.5}$$

Also, since (3.3) holds, we find that

$$J_2 \leqslant C\|f\|_{\bar{p},v,\bar{q}}\,. \tag{3.6}$$

Now, (3.4), (3.5) and (3.6) give sufficiency.

In order to prove the necessity, define for any non-negative sequence $\{a_k\}$, a function

$$f = \sum_{k\in\mathbb{Z}} a_k v^{1-\bar{p}'}\chi_k$$

so that if we write $B_k = a_{k-1}\int_{k-1}^{k} v^{1-\bar{p}'}$, we have for $n \leqslant x < n+1$;

$$|(Tf)(x)| = \left|\int_{-\infty}^{n} f + \int_{n}^{x} f\right| \geqslant \left(\sum_{k=-\infty}^{n} B_k\right)$$

which gives

$$\|Tf\|_{\ell^q(X_u)} \geqslant \left(\sum_{n\in\mathbb{Z}}\left(\sum_{k=-\infty}^{n} B_k\right)^{q}\|\chi_n\|_{X_u}^{q}\right)^{\frac{1}{q}}$$

and also for this f

$$\|f\|_{\bar{p},v,\bar{q}} = \left(\sum_{n\in\mathbb{Z}} B_n^{\bar{q}} V_n\right)^{\frac{1}{\bar{q}}}.$$

Consequently, (3.1) reduces to

$$\left(\sum_{n\in\mathbb{Z}}\left(\sum_{k=-\infty}^{n} B_k\right)^{\frac{1}{q}}\|\chi_n\|_{X_u}^{q}\right)^{\frac{1}{q}} \leqslant C\left(\sum_{n\in\mathbb{Z}} B_n^{\bar{q}} V_n\right)^{\frac{1}{\bar{q}}}$$

for $\{B_k\} \in \ell^{\bar{q}}_{\{v_n\}}$ which proves the necessity of (3.2). Finally, in order to prove the necessity of (3.3), take for a fixed $m \in \mathbb{Z}$, $g \geqslant 0$ and $m \leqslant x < m+1$

$$f = g\chi_m\,;$$

so that the LHS of (3.1) can be estimated as

$$\|Tf\|_{\ell^q(X_u)} = \left\|\left(\int_{-\infty}^{x} f\right)\chi_m\right\|_{X_u} = \left\|\left(\int_{m}^{x} g\right)\chi_m\right\|_{X_u}$$

while

$$\|f\|_{\bar{p},v,\bar{q}} = \left(\int_{m}^{m+1} g^{\bar{p}} v\right)^{\frac{1}{\bar{p}}}$$

so that (3.1) becomes

$$\left\| \left(\int_m^x g \right) \chi_m \right\|_{X_u} \leqslant \left(\int_m^{m+1} g^{\bar{p}} v \right)^{\frac{1}{\bar{p}}}$$

for each $m \in \mathbb{Z}$ and the assertion follows. □

Berezhnoi [2] (also Lomakina and Stepanov [9, Theorem 4]) obtained a characterization for the boundedness of the Hardy operator $(Hf)(x) = \phi(x) \int_a^x \psi(y) f(y) dy$ between BFS X and Y, where ϕ, ψ are weight functions.

The following was proved:

THEOREM C. *Let $-\infty \leqslant a < b \leqslant \infty$ and X, Y be BFS satisfying the ℓ-condition. Then the inequality*

$$\|Hf\|_Y \leqslant C\|f\|_X \tag{3.7}$$

holds for all measurable functions f if and only if

$$A := \sup_{b>t>a} A(t) := \sup_{b>t>a} \|\chi_{[t,b]}\phi\|_Y \|\chi_{[a,t]}\psi\|_{X'} < \infty.$$

REMARK 1. In fact, Theorem C was proved for $a = 0$ and $b = \infty$. However, it remains valid for general a and b. This fact has also been confirmed through a personal communication with Professor V. D. Stepanov.

REMARK 2. The "ℓ-condition" mentioned in Theorem C was introduced by Berezhnoi [2] which provides an ordering in certain sense: Two BFS X and Y are said to satisfy an ℓ-condition, if there exists a BSS ℓ such that X is ℓ-concave and Y is ℓ-convex simultaneously. For the notions of ℓ-concavity and ℓ-convexity, one may refer to [2] (see also [9]). The ℓ-condition, for the L^p-L^q case, corresponds to the case $p \leqslant q$.

Theorem 1 characterizes the inequality (3.1) in terms of two further inequalities namely (3.2) and (3.3). Next, we provide a characterization of (3.1) in terms of the weights u and v. This is indeed simple once we have the weight characterizations of the inequalities (3.2) and (3.3). The corresponding characterizations for (3.2) is given by Theorem A while for (3.3), if we take $g \equiv f v^{\frac{1}{\bar{p}}}$, $u \equiv \psi$, $v^{\frac{-1}{\bar{p}}} \equiv \phi$, then (3.3) becomes equivalent to (3.7) and its characterization can be obtained by Theorem C. Precisely, we have proved the following:

THEOREM 2. *Let $1 < \bar{p}, q, \bar{q} < \infty$, u, v be weight functions and X_u be a weighted BFS. Let X_u and the Lebesgue space $L^{\bar{p}}$ satisfy ℓ-condition. Then the inequality*

$$\|Tf\|_{\ell^q(X_u)} \leqslant C\|f\|_{\bar{p},v,\bar{q}} \tag{3.8}$$

holds for all $f \in \ell^{\bar{q}}(L_v^{\bar{p}})$ if and only if

(a) *for* $\bar{p} \leqslant q$

$$A_1 := \sup_{m\in\mathbb{Z}} \left(\sum_{n=m}^{\infty} \|\chi_n u\|_X^q\right)^{\frac{1}{q}} \left(\sum_{n=-\infty}^{m-1} \left(\int_n^{n+1} v^{1-\bar{p}'}\right)^{\frac{\bar{q}'}{\bar{p}'}}\right)^{\frac{1}{\bar{q}'}} < \infty,$$

$$A_2 := \sup_{m\in\mathbb{Z}} \sup_{m<t<m+1} \left(\|\chi_{[t,m+1]} u^{\frac{1}{q}}\|_X\right) \left(\|\chi_{[m,t]} v^{\frac{-1}{p}}\|_{L^{p'}}\right) < \infty .$$

(b) *for* $q < \bar{q}$, $A_2 < \infty$ *and*

$$A_3 := \left(\sum_{k\in\mathbb{Z}} \left(\sum_{n=k}^{\infty} \|x_n u\|_X^q\right)^{\frac{\beta}{q}}\right.$$

$$\left. \times \left(\sum_{n=-\infty}^{k} \left(\int_{n-1}^{n} v^{1-\bar{p}'}\right)^{\frac{\bar{q}'}{\bar{p}'}}\right)^{\frac{\beta}{q'}} \left(\int_{k-1}^{k} v^{1-\bar{p}'}\right)^{\frac{\bar{q}'}{\bar{p}'}}\right)^{\frac{1}{\beta}} < \infty$$

where $\dfrac{1}{\beta} = \dfrac{1}{q} - \dfrac{1}{\bar{q}}$.

REMARK 3. In Theorem 2, the spaces X_u and $L^{\bar{p}}$ were assumed to satisfy ℓ-condition. If we replace the BFS X by X^p then the ℓ-condition for the pair of spaces X_u^p and $L^{\bar{p}}$ corresponds to the case $p \leqslant \bar{p}$. In such situation, Theorem 2 can be written for the boundedness of T between the amalgams $\ell^{\bar{q}}(L_v^{\bar{p}})$ and $\ell^q(X_u^p)$. We next provide the corresponding boundedness that covers the case $\bar{p} \leqslant p$ too. For this purpose, we need the following results from [5], [6].

THEOREM D. *Let* $1 < p$, $\bar{p} < \infty$, $-\infty \leqslant a < b \leqslant \infty$, u, v *be weight functions defined on* (a,b). *Then the inequality*

$$\left\|\left(\int_a^x f\right) u^{\frac{1}{p}}\right\|_{X^p} \leqslant C\left(\int_a^b f^{\bar{p}} v\right)^{\frac{1}{\bar{p}}}$$

holds for all measurable functions $f \geqslant 0$, *if and only if*

(a) *for* $p \leqslant \bar{p}$

$$\sup_{a<t<b} \|\chi_{[t,b]} u^{\frac{1}{\bar{p}}}\|_{X^{\bar{p}}} \left(\int_a^t v^{1-p'}\right)^{\frac{1}{p'}} < \infty \tag{3.9}$$

(b) *for* $p > \bar{p}$

$$\left(\int_a^b \left(\|\chi_{[t,b]} u^{\frac{1}{\bar{p}}}\|_{X^{\bar{p}}}\right)^{\alpha} \left(\int_a^x v^{1-p'}\right)^{\frac{\alpha}{q'}} v^{1-p'}(x)dx\right)^{\frac{1}{\alpha}} < \infty,$$

where $\dfrac{1}{\alpha} = \dfrac{1}{\bar{p}} - \dfrac{1}{p}$.

REMARK 4. In [5], it was proved that (3.9) is equivalent to the following condition, which was, in fact, proved in [5] for $a = 0$, $b = \infty$

$$\sup_{a<t<b} V_1^{\frac{(s-1)}{p}}(t)\|V_1^{\frac{(p-s)}{p}} u^{\frac{1}{p}} \chi_{[t,b]}\|_{X^{\bar{p}}} < \infty,$$

where $V_1(t) = \int_a^t v^{1-p'}$ and $s \in (1,p)$.

Now, in view of Theorems 2 and D, the boundedness of the operator T between the amalgams $\ell^q(X_u^p)$ and $\ell^{\bar{q}}(L_v^{\bar{p}})$ can be obtained immediately. The following is the result:

THEOREM 3. *Let* $1 < p, q, \bar{p}, \bar{q} < \infty$ *and* u, v *be weight functions. Then the inequality*

$$\|Tf\|_{\ell^q(X_u^p)} \leqslant C\|f\|_{\bar{p},v,\bar{q}}$$

holds for all $f \in \ell^{\bar{q}}(L_v^{\bar{p}})$ *if and only if*

(a) *for* $\bar{p} \leqslant p$, $\bar{q} \leqslant q$

$$B_1 := \sup_{m\in\mathbb{Z}} \left(\sum_{n=m}^{\infty} \|\chi_n u^{\frac{1}{p}}\|_{X^p}^q\right)^{\frac{1}{q}} \left(\sum_{n=-\infty}^{m-1} \left(\int_n^{n+1} v^{1-\bar{p}'}\right)^{\frac{\bar{q}'}{\bar{p}'}}\right)^{\frac{1}{\bar{q}'}} < \infty,$$

$$B_2 := \sup_{m\in\mathbb{Z}} \sup_{m<t<m+1} \left(\|\chi_{[t,m+1]} u^{\frac{1}{p}}\|_{X^p}\right) \left(\int_m^t v^{1-\bar{p}'}\right)^{\frac{1}{\bar{p}'}} < \infty.$$

(b) *for* $p < \bar{p}$, $\bar{q} \leqslant q$; $B_1 < \infty$ *and*

$$B_3 := \left(\int_n^{n+1} \|\chi_{[t,n+1]} u^{\frac{1}{p}}\|_{X^p}^{\alpha} \left(\int_n^t v^{1-\bar{p}'}\right)^{\frac{\alpha}{\bar{p}'}} v^{1-\bar{p}'}(t)dt\right)^{\frac{1}{\alpha}} < \infty,$$

where $\dfrac{1}{\alpha} = \dfrac{1}{p} - \dfrac{1}{\bar{p}}$.

(c) *for* $q < \bar{q}$, $\bar{p} \leqslant p$; $B_2 < \infty$ *and*

$$B_4 := \left(\sum_{k\in\mathbb{Z}} \left(\sum_{n=k}^{\infty} \|x_n u^{\frac{1}{p}}\|_{X^p}^q\right)^{\frac{\beta}{q}} \times \left(\sum_{n=-\infty}^{k} \left(\int_{n-1}^{n} v^{1-\bar{p}'}\right)^{\frac{\bar{q}'}{\bar{p}'}}\right)^{\frac{\beta}{q'}} \left(\int_{k-1}^{k} v^{1-\bar{p}'}\right)^{\frac{\bar{q}'}{\bar{p}'}}\right)^{\frac{1}{\beta}} < \infty$$

where $\dfrac{1}{\beta} = \dfrac{1}{q} - \dfrac{1}{\bar{q}}$.

(d) *for* $q < \bar{q}$, $p \leqslant \bar{p}$

$$B_1 < \infty \quad \text{and} \quad B_3 < \infty.$$

REMARK 5. Theorem 3 extends a result of Carton-Lebrun, Heinig and Hofmann [9] who proved it for $X = L^1$.

4. Boundedness of the operator T^*

This section is devoted to the boundedness of the adjoint operator $(T^*f)(x) = \int_x^\infty f(t)dt$ between $\ell^q(X_u)$ and $\ell^{\bar{q}}(L_v^{\bar{p}})$. It is known that the dual of $\ell^{\bar{q}}(L_v^{\bar{p}})$ is the space $\ell^{\bar{q}'}(L^{\bar{p}'}_{v^{1-\bar{p}'}})$. However, no concrete structure to the dual of $\ell^q(X_u)$ is known. Therefore the usual practice of deriving the boundedness of T^* by duality argument is ruled out here. So, we treat this case directly. However, the result is obtained on similar lines as that in Theorem 1. Precisely, we prove the following:

THEOREM 4. *Let u, v be weight functions, X_u be a weighted BFS and $1 < \bar{p}, \bar{q}, q < \infty$. Then the inequality*

$$\|T^*f\|_{\ell^q(X_u)} \leqslant C\|f\|_{\bar{p},v,\bar{q}} \tag{4.1}$$

holds for all $f \in \ell^{\bar{q}}(L_v^{\bar{p}})$ if and only if the following inequalities hold:

(a) *for all sequence $\{A_k\} \in \ell^{\bar{q}}_{\{\tilde{V}_n\}}$*

$$\left(\sum_{n\in\mathbb{Z}}\left(\sum_{k=n}^{\infty}\tilde{A}_k\right)^q U_n^q\right)^{\frac{1}{q}} \leqslant C\left(\sum_{n\in\mathbb{Z}}\tilde{A}_n^{\bar{q}}\tilde{V}_n\right)^{\frac{1}{\bar{q}}}, \tag{4.2}$$

where

$$\tilde{A}_n = \int_n^{n+1} f, \quad U_n = \|\chi_n\|_{X_u} \quad and \quad \tilde{V}_n = \left(\int_n^{n+1} v^{1-\bar{p}'}\right)^{\frac{-\bar{q}}{\bar{p}'}}$$

(b) *for each $n \in \mathbb{Z}$*

$$\left\|\left(\int_x^{n+1} f\right)\chi_n\right\|_{X_u} \leqslant C\left(\int_n^{n+1} f^{\bar{p}}v\right)^{\frac{1}{\bar{p}}}. \tag{4.3}$$

Proof. We prove sufficiency first. Without any loss of generality, we may assume that $|T^*f| \leqslant T^*|f|$. We have

$$\begin{aligned}
\|(T^*f)\chi_n\| &= \left\|\left(\int_x^{n+1} f + \int_{n+1}^{\infty} f\right)\chi_n\right\|_{X_u} \\
&\leqslant \left\|\left(\int_x^{n+1} f\right)\chi_n\right\|_{X_u} + \left\|\left(\sum_{k=n+1}^{\infty}\int_k^{k+1}\right)\chi_n\right\|_{X_u}.
\end{aligned}$$

So that

$$\begin{aligned}\|T^*f\|_{\ell^q(X_u)} &\leqslant \left(\sum_{n\in\mathbb{Z}}\left\|\left(\int_x^{n+1} f\right)\chi_n\right\|_{X_u}^q\right)^{\frac{1}{q}} + \left(\sum_{n\in\mathbb{Z}}\left(\sum_{k=n+1}^{\infty}\int_k^{k+1} f\right)^q \|\chi_n\|_{X_u}^q\right)^{\frac{1}{q}}\\ &= J_1^* + J_2^* . \end{aligned} \tag{4.4}$$

Since (4.3) holds, we find,

$$J_1^* \leqslant C\|f\|_{\bar{p},v,\bar{q}} . \tag{4.5}$$

Taking $U_n = \|\chi_n\|_{X_u}$, $\tilde{A}_k = \int_k^{k+1} f$, using (4.2) and applying analogous procedure as done in Theorem 1, we get

$$\begin{aligned}J_2^* &= \left(\sum_{n\in\mathbb{Z}}\left(\sum_{k=n+1}^{\infty}\tilde{A}_k\right)^q U_n^q\right)^{\frac{1}{q}}\\ &\leqslant C\left(\sum_{n\in\mathbb{Z}}\tilde{A}_n^{\bar{q}}\tilde{V}_n\right)^{\frac{1}{\bar{q}}}\\ &\leqslant C\|f\|_{\bar{p},v,\bar{q}} . \end{aligned} \tag{4.6}$$

Sufficiency now follows from (4.4), (4.5) and (4.6).

In order to prove the necessity, define for any non-negative sequence $\{a_n\}$, a function

$$f = \sum_{k\in\mathbb{Z}} a_k v^{1-\bar{p}'}\chi_k$$

so that if we write $\tilde{B}_k = \left(a_k\int_k^{k+1} v^{1-\bar{p}'}\right)$, we have for $n \leqslant x < n+1$;

$$|(T^*f)(x)| = \left|\int_x^n f + \int_n^{\infty} f\right| \geqslant \left(\sum_{k=n}^{\infty}\tilde{B}_k\right)$$

which gives

$$\|T^*f\|_{\ell^q(X_u)} \geqslant \left(\sum_{n\in\mathbb{Z}}\left(\sum_{k=n}\tilde{B}_k\right)^q \|\chi_n\|_{X_u}^q\right)^{\frac{1}{q}}$$

and also for this f

$$\|f\|_{\bar{p},v,\bar{q}} = \left(\sum_{n\in\mathbb{Z}}\tilde{B}_n^{\bar{q}}\tilde{V}_n\right)^{\frac{1}{\bar{q}}} .$$

Consequently, (4.1) becomes

$$\left(\sum_{n\in\mathbb{Z}}\left(\sum_{k=n}^{\infty}\tilde{B}_k\right)^q \|\chi_n\|_{X_u}^q\right)^{\frac{1}{q}} \leqslant C\left(\sum_{n\in\mathbb{Z}}\tilde{B}_n^{\bar{q}}\tilde{V}_n\right)^{\frac{1}{\bar{q}}}$$

for $\{\tilde{B}_k\} \in \ell^{\bar{q}}_{\{\tilde{V}_n\}}$ which proves the necessity of (4.2). Finally, in order to prove the necessity of (4.3), take for a fixed $m \in \mathbb{Z}$, $g \geqslant 0$ and $m < x \leqslant m+1$

$$f = g\chi_m ;$$

so that the LHS of (4.1) can be estimated as

$$\|T^* f\|_{\ell^q(X_u)} = \left\| \left(\int_x^\infty f \right) \chi_m \right\|_{X_u} \geqslant \left\| \left(\int_x^{m+1} g \right) \chi_m \right\|_{X_u}$$

while

$$\|f\|_{\bar{p},v,\bar{q}} = \left(\int_m^{m+1} g^{\bar{p}} v \right)^{\frac{1}{\bar{p}}}$$

and (4.1) becomes

$$\left\| \left(\int_x^{m+1} g \right) \chi_m \right\|_{X_u} \leqslant C \left(\int_m^{m+1} g^{\bar{p}} v \right)^{\frac{1}{\bar{p}}}$$

for each $m \in \mathbb{Z}$ and the assertion follows. □

REMARK 6. Naturally, one may expect a result corresponding to Theorem 2, i.e., precise weight conditions for the inequality (4.1) to hold. This involves precise weight conditions for the inequality (4.3) which, unfortunately, are not yet known. However, for the special case when X is taken as the space X^p, these conditions have been obtained in [5], [6]. Using these conditions, we can obtain the result corresponding to Theorem 3. This result reads:

THEOREM 5. *Let $1 < p, q, \bar{p}, \bar{q} < \infty$ and u, v be weight functions. Then the inequality*

$$|T^* f|_{\ell^q(X_u^p)} \leqslant C\|f\|_{\bar{p},v\bar{q}}$$

holds for all $f \in \ell^{\bar{q}}(L_v^{\bar{p}})$ if and only if

(a) *for $\bar{p} \leqslant p$ and $\bar{q} \leqslant q$*

$$C_1^* := \sup_{m\in\mathbb{Z}} \left\{ \sum_{n=-\infty}^{m-1} \|\chi_n u^{\frac{1}{p}}\|_{X_p}^q \right\}^{\frac{1}{q}} \left\{ \sum_{n=m}^{\infty} \left(\int_n^{n+1} v^{1-\bar{p}'} \right)^{\frac{\bar{q}'}{\bar{p}'}} \right\}^{\frac{1}{\bar{q}'}} < \infty$$

and

$$C_2^* := \sup_{m\in\mathbb{Z}} \sup_{m<t<m+1} \left(\|\chi_{[m,t]} u^{\frac{1}{p}}\|_{X_p} \right) \left(\int_t^{m+1} v^{1-\bar{p}'} \right)^{\frac{1}{\bar{p}'}} < \infty$$

(b) *for* $q < \bar{q}$ *and* $\bar{p} \leqslant p$; $C_2^* < \infty$ *and*

$$C_3^* := \left\{ \sum_{k\in\mathbb{Z}} \left[\sum_{n=-\infty}^{k} \|\chi_n u^{\frac{1}{p}}\|_{X^p}^q \right]^{\frac{\beta}{q}} \times \left[\sum_{n=k}^{\infty} \left(\int_{n-1}^{n} v^{1-\bar{p}'} \right)^{\frac{\bar{q}'}{\bar{p}'}} \right]^{\frac{\beta}{q'}} \|\chi_k u^{\frac{1}{p}}\|_{X_p}^q \right\}^{\frac{1}{\beta}} < \infty$$

where $\dfrac{1}{\beta} = \dfrac{1}{q} - \dfrac{1}{\bar{q}}$.

(c) *for* $q < \bar{q}$ *and* $p \leqslant \bar{p}$; $C_3^* < \infty$ *and*

$$C_4^* := \left(\int_n^{n+1} \|\chi_{[n,t]} u^{\frac{1}{p}}\|_{X^p}^{\alpha} \left(\int_t^{n+1} v^{1-\bar{p}'} \right)^{\frac{\alpha}{\bar{p}'}} \|\chi_{[n,t]} u^{\frac{1}{p}}\|_{X^p} \right)^{\frac{1}{\alpha}} < \infty$$

where $\dfrac{1}{\alpha} = \dfrac{1}{p} - \dfrac{1}{\bar{p}}$.

(d) *for* $\bar{q} < q$ *and* $p \leqslant \bar{p}$

$$C_1^* < \infty \quad \textit{and} \quad C_4^* < \infty .$$

REMARK 7. Theorem 5 extends a result of Carton-Lebrun, Heining and Hofmann [3] who proved it for $X = L^1$.

REMARK 8. In this paper, one of the amalgam spaces has been $\ell^{\bar{q}}(L_v^{\bar{p}})$. It is of interest if this space is taken also as a more general space such as $\ell^{\bar{q}}(X_v^{\bar{p}})$ or even $\ell^{\bar{p}}(Y_v)$, X_v, Y_v being weighted BFS different from X_u. In this direction, one needs to extend Theorems D and E to X^p - X^q boundedness of the operators T and T^*.

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