

ON CONVERGENCE ANALYSIS OF AN ITERATIVE ALGORITHM FOR FINDING COMMON SOLUTION OF GENERALIZED MIXED EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS

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Abstract. The purpose of this paper is to investigate nonemptiness of the solution set for generalized mixed quasi-equilibrium problems and investigate the asymptotic behavior of an iterative algorithm for finding common solution of generalized mixed equilibrium problems and fixed point problems of asymptotically nonexpansive mappings under mild conditions of iteration parameters. Our results improve and extend the recent known results of equilibrium problems, variational inequalities and fixed point theory.

1. Introduction and formulation of the problem

Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \rightarrow 2^H$ a multi-valued mapping. Let $\varphi : C \times C \rightarrow \mathbb{R}$ be a real-valued function and let $\Phi : H \times C \times C \rightarrow \mathbb{R}$ be the equilibrium-like function, i.e.,

$$(\Phi 0) \quad \Phi(w, u, v) + \Phi(w, v, u) = 0 \quad \text{for all } (w, u, v) \in H \times C \times C.$$

Several problems arising in optimization, such as fixed point problems, (Nash) economic equilibrium problems, complementarity problems, and generalized set valued mixed variational inequalities, for instance, have the same mathematical formulation, which may be stated as follows: Given a nonempty closed convex subset C of a real Hilbert space H , real-valued functions $\varphi : C \times C \rightarrow \mathbb{R}$, $\Phi : H \times C \times C \rightarrow \mathbb{R}$, $T : C \rightarrow 2^H$,

$$\begin{aligned}
 &\text{find } u \in C \text{ and } w \in T(u) \text{ such that} \\
 &\quad \Phi(w, u, v) + \varphi(v, u) - \varphi(u, u) \geq 0 \text{ for all } v \in C. \quad (1.1)
 \end{aligned}$$

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It is said to be *generalized mixed equilibrium problem* (for short, GMEP). If T is a single-valued mapping, then the problem (1.1) is to find $u \in C$ such that

$$\Phi(T(u), u, v) + \varphi(v, u) - \varphi(u, u) \geq 0 \quad \text{for all } v \in C. \quad (1.2)$$

We denote Ω for the set of solutions of GMEP(1.1).

Special cases.

(1) Given $\Phi(w, u, v) = F(u, v)$, where $F : C \times C \rightarrow \mathbb{R}$, then GMEP(1.1) reduces to the following equilibrium problem:

$$\text{find } u \in C \text{ such that } F(u, v) + \varphi(v, u) - \varphi(u, u) \geq 0 \quad \text{for all } v \in C.$$

Existence theorems for such problems were studied by Flores-Bazan [11].

(2) If $\varphi(x, y) = x$, then GMEP(1.1) reduces to the following generalized equilibrium problem:

$$\text{(GEP)} \quad \text{find } u \in C \text{ and } w \in T(u) \text{ such that } \Phi(w, u, v) + \varphi(v) - \varphi(u) \geq 0 \quad \text{for all } v \in C.$$

The GEP was studied by Ceng, Ansari and Yao [3].

(3) If $\varphi(x, y) = 0$ and $\Phi(w, u, v) = F(u, v)$, then GMEP(1.1) reduces to the following equilibrium problem:

$$\text{(EP)} \quad \text{find } u \in C \text{ such that } F(u, v) \geq 0 \quad \text{for all } v \in C.$$

(4) Given a mapping $N : H \times H \rightarrow H$, let $\varphi(x, y) = x$ and $\Phi(w, u, v) = \langle N(w, u), v - u \rangle$ for all $(w, u, v) \in H \times C \times C$, then GMEP(1.1) reduces to the following generalized set-valued strongly nonlinear mixed variational inequality:

$$\text{find } u \in C \text{ and } w \in T(u) \text{ such that} \\ \langle N(w, u), v - u \rangle + \varphi(v) - \varphi(u) \geq 0 \text{ for all } v \in C. \quad (1.3)$$

It has been studied in Zeng, Schaible and Yao [30] in the case when $C = H$.

When T is single-valued and $N(w, u) = T(u)$, problem (1.3) reduces to the following problem considered by Dien [8] and Noor [20]:

$$\text{find } u \in C \text{ such that } \langle T(u), v - u \rangle + \varphi(v) - \varphi(u) \geq 0 \quad \text{for all } v \in C.$$

(5) If $\varphi(x, y) = 0$ and $\Phi(T(u), u, v) = \langle T(u), v - u \rangle$, then GMEP(1.2) reduces to the classical variational inequality:

$$\text{find } u \in C \text{ such that } \langle T(u), v - u \rangle \geq 0 \quad \text{for all } v \in C. \quad (1.4)$$

In recent years, several numerical techniques including projection, resolvent and auxiliary principle have been developed and analyzed for solving variational inequalities. It is well-known that projection and resolvent type techniques can not be extended for equilibrium problems. To overcome this difficulty, several authors (see, e.g. [2, 3, 4,

10, 21]) have applied the auxiliary principle technique introduced by Glowinski, Lions, and Tremolieres [12] for approximating solutions of equilibrium problems.

The viscosity approximation method is one of the important methods for approximation fixed points of nonexpansive type mappings. The viscosity approximation method was first discussed by Moudafi [18]. There are already several viscosity-like methods and the research is intensively continued which are very useful for finding a common element of set of fixed points of nonexpansive type mapping and set of solutions of variational inequality (1.4), when $T = I - f$ and f is a contraction mapping. Recently, Hirstoaga [16] and Takahashi and Takahashi [27] applied viscosity approximation technique for finding a common element of set of solutions of an equilibrium problem (EP) and set of fixed points of a nonexpansive mapping. In literature, there exists a class of mappings which is essentially wider than the class of nonexpansive mappings.

A mapping $S : C \rightarrow C$ is said to be *asymptotically nonexpansive* (cf. [13]) if for each $n \in \mathbb{N}$, there exists a number $k_n \geq 1$ with $\lim_{n \rightarrow \infty} k_n = 1$ such that

$$\|S^n(x) - S^n(y)\| \leq k_n \|x - y\| \quad \text{for all } x, y \in C.$$

The following example shows that asymptotically nonexpansive mappings are not necessarily nonexpansive.

EXAMPLE 1.1. (cf. [13]) Let B_H be the closed unit ball in the Hilbert space $H = \ell_2$ and $S : B_H \rightarrow B_H$ a mapping defined by

$$S(x_1, x_2, x_3, \dots) = (0, x_1^2, a_2 x_2, a_3 x_3, \dots),$$

where $\{a_i\}$ is a sequence of real numbers such that $0 < a_i < 1$ and $\prod_{i=2}^{\infty} a_i = 1/2$. Then

$$\|S(x) - S(y)\| \leq 2 \|x - y\| \quad \text{for all } x, y \in B_H,$$

i.e., S is Lipschitzian, but not nonexpansive. Observe that

$$\|S^n(x) - S^n(y)\| \leq 2 \prod_{i=2}^n a_i \|x - y\| \quad \text{for all } x, y \in B_H \quad \text{and } n \geq 2.$$

Here $k_n = 2 \prod_{i=2}^n a_i \rightarrow 1$ as $n \rightarrow \infty$. Therefore, S is asymptotically nonexpansive, but not nonexpansive.

Note that every nonempty closed convex bounded subset of a Hilbert space H has a fixed point for asymptotically nonexpansive mappings (see [13]). An iterative method for approximation of fixed points of asymptotically nonexpansive mappings was developed by Schu [24]. Indeed, he modified well known Mann iteration process for asymptotically nonexpansive mappings and proved the following weak convergence theorem:

THEOREM 1.2. (Schu [24]) *Let C be a nonempty closed convex bounded subset of a Hilbert space H and $S : C \rightarrow C$ an asymptotically nonexpansive with sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ satisfying the*

condition $\varepsilon \leq \alpha_n \leq 1 - \varepsilon$ for all $n \in \mathbb{N}$ and for some $\varepsilon > 0$. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in C$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S^n(x_n) \quad \text{for all } n \in \mathbb{N} \tag{1.5}$$

converges weakly to a fixed point of S .

Iterative methods for approximation of fixed points of asymptotically nonexpansive mappings have been further studied by authors (see e.g. [1, 5, 7, 17, 22, 23, 25, 28] and references therein).

Motivated by Moudafi [18], Shahzad and Udomene [26] extended strong convergence theorem of Chidume, Liu and Udomene [6] as follows.

THEOREM 1.3. (Theorem 3.1, Shahzad and Udomene [26]) *Let C be a nonempty closed convex bounded subset of a real Hilbert space, $f : C \rightarrow C$ a contraction mapping with constant $\alpha \in [0, 1)$ and $S : C \rightarrow C$ an asymptotically nonexpansive mapping with sequence $\{k_n\}$ in $[1, \infty)$. Let $\{t_n\}$ a sequence in $(0, 1)$ such that $\lim_{n \rightarrow \infty} t_n = 1$, $t_n k_n \leq 1$, $\sum_{n=1}^{\infty} t_n(1 - t_n) = \infty$ and $\lim_{n \rightarrow \infty} \frac{k_n - 1}{k_n - t_n} = 0$. Define the sequence $\{z_n\}$ iteratively by $z_1 \in C$,*

$$z_{n+1} = \left(1 - \frac{t_n}{k_n}\right)f(z_n) + \frac{t_n}{k_n}S^n(z_n) \quad \text{for all } n \in \mathbb{N}. \tag{1.6}$$

Then, we have the following:

(a) for each $n \in \mathbb{N}$, there is a unique $x_n \in C$ such that

$$x_n = \left(1 - \frac{t_n}{k_n}\right)f(x_n) + \frac{t_n}{k_n}S^n(x_n), \tag{1.7}$$

(b) if in addition

$$\lim_{n \rightarrow \infty} \|x_n - S(x_n)\| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|z_n - S(z_n)\| = 0, \tag{1.8}$$

it follows that $\{z_n\}$ converges strongly to a fixed point of S .

In Theorem 1.3, the strong convergence of the almost fixed points

$$x_n = \left(1 - \frac{t_n}{k_n}\right)u + \frac{t_n}{k_n}S^n(x_n)$$

is applied for convergence of the iteration process (1.6). Now our concern is the following:

QUESTION 1.4. *Is it possible to develop an iterative algorithm for finding a common element of set of solutions of GMEP(1.1) and set of fixed points of an asymptotically nonexpansive mapping S ?*

The purpose of this paper is to provide necessary conditions for the solution set Ω of GMEP(1.1) to be nonempty and develop an iterative algorithm so that it can generate strongly convergent sequences. In Section 2, we will recall the useful definitions and lemmas. Section 3 is devoted to deal with the problem of existence of solutions of GMEP(1.1). Section 4 is devoted to develop an iterative algorithm for finding a common element of set of solutions of GMEP(1.1) and set of fixed points of an asymptotically nonexpansive mapping. The strong convergence of sequence $\{x_n\}$ defined by (1.7) is not applied in our results. Our theorems significantly improve and extend corresponding results of Ceng, Ansari and Yao [3], Ceng and Yao [4], Flores-Bazan [11], Noor [21], Shahzad and Udomene [26] and Takahashi and Takahashi [27] and also provide an affirmative answer to Question 1.4.

2. Basic definitions and preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively and let C be a closed convex subset of H . For every point $x \in H$, there exists a unique nearest point in C , denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the *metric projection* of H onto C . We know that P_C is a nonexpansive mapping of H onto C . It is also known that P_C satisfies

$$\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y \rangle \quad \text{for every } x, y \in H.$$

Furthermore, for $x \in H$ and $u \in C$,

$$u = P_C(x) \Leftrightarrow \langle x - u, u - y \rangle \geq 0 \quad \text{for all } y \in C.$$

Let C be a nonempty subset of real Hilbert space H and let $T : C \rightarrow H$ and $\eta : C \times C \rightarrow H$ be two mappings. Then T is called:

(i) η - *monotone* if

$$\langle T(x) - T(y), \eta(x, y) \rangle \geq 0 \quad \text{for all } x, y \in C;$$

(ii) η - *strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle T(x) - T(y), \eta(x, y) \rangle \geq \alpha \|x - y\|^2 \quad \text{for all } x, y \in C;$$

(iii) *Lipschitz continuous* if there exists a constant $\beta > 0$ such that

$$\|T(x) - T(y)\| \leq \beta \|x - y\| \quad \text{for all } x, y \in C.$$

When $\eta(x, y) = x - y$ for all $x, y \in C$, then the definitions (i) and (ii) reduce to the definitions of monotonicity and strong monotonicity, respectively.

The mapping $\eta : C \times C \rightarrow H$ is said to be *Lipschitz continuous* if there exists a constant $\lambda > 0$ such that

$$\|\eta(x, y)\| \leq \lambda \|x - y\| \quad \text{for all } x, y \in C.$$

Let C be a convex subset of a real Hilbert space H and $\kappa : C \rightarrow \mathbb{R}$ a Fréchet differential function. Then κ is said to be

(i) η - convex [15] if

$$\kappa(y) - \kappa(x) \geq \langle \kappa'(x), \eta(y, x) \rangle \quad \text{for all } x, y \in C,$$

where $\kappa'(x)$ is the Fréchet derivative of κ at x ;

(ii) η - convex strongly convex [20] if there exists a constant $\mu > 0$ such that

$$\kappa(y) - \kappa(x) - \langle \kappa'(x), \eta(y, x) \rangle \geq \frac{\mu}{2} \|x - y\|^2 \quad \text{for all } x, y \in C.$$

In particular, if $\eta(y, x) = y - x$ for all $y, x \in C$, then κ is said to be strongly convex. The following result is proved in [4].

PROPOSITION 2.1. (Proposition 2.1, Ceng and Yao [4]) *Let C be convex subset of a real Hilbert space H and $\eta : C \times C \rightarrow H$ a mapping such that $\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in C$. If $\kappa : C \rightarrow \mathbb{R}$ is a differentiable η -strongly convex functional with constant $\mu > 0$, then κ' is η -strongly monotone with constant $\mu > 0$.*

Let C be a nonempty subset of real Hilbert space H . The bi-function $\varphi(\cdot, \cdot) : C \times C \rightarrow \mathbb{R}$ is said to be skew-symmetric if

$$\varphi(u, v) + \varphi(v, u) - \varphi(u, u) - \varphi(v, v) \leq 0 \quad \text{for all } u, v \in C. \quad (2.1)$$

If the skew-symmetric bi-function $\varphi(\cdot, \cdot)$ is linear in both arguments, then

$$\varphi(u, u) \geq 0 \quad \text{for all } u \in C.$$

A function $\psi : C \times C \rightarrow \mathbb{R}$ is called *weakly sequentially continuous* at $(x_0, y_0) \in C \times C$, if $\psi(x_n, y_n) \rightarrow \psi(x_0, y_0)$ as $n \rightarrow \infty$ for each sequence $\{(x_n, y_n)\}$ in $C \times C$ converging weakly to (x_0, y_0) . The function $\psi(\cdot, \cdot)$ is called *weakly sequentially continuous on $C \times C$* , if it is weakly sequentially continuous at each point of $C \times C$.

We will adopt the following notations:

1. \rightharpoonup for weak convergence and \rightarrow for strong convergence.
2. $F(T) = \{x \in C : Tx = x\}$ denotes the set of fixed points of a self-mapping T on a set C .

Let A be a nonempty subset of a metric space X . For $x \in X$, define

$$d(x, A) = \inf\{d(x, y) : y \in A\}.$$

Let $CB(X)$ denote the set of nonempty closed bounded subsets of X .

For $A, B \in CB(X)$, define

$$\begin{aligned} \delta(A, B) &= \sup\{d(x, B) : x \in A\}, \\ \mathcal{H}(A, B) &= \max\{\delta(A, B), \delta(B, A)\} = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}. \end{aligned}$$

\mathcal{H} is called the Hausdorff metric on $CB(X)$.

REMARK 2.2. Let $A, B \in CB(X)$ and $a \in A$. Then $d(a, B) \leq \mathcal{H}(A, B)$. Indeed,

$$d(a, B) \leq \max\{d(a, B), \delta(B, A)\} \leq \max\{\delta(A, B), \delta(B, A)\} = \mathcal{H}(A, B).$$

LEMMA 2.3. (Nadler’s theorem, Nadler [19]) *Let $A, B \in CB(X)$ and $a \in A$. Then for $\varrho > 1$, there must exist a point $b \in B$ such that $d(a, b) \leq \varrho H(A, B)$.*

Let E be a topological vector space. The set of all nonempty subsets of E will be denoted by 2^E . For subset D of E , we denote by $co(D)$, convex hull of D .

DEFINITION 2.4. *Let C be nonempty subset of a topological vector space E . A multi-valued mapping $G : C \rightarrow 2^E$ is called a KKM-mapping if for every finite subset $\{v_1, v_2, \dots, v_n\}$ of C ,*

$$co(\{v_1, v_2, \dots, v_n\}) \subseteq \bigcup_{i=1}^n G(v_i).$$

The following lemma was given in Fan [9].

LEMMA 2.5. *Let C be nonempty subset of a Hausdorff topological vector space E and $G : C \rightarrow 2^E$ a closed-valued KKM-mapping. If $G(x_0)$ is compact for at least one $x_0 \in C$, then $\bigcap_{v \in C} G(v) \neq \emptyset$.*

In what follows, we shall make use of the following lemmas.

LEMMA 2.6. *Let H be a real Hilbert space. Then*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle \quad \text{for all } x, y \in H.$$

LEMMA 2.7. (Proposition 5.3, Goebel and Reich [14]) *Let C be a nonempty closed convex subset of a strictly convex Banach space X and $T : C \rightarrow C$ a nonexpansive mapping with $F(T) \neq \emptyset$. Then $F(T)$ is closed and convex.*

LEMMA 2.8. (Theorem 1, Goebel and Kirk [13]) *If C is a nonempty closed convex bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive mapping $T : C \rightarrow C$ has a fixed point in C .*

LEMMA 2.9. (Lemma 1, Osilike and Aniagbosor [22]) *Let $\{\delta_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three sequences of nonnegative numbers such that*

$$\beta_n \geq 1 \text{ and } \delta_{n+1} \leq \beta_n \delta_n + \gamma_n \quad \text{for all } n \in \mathbb{N}.$$

If $\sum_{n=1}^\infty (\beta_n - 1) < \infty$ and $\sum_{n=1}^\infty \gamma_n < \infty$, then $\lim_{n \rightarrow \infty} \delta_n$ exists.

LEMMA 2.10. (Lemma 2.5, Xu [29]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n \sigma_n + \xi_n \quad \text{for all } n \in \mathbb{N}, \tag{2.2}$$

where $\{\lambda_n\}$, $\{\sigma_n\}$ and $\{\xi_n\}$ are sequences of reals numbers satisfy the conditions:

(i) $\{\lambda_n\} \subset [0, 1]$, $\lim_{n \rightarrow \infty} \lambda_n = 0$ and $\sum_{n=1}^\infty \lambda_n = \infty$;

(ii) $\limsup_{n \rightarrow \infty} \sigma_n \leq 0$;

(iii) $\xi_n \geq 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^\infty \xi_n < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

LEMMA 2.11. Let $\{a_n\}$ be a sequence of nonnegative real numbers such that

$$a_{n+1} \leq \theta a_n b_n + c_n \quad \text{for all } n \in \mathbb{N}, \quad (2.3)$$

where $\theta \in (0, 1)$ and $\{b_n\}$ and $\{c_n\}$ are sequences satisfy the conditions:

(i) $b_n \geq 1$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} (b_n - 1) < \infty$,

(ii) $c_n \geq 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} c_n < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

Proof. The inequality (2.3) reduces to the following inequality:

$$a_{n+1} \leq a_n b_n + c_n \quad \text{for all } n \in \mathbb{N}.$$

Note that Lemma 2.9 implies that $\lim_{n \rightarrow \infty} a_n$ exists. Suppose $\lim_{n \rightarrow \infty} a_n = d$ for some $d > 0$. The conditions (i) and (ii) imply that $\lim_{n \rightarrow \infty} b_n = 1$ and $\lim_{n \rightarrow \infty} c_n = 0$. It follows from (2.3) that

$$d \leq \theta d.$$

Therefore, $\lim_{n \rightarrow \infty} a_n = 0$. \square

LEMMA 2.12. Let $\{x_n\}$ be a sequence in a normed space $(X, \|\cdot\|)$ such that

$$\|x_{n+1} - x_{n+2}\| \leq \theta \|x_n - x_{n+1}\| b_n + c_n \quad \text{for all } n \in \mathbb{N}, \quad (2.4)$$

where $\theta \in (0, 1)$ and $\{b_n\}$ and $\{c_n\}$ are sequences satisfy the conditions:

(i) $b_n \geq 1$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} (b_n - 1) < \infty$,

(ii) $c_n \geq 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} c_n < \infty$.

Then $\{x_n\}$ is a Cauchy sequence.

Proof. Set $a_n := \|x_n - x_{n+1}\|$. From Lemma 2.11, we have $\lim_{n \rightarrow \infty} a_n = 0$. From (2.4), we have, for $m > n \geq 1$

$$\begin{aligned} \sum_{i=n}^{m-1} a_i &\leq \sum_{i=n}^{m-1} (\theta a_{i-1} b_{i-1} + c_{i-1}) \\ &\leq \theta (a_{n-1} b_{n-1} + a_n b_n + \dots + a_{m-2} b_{m-2}) + \sum_{i=n}^{m-1} c_{i-1} \\ &\leq \theta (a_n b_n + \dots + a_{m-2} b_{m-2} + a_{m-1} b_{m-1} + a_{n-1} b_{n-1} - a_{m-1} b_{m-1}) + \sum_{i=n}^{m-1} c_{i-1} \\ &\leq \theta (a_n b_n + \dots + a_{m-2} b_{m-2} + a_{m-1} b_{m-1}) + \theta a_{n-1} b_{n-1} + \sum_{i=n}^{m-1} c_{i-1} \end{aligned}$$

$$\begin{aligned} &\leq \theta \left(\prod_{i=n}^{m-1} b_i \right) (a_n + \dots + a_{m-2} + a_{m-1}) + \theta a_{n-1} b_{n-1} + \sum_{i=n}^{\infty} c_{i-1} \\ &\leq \theta \left(\prod_{i=n}^{\infty} b_i \right) \left(\sum_{i=n}^{m-1} a_i \right) + \theta a_{n-1} b_{n-1} + \sum_{i=n}^{\infty} c_{i-1}, \end{aligned}$$

which implies that

$$\sum_{i=n}^{m-1} a_i \leq \frac{\theta a_{n-1} b_{n-1}}{1 - \theta \left(\prod_{i=n}^{\infty} b_i \right)} + \frac{\sum_{i=n}^{\infty} c_{i-1}}{1 - \theta \left(\prod_{i=n}^{\infty} b_i \right)}. \tag{2.5}$$

Note that the conditions (i) and (ii) imply that $\lim_{n \rightarrow \infty} b_n = 1$, $\lim_{n \rightarrow \infty} \prod_{i=n}^{\infty} b_i = 1$, $\lim_{n \rightarrow \infty} c_n = 0$ and $\lim_{n \rightarrow \infty} \sum_{i=n}^{\infty} c_i = 0$. Observe that

$$\begin{aligned} \|x_m - x_n\| &\leq \sum_{i=n}^{m-1} \|x_i - x_{i+1}\| = \sum_{i=n}^{m-1} a_i \\ &\leq \frac{\theta a_{n-1} b_{n-1}}{1 - \theta \left(\prod_{i=n}^{\infty} b_i \right)} + \frac{\sum_{i=n}^{\infty} c_{i-1}}{1 - \theta \left(\prod_{i=n}^{\infty} b_i \right)} \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Therefore, $\lim_{m,n \rightarrow \infty} \|x_m - x_n\| = 0$. \square

3. Existence and uniqueness of solutions of auxiliary problems

Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \rightarrow 2^H$ a multi-valued mapping. For $x \in C$, let $w \in T(x)$. Let $\varphi : C \times C \rightarrow \mathbb{R}$ be a real-valued function satisfying:

- ($\varphi 1$) $\varphi(\cdot, \cdot)$ is skew symmetric;
- ($\varphi 2$) for each fixed $y \in C$, $\varphi(\cdot, y)$ is convex and upper semicontinuous;
- ($\varphi 3$) $\varphi(\cdot, \cdot)$ is weakly continuous on $C \times C$.

Let $\kappa : C \rightarrow \mathbb{R}$ be a differentiable functional with Fréchet derivative $\kappa'(x)$ at x satisfying

- ($\kappa 1$) κ' is sequentially continuous from the weak topology to the strong topology,
- ($\kappa 2$) κ' is Lipschitz continuous with Lipschitz constant $v > 0$.

Let $\eta : C \times C \rightarrow H$ be a function satisfying:

- ($\eta 1$) $\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in C$;
- ($\eta 2$) $\eta(\cdot, \cdot)$ is affine in the first coordinate variable;
- ($\eta 3$) for each fixed $y \in C$, $x \mapsto \eta(y, x)$ is sequentially continuous from the weak topology to the weak topology.

To solve the generalized mixed equilibrium problem (1.1), let us consider the equilibrium-like function $\Phi : H \times C \times C \rightarrow \mathbb{R}$ which satisfies the following conditions with respect to the multi-valued mapping $T : C \rightarrow 2^H$:

($\Phi 1$) for each fixed $v \in C$, $(w, u) \mapsto \Phi(w, u, v)$ is an upper semicontinuous function from $H \times C$ to \mathbb{R} , i.e.,

$$w_n \rightarrow w \text{ and } u_n \rightarrow u \text{ imply } \limsup_{n \rightarrow \infty} \Phi(w_n, u_n, v) \leq \Phi(w, u, v);$$

($\Phi 2$) for each fixed $(w, v) \in H \times C$, $u \mapsto \Phi(w, u, v)$ is a concave function;

($\Phi 3$) for each fixed $(w, u) \in H \times C$, $v \mapsto \Phi(w, u, v)$ is a convex function.

Let r be a positive parameter. For a given element $x \in C$ and $w_x \in T(x)$, consider the following auxiliary problem for GMEP(1.1):

$$\begin{cases} \text{find } u \in C \text{ such that} \\ \Phi(w_x, u, v) + \varphi(v, u) - \varphi(u, u) + \frac{1}{r} \langle \kappa'(u) - \kappa'(x), \eta(v, u) \rangle \geq 0 \text{ for all } v \in C. \end{cases} \quad (3.1)$$

It is easy to see that if $u = x$, then u is a solution of GMEP(1.1).

For the case of single-valued mapping T , for given $x \in C$, we consider the following auxiliary problem for GMEP(1.2):

$$\begin{cases} \text{find } u \in C \text{ such that} \\ \Phi(T(x), u, v) + \varphi(v, u) - \varphi(u, u) + \frac{1}{r} \langle \kappa'(u) - \kappa'(x), \eta(v, u) \rangle \geq 0 \text{ for all } v \in C, \end{cases} \quad (3.2)$$

where $r > 0$ is a constant.

We now establish main existence theorem of this section.

THEOREM 3.1. *Let C be a nonempty closed convex bounded subset of a real Hilbert space H and $\varphi : C \times C \rightarrow \mathbb{R}$ a real-valued function satisfying the conditions ($\varphi 1$) \sim ($\varphi 3$). Let $T : C \rightarrow 2^H$ be a multi-valued mapping and $\Phi : H \times C \times C \rightarrow \mathbb{R}$ be the equilibrium-like function satisfying the conditions ($\Phi 1$) \sim ($\Phi 3$). Assume that $\eta : C \times C \rightarrow H$ is a Lipschitz function with Lipschitz constant $\lambda > 0$ which satisfies the conditions ($\eta 1$) \sim ($\eta 3$). Let $\kappa : C \rightarrow \mathbb{R}$ be a η -strongly convex function with constant $\mu > 0$ which satisfies the conditions ($\kappa 1$) \sim ($\kappa 1$). For each $x \in C$, let $w_x \in T(x)$. For $r > 0$, define a mapping $T_r : C \rightarrow C$ by*

$$T_r(x) = \{u \in C : \Phi(w_x, u, v) + \varphi(v, u) - \varphi(u, u) + \frac{1}{r} \langle \kappa'(u) - \kappa'(x), \eta(v, u) \rangle \geq 0 \text{ for all } v \in C\}.$$

Then we have the following:

(a) The auxiliary problem (3.1) has a unique solution;

(b) T_r is single-valued;

(c) if $\frac{\lambda v}{\mu} \leq 1$ and $\Phi(w_1, T_r(x_1), T_r(x_2)) + \Phi(w_2, T_r(x_2), T_r(x_1)) \leq 0$ for all $x_1, x_2 \in C$ and all $w_1 \in T(x_1), w_2 \in T(x_2)$, it follows that T_r is nonexpansive;

(d) $F(T_r) = \Omega$;

(e) Ω is closed and convex.

Proof. (a) We divide the proof into the following steps.

Step 1. *Existence of solutions of auxiliary problem (3.1).*

The auxiliary problem (3.1) can be written as follows: to find $u \in C$ such that

$$r[\Phi(w_x, u, v) + \varphi(v, u) - \varphi(u, u)] + \langle \kappa'(u) - \kappa'(x), \eta(v, u) \rangle \geq 0 \quad \text{for all } v \in C.$$

For each $v \in C$, we define

$$G(v) = \{z \in C : r[\Phi(w_x, z, v) + \varphi(v, z) - \varphi(z, z)] + \langle \kappa'(z) - \kappa'(x), \eta(v, z) \rangle \geq 0\}. \tag{3.3}$$

Note that for each $v \in C$, $G(v)$ is nonempty since $v \in G(v)$. First, we show that G is a KKM mapping. Suppose, for contradiction, that there exists a finite subset $\{v_1, v_2, \dots, v_n\}$ of C and $\alpha_i \geq 0$ for all $i = 1, 2, \dots, n$ with $\sum_{i=1}^n \alpha_i = 1$ such that

$$\hat{v} = \sum_{i=1}^n \alpha_i v_i \notin G(v_i) \quad \text{for all } i = 1, 2, \dots, n.$$

By the definition of G , we have

$$r[\Phi(w_x, \hat{v}, v_i) + \varphi(v_i, \hat{v}) - \varphi(\hat{v}, \hat{v})] + \langle \kappa'(\hat{v}) - \kappa'(x), \eta(v_i, \hat{v}) \rangle < 0 \quad \text{for all } i = 1, 2, \dots, n;$$

which implies that

$$\sum_{i=1}^n \alpha_i r[\Phi(w_x, \hat{v}, v_i) + \varphi(v_i, \hat{v}) - \varphi(\hat{v}, \hat{v})] + \langle \kappa'(\hat{v}) - \kappa'(x), \sum_{i=1}^n \alpha_i \eta(v_i, \hat{v}) \rangle < 0.$$

Since Φ is an equilibrium-like function, it follows that $\Phi(w_x, \hat{v}, \hat{v}) = 0$. Note that $(\Phi 3)$ implies the convexity of functional $v \mapsto \Phi(w_x, \hat{v}, v)$. Since $\varphi(\cdot, \hat{v})$ is convex and $\eta(\cdot, \hat{v})$ is affine, we have

$$\begin{aligned} 0 &= r[\Phi(w_x, \hat{v}, \hat{v}) + \varphi(\hat{v}, \hat{v}) - \varphi(\hat{v}, \hat{v})] + \langle \kappa'(\hat{v}) - \kappa'(x), \eta(\hat{v}, \hat{v}) \rangle \\ &\leq \sum_{i=1}^n \alpha_i r[\Phi(w_x, \hat{v}, v_i) + \varphi(v_i, \hat{v}) - \varphi(\hat{v}, \hat{v})] + \langle \kappa'(\hat{v}) - \kappa'(x), \sum_{i=1}^n \alpha_i \eta(v_i, \hat{v}) \rangle \\ &< 0, \end{aligned}$$

a contradiction. Hence G is a KKM mapping.

Since $\overline{G(v)}^w$ is a weakly closed subset of the bounded set C in a Hilbert space H for each $v \in C$, it follows that $\overline{G(v)}^w$ is weakly compact. Using Lemma 2.5, we obtain that $\bigcap_{v \in C} \overline{G(v)}^w \neq \emptyset$. Suppose $x^* \in \bigcap_{v \in C} \overline{G(v)}^w \neq \emptyset$. We now show that $\overline{G(v)}^w = G(v)$ for each $v \in C$, i.e., $G(v)$ is weakly closed. Let $u \in \overline{G(v)}^w$ and $\{z_i\}$ a sequence in $G(v)$ such that $z_i \rightharpoonup u \in C$. From (3.3), we have

$$r[\Phi(w_x, z_i, v) + \varphi(v, z_i) - \varphi(z_i, z_i)] + \langle \kappa'(z_i) - \kappa'(x), \eta(v, z_i) \rangle \geq 0. \tag{3.4}$$

By $(\eta 3)$, we obtain that $\{\eta(v, z_i)\}$ converges weakly to $\{\eta(v, u)\}$ for each fixed $v \in C$. By $(\kappa 2)$, one can see that $\|\kappa'(z_i) - \kappa'(u)\| \rightarrow 0$ as $n \rightarrow \infty$. So, we have

$$\begin{aligned} & | \langle \kappa'(z_i) - \kappa'(x), \eta(v, z_i) \rangle - \langle \kappa'(u) - \kappa'(x), \eta(v, u) \rangle | \\ &= | \langle \kappa'(z_i) - \kappa'(u), \eta(v, z_i) \rangle + \langle \kappa'(u) - \kappa'(x), \eta(v, z_i) - \eta(v, u) \rangle | \\ &\leq \| \kappa'(z_i) - \kappa'(u) \| \| \eta(v, z_i) \| + | \langle \kappa'(u) - \kappa'(x), \eta(v, z_i) - \eta(v, u) \rangle | \rightarrow 0 \\ & \hspace{15em} \text{as } n \rightarrow \infty. \end{aligned}$$

Since $\varphi(\cdot, \cdot)$ is weakly continuous on $C \times C$, we have

$$z_i \rightharpoonup u \quad \Rightarrow \quad \lim_{i \rightarrow \infty} \varphi(z_i, z_i) = \varphi(u, u).$$

Further, $(\Phi 1) \sim (\Phi 2)$ imply that the weak upper semicontinuity of the functional $z \mapsto \Phi(w_x, z, v)$. It follows that $\Phi(w_x, z, v) \geq \limsup_{n \rightarrow \infty} \Phi(w_x, z_i, v)$. Thus, from (3.4), we have

$$\begin{aligned} & r[\Phi(w_x, u, v) + \varphi(v, u) - \varphi(u, u)] + \langle \kappa'(u) - \kappa'(x), \eta(v, u) \rangle \\ & \geq r[\limsup_{i \rightarrow \infty} \Phi(N(w_x, z_i, v) + \limsup_{i \rightarrow \infty} \varphi(v, z_i) - \liminf_{i \rightarrow \infty} \varphi(z_i, z_i)] \\ & \quad + \limsup_{i \rightarrow \infty} \langle \kappa'(z_i) - \kappa'(x), \eta(v, z_i) \rangle \\ & \geq \limsup_{i \rightarrow \infty} [r[\Phi(N(w_x, z_i, v) + \varphi(v, z_i) - \varphi(z_i, z_i)] + \langle \kappa'(z_i) - \kappa'(x), \eta(v, z_i) \rangle] \\ & \geq 0. \end{aligned}$$

This shows that $u \in G(v)$. Hence $G(v)$ is a weakly closed, i.e., $\overline{G(v)}^w = G(v)$. So, $x^* \in \bigcap_{v \in C} G(v)$, i.e., x^* is a solution of the auxiliary problem (3.1).

Step 2. *Uniqueness of solutions of auxiliary problem (3.1).*

Let u_1 and u_2 be two distinct solutions of the auxiliary problem (3.1). Then

$$r[\Phi(w_x, u_1, v) + \varphi(v, u_1) - \varphi(u_1, u_1)] + \langle \kappa'(u_1) - \kappa'(x), \eta(v, u_1) \rangle \geq 0 \quad (3.5)$$

and

$$r[\Phi(w_x, u_2, v) + \varphi(v, u_2) - \varphi(u_2, u_2)] + \langle \kappa'(u_2) - \kappa'(x), \eta(v, u_2) \rangle \geq 0 \quad \text{for all } v \in C. \quad (3.6)$$

Taking $v = u_2$ in (3.5) and $v = u_1$ in (3.6) and adding up these inequalities, we obtain

$$\begin{aligned} & r[\Phi(w_x, u_1, u_2) + \Phi(w_x, u_2, u_1) + \varphi(u_2, u_1) - \varphi(u_1, u_1) + \varphi(u_1, u_2) - \varphi(u_2, u_2)] \\ & \quad + \langle \kappa'(u_1) - \kappa'(x), \eta(u_2, u_1) \rangle + \langle \kappa'(u_2) - \kappa'(x), \eta(u_1, u_2) \rangle \geq 0. \quad (3.7) \end{aligned}$$

From $(\Phi 0)$, $(\varphi 1)$ and $(\eta 1)$, we have

$$\Phi(w_x, u_1, u_2) + \Phi(w_x, u_2, u_1) = 0,$$

$$\varphi(u_2, u_1) - \varphi(u_1, u_1) + \varphi(u_1, u_2) - \varphi(u_2, u_2) \leq 0$$

and

$$\eta(u_1, u_2) + \eta(u_2, u_1) = 0.$$

It follows from (3.7) that

$$-\langle \kappa'(u_1) - \kappa'(x), \eta(u_1, u_2) \rangle + \langle \kappa'(u_2) - \kappa'(x), \eta(u_1, u_2) \rangle \geq 0,$$

which implies that

$$\langle \kappa'(u_1) - \kappa'(u_2), \eta(u_1, u_2) \rangle \leq 0.$$

Since $\kappa' : C \rightarrow H$ is η -strongly monotone with constant $\mu > 0$, we obtain

$$\mu \|u_1 - u_2\|^2 \leq \langle \kappa'(u_1) - \kappa'(u_2), \eta(u_1, u_2) \rangle \leq 0.$$

Hence solution of the auxiliary problem (3.1) is unique.

(b) It is easy to see from Steps 1 and 2 of part (a) that T_r is a single-valued.

(c) We claim that T_r is nonexpansive. For this, let us consider x_1 and x_2 be two elements in C such that $w_{x_i} \in T(x_i)$ for $i = 1, 2$. By the definition of T_r , we have

$$\begin{aligned} r[\Phi(w_{x_1}, T_r(x_1), v) + \varphi(v, T_r(x_1)) - \varphi(T_r(x_1), T_r(x_1))] \\ + \langle \kappa'(T_r(x_1)) - \kappa'(x_1), \eta(v, T_r(x_1)) \rangle \geq 0 \end{aligned} \quad (3.8)$$

and

$$\begin{aligned} r[\Phi(w_{x_2}, T_r(x_2), v) + \varphi(v, T_r(x_2)) - \varphi(T_r(x_2), T_r(x_2))] \\ + \langle \kappa'(T_r(x_2)) - \kappa'(x_2), \eta(v, T_r(x_2)) \rangle \geq 0 \end{aligned} \quad (3.9)$$

for all $v \in C$. Taking $v = T_r(x_2)$ in (3.8) and $v = T_r(x_1)$ in (3.9) and adding up these inequalities, we obtain

$$\begin{aligned} r[\Phi(w_{x_1}, T_r(x_1), T_r(x_2)) + \Phi(w_{x_2}, T_r(x_2), T_r(x_1)) + \varphi(T_r(x_2), T_r(x_1)) \\ - \varphi(T_r(x_1), T_r(x_1)) + \varphi(T_r(x_1), T_r(x_2)) - \varphi(T_r(x_2), T_r(x_2))] \\ + \langle \kappa'(T_r(x_1)) - \kappa'(x_1), \eta(T_r(x_2), T_r(x_1)) \rangle \\ + \langle \kappa'(T_r(x_2)) - \kappa'(x_2), \eta(T_r(x_1), T_r(x_2)) \rangle \geq 0. \end{aligned} \quad (3.10)$$

By assumption, we have

$$\Phi(w_{x_1}, T_r(x_1), T_r(x_2)) + \Phi(w_{x_2}, T_r(x_2), T_r(x_1)) \leq 0.$$

From $(\varphi 1)$ and $(\eta 1)$, we have

$$\varphi(T_r(x_2), T_r(x_1)) - \varphi(T_r(x_1), T_r(x_1)) + \varphi(T_r(x_1), T_r(x_2)) - \varphi(T_r(x_2), T_r(x_2)) \leq 0$$

and

$$\eta(T_r(x_1), T_r(x_2)) + \eta(T_r(x_2), T_r(x_1)) = 0.$$

It follows from (3.10) that

$$0 \leq -\langle \kappa'(T_r(x_1)) - \kappa'(x_1), \eta(T_r(x_1), T_r(x_2)) \rangle + \langle \kappa'(T_r(x_2)) - \kappa'(x_2), \eta(T_r(x_1), T_r(x_2)) \rangle,$$

which implies that

$$\langle \kappa'(T_r(x_1)) - \kappa'(T_r(x_2)), \eta(T_r(x_1), T_r(x_2)) \rangle \leq \langle \kappa'(x_1) - \kappa'(x_2), \eta(T_r(x_1), T_r(x_2)) \rangle.$$

Since $\kappa' : C \rightarrow H$ is η -strongly monotone with constant $\mu > 0$, we obtain

$$\begin{aligned} \mu \|T_r(x_1) - T_r(x_2)\|^2 &\leq \langle \kappa'(T_r(x_1)) - \kappa'(T_r(x_2)), \eta(T_r(x_1), T_r(x_2)) \rangle \\ &\leq \langle \kappa'(x_1) - \kappa'(x_2), \eta(T_r(x_1), T_r(x_2)) \rangle. \end{aligned} \tag{3.11}$$

Since κ' and η are Lipschitz continuous with constants μ and ν , respectively; we conclude that

$$\|T_r(x_1) - T_r(x_2)\|^2 \leq \frac{\lambda \nu}{\mu} \|x_1 - x_2\|^2 \leq \|x_1 - x_2\|^2,$$

since $\frac{\lambda \nu}{\mu} \leq 1$. Therefore, T_r is nonexpansive.

(d) Observe that

$$\begin{aligned} x \in F(T_r) &\Leftrightarrow T_r x = x \\ &\Leftrightarrow r[\Phi(w_x, x, \nu) + \varphi(\nu, x) - \varphi(x, x)] \geq 0 \quad \text{for all } \nu \in C \\ &\Leftrightarrow \Phi(w_x, x, \nu) + \varphi(\nu, x) - \varphi(x, x) \geq 0 \quad \text{for all } \nu \in C \\ &\Leftrightarrow x \in \Omega. \end{aligned}$$

(e) Since $F(T_r) = \Omega$, it follows from Lemma 2.7 that Ω is closed and convex. \square

COROLLARY 3.2. *Let C be a nonempty closed convex bounded subset of a real Hilbert space H and $\varphi : C \times C \rightarrow \mathbb{R}$ a real-valued function satisfying the conditions $(\varphi 1) \sim (\varphi 3)$. Let $T : C \rightarrow H$ be a mapping and $\Phi : H \times C \times C \rightarrow \mathbb{R}$ be the equilibrium-like function satisfying the conditions $(\Phi 1) \sim (\Phi 3)$. Assume that $\eta : C \times C \rightarrow H$ is a Lipschitz function with Lipschitz constant $\lambda > 0$ which satisfies the conditions $(\eta 1) \sim (\eta 3)$. Let $\kappa : C \rightarrow \mathbb{R}$ be a η -strongly convex function with constant $\mu > 0$ which satisfies the conditions $(\kappa 1) \sim (\kappa 2)$. For $r > 0$, define a mapping $T_r : C \rightarrow C$ by*

$$T_r(x) = \{u \in C : \Phi(T(x), u, \nu) + \varphi(\nu, u) - \varphi(u, u) + \frac{1}{r} \langle \kappa'(u) - \kappa'(x), \eta(\nu, u) \rangle \geq 0 \text{ for all } \nu \in C\}, \quad x \in C.$$

Then we have the following:

- (a) The auxiliary problem (3.2) has a unique solution;
- (b) T_r is single-valued;
- (c) if $\frac{\lambda \nu}{\mu} \leq 1$ and $\Phi(w_1, T_r(x_1), T_r(x_2)) + \Phi(w_2, T_r(x_2), T_r(x_1)) \leq 0$ for all $x_1, x_2 \in C$ and all $w_1 \in T(x_1), w_2 \in T(x_2)$, it follows that T_r is nonexpansive;
- (d) $F(T_r) = \Omega$;
- (e) Ω is closed and convex.

4. Iterative algorithm and convergence analysis

Let C be a nonempty closed convex subset of a real Hilbert space H , $T : C \rightarrow CB(H)$ a multi-valued mapping, $f : C \rightarrow C$ a contraction mapping with Lipschitz constant $\alpha \in [0, 1)$ and $S : C \rightarrow C$ an asymptotically nonexpansive mapping with sequence $\{k_n\}$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ and $\{r_n\}$ a sequence in $(0, \infty)$. For given elements $x_1 \in C$ and $w_1 \in T(x_1)$, from Theorem 3.1, we know that the auxiliary problem (3.1) has a unique solution $u_1 = T_{r_1}(x_1) \in C$, i.e.,

$$\Phi(w_1, u_1, v) + \varphi(v, u_1) - \varphi(u_1, u_1) + \frac{1}{r_1} \langle \kappa'(u_1) - \kappa'(x_1), \eta(v, u_1) \rangle \geq 0 \quad \text{for all } v \in C.$$

For $u_1 \in C$, we define

$$x_2 = \alpha_1 f(x_1) + (1 - \alpha_1) S(u_1).$$

Since $w_1 \in T(x_1)$, by Nadler’s theorem, there exists $w_2 \in T(x_2)$ such that

$$\|w_1 - w_2\| \leq (1 + 1) \mathcal{H}(T(x_1), T(x_2)).$$

For element $x_2 \in C$ and $w_2 \in T(x_2)$, again from Theorem 3.1, we know that the auxiliary problem (3.1) has a unique solution $u_2 = T_{r_2}(x_2) \in C$, i.e.,

$$\Phi(w_2, u_2, v) + \varphi(v, u_2) - \varphi(u_2, u_2) + \frac{1}{r_2} \langle \kappa'(u_2) - \kappa'(x_2), \eta(v, u_2) \rangle \geq 0 \quad \text{for all } v \in C.$$

For $u_2 \in C$, we define

$$x_3 = \alpha_2 f(x_2) + (1 - \alpha_2) S^2(u_2).$$

Since $w_2 \in T(x_2)$, by Nadler’s theorem, there exists $w_3 \in T(x_3)$ such that

$$\|w_2 - w_3\| \leq (1 + 1/2) \mathcal{H}(T(x_2), T(x_3)).$$

Inductively, we can develop the following algorithm for finding a common element of set of fixed points of asymptotically nonexpansive mapping S and set of solutions of GMEP.

ALGORITHM 4.1. For given $x_1 \in C$ and $w_1 \in T(x_1)$, there exist a sequence $\{w_n\}$ in H and two sequences $\{x_n\}$ and $\{u_n\}$ in C such that

$$\left\{ \begin{array}{l} w_n \in T(x_n), \|w_n - w_{n+1}\| \leq (1 + \frac{1}{n}) \mathcal{H}(T(x_n), T(x_{n+1})); \\ \Phi(w_n, u_n, v) + \varphi(v, u_n) - \varphi(u_n, u_n) \\ \quad + \frac{1}{r_n} \langle \kappa'(u_n) - \kappa'(x_n), \eta(v, u_n) \rangle \geq 0 \quad \text{for all } v \in C; \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S^n(u_n) \quad \text{for all } n \in \mathbb{N}. \end{array} \right. \quad (4.1)$$

If $S = I$, the identity mapping and $r_n = r$ for all $n \in \mathbb{N}$, then Algorithm 4.1 reduces to the following algorithm.

ALGORITHM 4.2. For given $x_1 \in C$ and $w_1 \in T(x_1)$, there exist a sequence $\{w_n\}$ in H and two sequences $\{x_n\}$ and $\{u_n\}$ in C such that

$$\begin{cases} w_n \in T(x_n), \|w_n - w_{n+1}\| \leq (1 + \frac{1}{n})\mathcal{H}(T(x_n), T(x_{n+1})); \\ \Phi(w_n, u_n, v) + \varphi(v, u_n) - \varphi(u_n, u_n) \\ \quad + \frac{1}{r} \langle \kappa'(u_n) - \kappa'(x_n), \eta(v, u_n) \rangle \geq 0 \quad \text{for all } v \in C; \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)u_n \quad \text{for all } n \in \mathbb{N}. \end{cases} \tag{4.2}$$

We now prove strong convergence of iterative sequences $\{x_n\}$, $\{u_n\}$ and $\{w_n\}$ generated by Algorithm 4.1.

THEOREM 4.3. Let C be a nonempty closed convex bounded subset of a real Hilbert space H , $T : C \rightarrow CB(H)$ a multi-valued \mathcal{H} -Lipschitz continuous mapping with constant $L > 0$, $f : C \rightarrow C$ a contraction mapping with Lipschitz constant $\alpha \in [0, 1)$. Let $\varphi : C \times C \rightarrow \mathbb{R}$ be a real-valued function satisfying the conditions $(\varphi 1) \sim (\varphi 3)$ and let $\Phi : H \times C \times C \rightarrow \mathbb{R}$ be the equilibrium-like function satisfying the conditions $(\Phi 1) \sim (\Phi 3)$ and $(\Phi 4)$:

$$(\Phi 4) \quad \Phi(w, T_r(x), T_s(y)) + \Phi(\tilde{w}, T_s(y), T_r(x)) \leq -\gamma \|T_r(x) - T_s(y)\|^2$$

for all $x, y \in C$ and $r, s \in (0, \infty)$, where $\gamma > 0$, $w \in T_r(x)$ and $\tilde{w} \in T_s(y)$. Assume that $\eta : C \times C \rightarrow H$ is a Lipschitz function with Lipschitz constant $\lambda > 0$ satisfying $(\eta 1) \sim (\eta 3)$. Let $\kappa : C \rightarrow \mathbb{R}$ be a η -strongly convex function with constant $\mu > 0$ satisfying $(\kappa 1) \sim (\kappa 2)$ and $\frac{\lambda v}{\mu} \leq 1$. Let $S : C \rightarrow C$ be an asymptotically nonexpansive mapping with sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and $F(S) \cap \Omega \neq \emptyset$. Let $\{x_n\}$, $\{u_n\}$ and $\{w_n\}$ be three sequences generated by Algorithm 4.1, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{r_n\}$ is another sequence in $(0, \infty)$ satisfying conditions:

$$(C1) \quad \lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty;$$

$$(C2) \quad \liminf_{n \rightarrow \infty} r_n > 0 \text{ and } \sum_{n=1}^{\infty} |r_n - r_{n+1}| < \infty;$$

$$(C3) \quad \sum_{n=1}^{\infty} (1 - \alpha_n)\varepsilon_n < \infty, \text{ where } \varepsilon_n = \sup_{x \in C} \|S^n(x) - S^{n+1}(x)\|.$$

If T_r is firmly nonexpansive, then we have the following:

(a) there exist $x^* \in F(S) \cap \Omega$ and $\hat{w} \in T(x^*)$ such that

$$x_n \rightarrow x^*, u_n \rightarrow x^* \quad \text{and} \quad w_n \rightarrow \hat{w} \quad \text{as } n \rightarrow \infty,$$

(b) x^* is the unique solution of variational inequality:

$$\text{find } x \in F(S) \cap \Omega \quad \text{such that} \quad \langle (I - f)(x), x - p \rangle \leq 0 \quad \text{for all } p \in F(S) \cap \Omega.$$

Proof. (a) It is easy to see from $(\Phi 4)$ that

$$\Phi(w, T_r(x), T_s(y)) + \Phi(\tilde{w}, T_s(y), T_r(x)) \leq -\gamma \|T_r(x) - T_s(y)\|^2 \leq 0$$

for all $x, y \in C$ and $r, s \in (0, \infty)$, where $\gamma > 0$, $w \in T_r(x)$ and $\tilde{w} \in T_s(y)$. Since all the assumptions of Theorem 3.1 are satisfied, all the conclusions (a) \sim (e) of Theorem 3.1 hold. Since $F(S) \cap \Omega$ is a nonempty closed convex subset of C , there exists the

metric projection mapping $P_{F(S)\cap\Omega}$ from C onto $F(S) \cap \Omega$. Since f is contraction, it follows that $P_{F(S)\cap\Omega}f : C \rightarrow F(S) \cap \Omega \subseteq C$ is also a contraction mapping. In fact,

$$\|P_{F(S)\cap\Omega}f(x) - P_{F(S)\cap\Omega}f(y)\| \leq \|f(x) - f(y)\| \leq \alpha\|x - y\| \quad \text{for all } x, y \in C.$$

Hence there exists a unique element $q \in C$ such that $q = P_{F(S)\cap\Omega}f(q)$. Note that $f(q) \in C$ and $P_{F(S)\cap\Omega}f(q) \in F(S) \cap \Omega$, it follows that $q \in F(S) \cap \Omega$.

We now divide the proof into the following three steps:

Step 1. $\|x_n - x_{n+1}\| \rightarrow 0$ and $\|u_n - u_{n+1}\| \rightarrow 0$ as $n \rightarrow \infty$.

Observe that

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\alpha_n f(x_n) + (1 - \alpha_n)S^n(u_n) - \alpha_{n-1}f(x_{n-1}) - (1 - \alpha_{n-1})S^{n-1}(u_{n-1})\| \\ &= \|\alpha_n f(x_n) - \alpha_n f(x_{n-1}) + \alpha_n f(x_{n-1}) - \alpha_{n-1}f(x_{n-1}) \\ &\quad + (1 - \alpha_n)S^n(u_n) - (1 - \alpha_n)S^{n-1}(u_{n-1}) \\ &\quad + (1 - \alpha_n)S^{n-1}(u_{n-1}) - (1 - \alpha_{n-1})S^{n-1}(u_{n-1})\| \\ &\leq \alpha_n \|f(x_n) - f(x_{n-1})\| + |\alpha_n - \alpha_{n-1}|(\|f(x_{n-1})\| + \|S^{n-1}(u_{n-1})\|) \\ &\quad + (1 - \alpha_n)\|S^n(u_n) - S^{n-1}(u_{n-1})\| \\ &\leq \alpha\alpha_n \|x_n - x_{n-1}\| + 2|\alpha_n - \alpha_{n-1}|\text{diam}(C) \\ &\quad + (1 - \alpha_n)(\|S^n(u_n) - S^{n-1}(u_n)\| + \|S^{n-1}(u_n) - S^{n-1}(u_{n-1})\|) \\ &\leq \alpha\alpha_n \|x_n - x_{n-1}\| + 2|\alpha_n - \alpha_{n-1}|\text{diam}(C) \\ &\quad + (1 - \alpha_n)(k_{n-1}\|u_n - u_{n-1}\| + \varepsilon_{n-1}). \end{aligned} \tag{4.3}$$

Note that $u_n = T_{r_n}x_n$ and $u_{n+1} = T_{r_{n+1}}x_{n+1}$. Hence from (4.1), we have

$$\Phi(w_n, u_n, v) + \varphi(v, u_n) - \varphi(u_n, u_n) + \frac{1}{r_n}\langle \kappa'(u_n) - \kappa'(x_n), \eta(v, u_n) \rangle \geq 0 \tag{4.4}$$

and

$$\begin{aligned} \Phi(w_{n+1}, u_{n+1}, v) &+ \varphi(v, u_{n+1}) - \varphi(u_{n+1}, u_{n+1}) \\ &+ \frac{1}{r_{n+1}}\langle \kappa'(u_{n+1}) - \kappa'(x_{n+1}), \eta(v, u_{n+1}) \rangle \geq 0 \end{aligned} \tag{4.5}$$

for all $v \in C$. Taking $v = u_{n+1}$ in (4.4) and $v = u_n$ in (4.5) and adding up these inequalities, we obtain from (2.1) and (Φ4) that

$$\begin{aligned} -\gamma\|u_n - u_{n+1}\|^2 &- \frac{1}{r_n}\langle \kappa'(u_n) - \kappa'(x_n), \eta(u_n, u_{n+1}) \rangle \\ &+ \frac{1}{r_{n+1}}\langle \kappa'(u_{n+1}) - \kappa'(x_{n+1}), \eta(u_n, u_{n+1}) \rangle \geq 0, \end{aligned}$$

it follows that

$$\begin{aligned}
\gamma r_n \|u_n - u_{n+1}\|^2 &\leq \langle \kappa'(u_n) - \kappa'(x_n) - \frac{r_n}{r_{n+1}}(\kappa'(u_{n+1}) - \kappa'(x_{n+1})), \eta(u_{n+1}, u_n) \rangle \\
&\leq \langle \kappa'(u_n) - \kappa'(u_{n+1}), \eta(u_{n+1}, u_n) \rangle + \langle \kappa'(u_{n+1}) - \kappa'(x_n) \\
&\quad - \frac{r_n}{r_{n+1}}(\kappa'(u_{n+1}) - \kappa'(x_{n+1})), \eta(u_{n+1}, u_n) \rangle \\
&\leq -\mu \|u_n - u_{n+1}\|^2 + \langle \kappa'(u_{n+1}) - \kappa'(x_{n+1}) + \kappa'(x_{n+1}) - \kappa'(x_n) \\
&\quad - \frac{r_n}{r_{n+1}}(\kappa'(u_{n+1}) - \kappa'(x_{n+1})), \eta(u_{n+1}, u_n) \rangle \\
&\leq -\mu \|u_n - u_{n+1}\|^2 + (\|\kappa'(x_{n+1}) - \kappa'(x_n)\| \\
&\quad + \left|1 - \frac{r_n}{r_{n+1}}\right| \|\kappa'(u_{n+1}) - \kappa'(x_{n+1})\|) \|\eta(u_{n+1}, u_n)\| \\
&\leq -\mu \|u_n - u_{n+1}\|^2 + \lambda v (\|x_{n+1} - x_n\| \\
&\quad + \frac{|r_{n+1} - r_n|}{r_{n+1}} \|u_{n+1} - x_{n+1}\|) \|u_n - u_{n+1}\|, \tag{4.6}
\end{aligned}$$

since η and κ' are Lipschitz continuous with Lipschitz constants λ and v , respectively. Note that $\liminf_{n \rightarrow \infty} r_n > 0$, there exists a constant $\bar{r} > 0$ such that $r_n \geq \bar{r} > 0$ for all $n \in \mathbb{N}$. From (4.6), we have

$$\gamma \bar{r} \|u_n - u_{n+1}\| \leq -\mu \|u_n - u_{n+1}\| + \lambda v \left(\|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{\bar{r}} \|u_{n+1} - x_{n+1}\| \right),$$

which implies that

$$\left(1 + \frac{\gamma \bar{r}}{\mu}\right) \|u_n - u_{n+1}\| \leq \|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{\bar{r}} \text{diam}(C). \tag{4.7}$$

From (4.7), we have

$$\|u_n - u_{n+1}\| \leq \delta \|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{\bar{r}} \delta \text{diam}(C), \tag{4.8}$$

where $\delta = \frac{1}{1 + \frac{\gamma \bar{r}}{\mu}}$. Set $\theta := \max\{\alpha, \delta\} = \max\left\{\alpha, \frac{1}{1 + \frac{\gamma \bar{r}}{\mu}}\right\}$. From (4.3), one can see that

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \alpha \alpha_n \|x_n - x_{n-1}\| + (1 - \alpha_n) k_{n-1} \left(\delta \|x_n - x_{n-1}\| + \frac{|r_n - r_{n-1}|}{\bar{r}} \delta \text{diam}(C) \right) \\
&\quad + 2|\alpha_n - \alpha_{n-1}| \text{diam}(C) + (1 - \alpha_n) \varepsilon_{n-1} \\
&\leq (\alpha \alpha_n + (1 - \alpha_n) \delta k_{n-1}) \|x_n - x_{n-1}\| + (1 - \alpha_n) k_{n-1} \frac{|r_n - r_{n-1}|}{\bar{r}} \delta \text{diam}(C) \\
&\quad + 2|\alpha_n - \alpha_{n-1}| \text{diam}(C) + (1 - \alpha_n) \varepsilon_{n-1}
\end{aligned}$$

$$\begin{aligned}
 &\leq \theta(\alpha_n + (1 - \alpha_n)k_{n-1})\|x_n - x_{n-1}\| + (1 - \alpha_n)k_{n-1} \frac{|r_n - r_{n-1}|}{\bar{r}} \delta \operatorname{diam}(C) \\
 &\quad + 2|\alpha_n - \alpha_{n-1}|\operatorname{diam}(C) + (1 - \alpha_n)\varepsilon_{n-1} \\
 &\leq \theta k_{n-1}\|x_n - x_{n-1}\| + k_{n-1} \frac{|r_n - r_{n-1}|}{\bar{r}} \delta \operatorname{diam}(C) \\
 &\quad + 2|\alpha_n - \alpha_{n-1}|\operatorname{diam}(C) + (1 - \alpha_n)\varepsilon_{n-1}.
 \end{aligned} \tag{4.9}$$

From conditions (C1) and (C3), we have

$$\begin{aligned}
 \sum_{n=1}^{\infty} (1 - \alpha_{n+1})\varepsilon_n &= \sum_{n=1}^{\infty} [(1 - \alpha_n)\varepsilon_n + (\alpha_n - \alpha_{n+1})\varepsilon_n] \\
 &\leq \sum_{n=1}^{\infty} [(1 - \alpha_n)\varepsilon_n + |\alpha_n - \alpha_{n+1}| \sup_{n \in \mathbb{N}} \varepsilon_n] < \infty.
 \end{aligned}$$

Set $a_n := \|x_n - x_{n+1}\|$, $b_n := k_n$ and

$$c_n := \frac{|r_n - r_{n+1}|}{\bar{r}} \delta \operatorname{diam}(C) \sup_{n \in \mathbb{N}} k_n + 2|\alpha_n - \alpha_{n+1}|\operatorname{diam}(C) + (1 - \alpha_{n+1})\varepsilon_n.$$

Thus, the inequality (4.9) reduces to

$$a_{n+1} \leq a_n b_n + c_n \quad \text{for all } n \in \mathbb{N}. \tag{4.10}$$

Note that the conditions $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$, $\sum_{n=1}^{\infty} |r_n - r_{n+1}| < \infty$ and $\sum_{n=1}^{\infty} (1 - \alpha_{n+1})\varepsilon_n < \infty$ imply that $\sum_{n=1}^{\infty} c_n < \infty$. Applying Lemma 2.11, we obtain that $\lim_{n \rightarrow \infty} \|x_n - x_{n+1}\| = 0$. Hence from (4.8), we get $\lim_{n \rightarrow \infty} \|u_n - u_{n+1}\| = 0$.

Step 2. *There exists $x^* \in C$ such that $x_n \rightarrow x^*$, $u_n \rightarrow x^*$ and $w_n \rightarrow \hat{w}$ as $n \rightarrow \infty$, where $\hat{w} \in T(x^*)$.*

Since $\lim_{n \rightarrow \infty} \alpha_n = 0$, we know from (C3) that $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. It follows that

$$\begin{aligned}
 \|x_{n+1} - S^{n+1}(u_{n+1})\| &\leq \|x_{n+1} - S^n(u_n)\| + \|S^n(u_n) - S^{n+1}(u_{n+1})\| \\
 &\leq \alpha_n \|f(x_n) - S^n(u_n)\| + \|S^n(u_n) - S^n(u_{n+1})\| \\
 &\quad + \|S^n(u_{n+1}) - S^{n+1}(u_{n+1})\| \\
 &\leq \alpha_n \operatorname{diam}(C) + k_n \|u_n - u_{n+1}\| + \varepsilon_n \rightarrow 0 \quad \text{as } n \rightarrow \infty.
 \end{aligned}$$

For $p \in F(S) \cap \Omega$, we have

$$\begin{aligned}
 \|u_n - p\|^2 &= \|T_{r_n}(x_n) - T_{r_n}(p)\|^2 \\
 &\leq \langle T_{r_n}(x_n) - T_{r_n}(p), x_n - p \rangle \\
 &\leq \langle u_n - p, x_n - p \rangle \\
 &\leq \frac{1}{2} (\|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n\|^2),
 \end{aligned}$$

and hence

$$\begin{aligned}
 \|u_n - p\|^2 &\leq \|x_n - p\|^2 - \|u_n - x_n\|^2 \\
 &\leq \|x_n - p\|^2
 \end{aligned} \tag{4.11}$$

By the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \alpha_n \|f(x_n) - p\|^2 + (1 - \alpha_n) \|S^n(u_n) - p\|^2 \\ &\leq \alpha_n \|f(x_n) - p\|^2 + (1 - \alpha_n) k_n \|u_n - p\|^2 \\ &\leq \alpha_n \text{diam}(C)^2 + k_n (\|x_n - p\|^2 - \|u_n - x_n\|^2), \end{aligned}$$

it follows that

$$\begin{aligned} k_n \|u_n - x_n\|^2 &\leq \alpha_n \text{diam}(C)^2 + (k_n - 1) \|x_n - p\|^2 + (\|x_n - p\|^2 - \|x_{n+1} - p\|^2) \\ &\leq (\alpha_n + (k_n - 1)) \text{diam}(C)^2 + (\|x_n - p\| + \|x_{n+1} - p\|) \|x_n - x_{n+1}\| \\ &\leq (\alpha_n + (k_n - 1)) \text{diam}(C)^2 + 2 \|x_n - x_{n+1}\| \text{diam}(C). \end{aligned}$$

Since $\alpha_n \rightarrow 0$, $k_n \rightarrow 1$ and $\|x_n - x_{n+1}\| \rightarrow 0$ as $n \rightarrow \infty$, we have $\|u_n - x_n\| \rightarrow 0$ as $n \rightarrow \infty$. By (4.10) and Lemma 2.12, we see that $\{x_n\}$ is a Cauchy sequence in C and there exists an element $x^* \in C$ such that $x_n \rightarrow x^*$. Since $\|u_n - x_n\| \rightarrow 0$ as $n \rightarrow \infty$, it follows that $u_n \rightarrow x^*$.

We now show that $\{w_n\}$ is a Cauchy sequence. From (4.1), we have

$$\begin{aligned} \|w_n - w_{n+1}\| &\leq \left(1 + \frac{1}{n}\right) \mathcal{H}(T(x_n), T(x_{n+1})) \\ &\leq 2 \mathcal{H}(T(x_n), T(x_{n+1})) \\ &\leq 2L \|x_n - x_{n+1}\|. \end{aligned}$$

For $m > n \geq 1$, we have from (2.5) that

$$\begin{aligned} \|w_m - w_n\| &\leq \sum_{i=n}^{m-1} \|w_i - w_{i+1}\| \\ &\leq 2L \sum_{i=n}^{m-1} \|x_i - x_{i+1}\| \\ &\leq 2L \left(\frac{\theta a_{n-1} b_{n-1}}{1 - \theta \left(\prod_{i=n}^{\infty} b_i\right)} + \frac{\sum_{i=n}^{\infty} c_{i-1}}{1 - \theta \left(\prod_{i=n}^{\infty} b_i\right)} \right) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Thus, $\lim_{m,n \rightarrow \infty} \|w_m - w_n\| = 0$, i.e., $\{w_n\}$ is a Cauchy sequence in H and hence there exists an element \hat{w} in H such that $\lim_{n \rightarrow \infty} w_n = \hat{w}$. Since $w_n \in T(x_n)$, we obtain from Remark 2.2 that

$$\begin{aligned} d(\hat{w}, T(x^*)) &\leq \|\hat{w} - w_n\| + d(w_n, T(x^*)) \\ &\leq \|\hat{w} - w_n\| + \mathcal{H}(T(w_n), T(x^*)) \\ &\leq \|\hat{w} - w_n\| + L \|w_n - x^*\| \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

i.e., $d(\hat{w}, T(x^*)) = 0$. Since $T(x^*) \in CB(H)$, we conclude that $\hat{w} \in T(x^*)$.

Step 3. $x^* \in F(S) \cap \Omega$.

Since

$$\begin{aligned} \|x^* - S^n(x^*)\| &\leq \|x^* - u_n\| + \|u_n - x_n\| + \|x_n - S^n(u_n)\| + \|S^n(u_n) - S^n(x^*)\| \\ &\leq (1 + k_n)\|x^* - u_n\| + \|u_n - x_n\| + \|x_n - S^n(u_n)\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

By the continuity of S , one can easily see that $x^* = S(x^*)$. Since $u_n = T_{r_n}(x_n)$, we have

$$\Phi(w_n, u_n, v) + \varphi(v, u_n) - \varphi(u_n, u_n) + \frac{1}{r} \langle \kappa'(u_n) - \kappa'(x_n), \eta(v, u_n) \rangle \geq 0.$$

Since $x_n, u_n \rightarrow x^*$; it follows that $\kappa'(u_n) - \kappa'(x_n) \rightarrow 0$. Hence from $(\Phi 1)$ and $(\varphi 1)$, we have

$$\Phi(w, x^*, v) + \varphi(v, x^*) - \varphi(x^*, x^*) \geq 0,$$

i.e., $x^* \in \Omega$. Therefore, $x^* \in F(S) \cap \Omega$.

Step 4. $x^* = q$.

Since $q = P_{F(S) \cap \Omega} f(q)$, we have $\langle f(q) - q, p - q \rangle \leq 0$ for all $p \in F(S) \cap \Omega$. Note that $x_n \rightarrow x^*$, we have

$$\lim_{n \rightarrow \infty} \langle f(q) - q, x_n - q \rangle = \langle f(q) - q, x^* - q \rangle \leq 0.$$

By Lemma 2.6, we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq (1 - \alpha_n)^2 \|S^n(u_n) - q\|^2 + 2\alpha_n \langle f(x_n) - q, x_{n+1} - q \rangle \\ &\leq (1 - \alpha_n)^2 k_n \|u_n - q\|^2 + 2\alpha_n \langle f(x_n) - f(q) + f(q) - q, x_{n+1} - q \rangle \\ &\leq (1 - \alpha_n)^2 k_n \|u_n - q\|^2 + 2\alpha \alpha_n \|x_n - q\| \|x_{n+1} - q\| \\ &\quad + 2\alpha_n \langle f(q) - q, x_{n+1} - q \rangle \\ &\leq (1 - \alpha_n)^2 k_n \|u_n - q\|^2 + \alpha \alpha_n (\|x_n - q\|^2 + \|x_{n+1} - q\|^2) \\ &\quad + 2\alpha_n \langle f(q) - q, x_{n+1} - q \rangle, \end{aligned}$$

it follows from (4.11) that

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq \frac{(1 - \alpha_n)^2 k_n + \alpha \alpha_n}{1 - \alpha \alpha_n} \|x_n - q\|^2 + \frac{2\alpha_n}{1 - \alpha \alpha_n} \langle f(q) - q, x_{n+1} - q \rangle \\ &\leq \left(1 - \frac{2\alpha_n(k_n - \alpha)}{1 - \alpha \alpha_n} \right) \|x_n - q\|^2 + \frac{2\alpha_n(k_n - \alpha)}{1 - \alpha \alpha_n} \left[\frac{\alpha_n k_n}{k_n - \alpha} \sup_{n \in \mathbb{N}} \|x_n - q\|^2 \right. \\ &\quad \left. + \frac{1}{k_n - \alpha} \langle f(q) - q, x_{n+1} - q \rangle \right] + (k_n - 1) \sup_{n \in \mathbb{N}} \frac{\|x_n - q\|^2}{k_n - \alpha}. \end{aligned}$$

Set

$$\lambda_n := \frac{2\alpha_n(k_n - \alpha)}{1 - \alpha \alpha_n},$$

$$\sigma_n := \frac{\alpha_n k_n}{k_n - \alpha} \sup_{n \in \mathbb{N}} \|x_n - q\|^2 + \frac{1}{k_n - \alpha} \langle f(q) - q, x_{n+1} - q \rangle,$$

$$\xi_n := (k_n - 1) \sup_{n \in \mathbb{N}} \frac{\|x_n - q\|^2}{k_n - \alpha}.$$

Then $\lim_{n \rightarrow \infty} \lambda_n = 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, $\limsup_{n \rightarrow \infty} \sigma_n \leq 0$ and $\sum_{n=1}^{\infty} \xi_n < \infty$. Applying Lemma 2.10, we obtain that $\lim_{n \rightarrow \infty} x_n = q$. By the uniqueness of strong limits of $\{x_n\}$, we conclude that $x^* = q$. This completes the proof. \square

REMARK 4.4. In Theorem 4.3 different technique from Shahazad and Udomene [26] is used and strong convergence of almost fixed points defined by (1.7) is not applied. Theorem 4.3 provides an affirmative answer of Question 1.4.

In case of nonexpansiveness of S , the condition (C3) is not required. Indeed, we have

COROLLARY 4.5. *Let C be a nonempty closed convex bounded subset of a real Hilbert space H , $T : C \rightarrow CB(H)$ a multi-valued \mathcal{H} -Lipschitz continuous mapping with constant $L > 0$, $f : C \rightarrow C$ a contraction mapping with Lipschitz constant $\alpha \in [0, 1)$. Let $\varphi : C \times C \rightarrow \mathbb{R}$ be a real-valued function satisfying the conditions $(\varphi 1) \sim (\varphi 3)$ and let $\Phi : H \times C \times C \rightarrow \mathbb{R}$ be the equilibrium-like function satisfying the conditions $(\Phi 1) \sim (\Phi 3)$ and $(\Phi 4)$:*

$$(\Phi 4) \quad \Phi(w, T_r(x), T_s(y)) + \Phi(\tilde{w}, T_s(y), T_r(x)) \leq -\gamma \|T_r(x) - T_s(y)\|^2$$

for all $x, y \in C$ and $r, s \in (0, \infty)$, where $\gamma > 0$, $w \in T_r(x)$ and $\tilde{w} \in T_s(y)$. Assume that $\eta : C \times C \rightarrow H$ is a Lipschitz function with Lipschitz constant $\lambda > 0$ satisfying $(\eta 1) \sim (\eta 3)$. Let $\kappa : C \rightarrow \mathbb{R}$ be a η -strongly convex function with constant $\mu > 0$ satisfying $(\kappa 1) \sim (\kappa 2)$ and $\frac{\lambda v}{\mu} \leq 1$. Let $S : C \rightarrow C$ be a nonexpansive mapping with $F(S) \cap \Omega \neq \emptyset$. Let $\{x_n\}$, $\{u_n\}$ and $\{w_n\}$ be three sequences generated by the following algorithm

$$\begin{cases} w_n \in T(x_n), \|w_n - w_{n+1}\| \leq (1 + \frac{1}{r_n}) \mathcal{H}(T(x_n), T(x_{n+1})); \\ \Phi(w_n, u_n, v) + \varphi(v, u_n) - \varphi(u_n, u_n) \\ \quad + \frac{1}{r_n} \langle \kappa'(u_n) - \kappa'(x_n), \eta(v, u_n) \rangle \geq 0 \quad \text{for all } v \in C; \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S(u_n) \quad \text{for all } n \in \mathbb{N}, \end{cases} \tag{4.12}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{r_n\}$ is another sequence in $(0, \infty)$ satisfying conditions:

- (C1) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$;
- (C2) $\liminf_{n \rightarrow \infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_n - r_{n+1}| < \infty$;

If T_r is firmly nonexpansive, then we have the following:

- (a) there exist $x^* \in F(S) \cap \Omega$ and $\hat{w} \in T(x^*)$ such that

$$x_n \rightarrow x^*, u_n \rightarrow x^* \quad \text{and} \quad w_n \rightarrow \hat{w} \quad \text{as } n \rightarrow \infty,$$

(b) x^* is the unique solution of variational inequality:

$$\text{find } x \in F(S) \cap \Omega \text{ such that } \langle (I - f)(x), x - p \rangle \leq 0 \quad \text{for all } p \in F(S) \cap \Omega.$$

COROLLARY 4.6. Let C be a nonempty closed convex bounded subset of a real Hilbert space H , $T : C \rightarrow CB(H)$ a multi-valued \mathcal{H} -Lipschitz continuous mapping with constant $L > 0$, $f : C \rightarrow C$ a contraction mapping with Lipschitz constant $\alpha \in [0, 1)$. Let $\varphi : C \times C \rightarrow \mathbb{R}$ be a real-valued function satisfying the conditions $(\varphi 1) \sim (\varphi 3)$ and let $\Phi : H \times C \times C \rightarrow \mathbb{R}$ be the equilibrium-like function satisfying the conditions $(\Phi 1) \sim (\Phi 3)$ and $(\Phi 4)$:

$$(\Phi 4) \quad \Phi(w, T_r(x), T_s(y)) + \Phi(\tilde{w}, T_s(y), T_r(x)) \leq -\gamma \|T_r(x) - T_s(y)\|^2$$

for all $x, y \in C$ and $r, s \in (0, \infty)$, where $\gamma > 0$, $w \in T_r(x)$ and $\tilde{w} \in T_s(y)$. Assume that $\eta : C \times C \rightarrow H$ is a Lipschitz function with Lipschitz constant $\lambda > 0$ satisfying $(\eta 1) \sim (\eta 3)$. Let $\kappa : C \rightarrow \mathbb{R}$ be a η -strongly convex function with constant $\mu > 0$ satisfying $(\kappa 1) \sim (\kappa 2)$ and $\frac{\lambda \nu}{\mu} \leq 1$. Let $S : C \rightarrow C$ be a nonexpansive mapping with $F(S) \cap \Omega \neq \emptyset$. Let $\{x_n\}$, $\{u_n\}$ and $\{w_n\}$ be three sequences generated by the following algorithm

$$\begin{cases} w_n \in T(x_n), \|w_n - w_{n+1}\| \leq (1 + \frac{1}{n})\mathcal{H}(T(x_n), T(x_{n+1})); \\ \Phi(w_n, u_n, v) + \varphi(v, u_n) - \varphi(u_n, u_n) \\ \quad + \frac{1}{r_n} \langle u_n - x_n, v - u_n \rangle \geq 0 \quad \text{for all } v \in C; \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S(u_n) \quad \text{for all } n \in \mathbb{N}, \end{cases} \quad (4.13)$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{r_n\}$ is another sequence in $(0, \infty)$ satisfying conditions:

$$(C1) \quad \lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty;$$

$$(C2) \quad \liminf_{n \rightarrow \infty} r_n > 0 \text{ and } \sum_{n=1}^{\infty} |r_n - r_{n+1}| < \infty;$$

Then we have the following:

(a) there exist $x^* \in F(S) \cap \Omega$ and $\hat{w} \in T(x^*)$ such that

$$x_n \rightarrow x^*, u_n \rightarrow x^* \quad \text{and} \quad w_n \rightarrow \hat{w} \quad \text{as } n \rightarrow \infty,$$

(b) x^* is the unique solution of variational inequality:

$$\text{find } x \in F(S) \cap \Omega \text{ such that } \langle (I - f)(x), x - p \rangle \leq 0 \quad \text{for all } p \in F(S) \cap \Omega.$$

Proof. Since T_r is firmly nonexpansive by Corollary ??, Corollary 4.6 follows from Corollary 4.5. \square

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