Mathematical
Inequalities
& Applications
Volume 12, Number 3 (2009), 651–662

# NON–HOMOGENEOUS BOUNDARY VALUE PROBLEM FOR ONE–DIMENSIONAL COMPRESSIBLE VISCOUS MICROPOLAR FLUID MODEL: A GLOBAL EXISTENCE THEOREM

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(*Communicated by J. Pečaric*)

*Abstract.* An initial-boundary value problem for 1-D flow of a compressible viscous heat-conducting micropolar fluid is considered; the fluid is assumed thermodynamically perfect and polytropic. By transforming the original problem into homogeneous one we prove a global-in-time existence theorem. The proof is based on a local existence theorem, obtained in the previous research paper [5].

## 1. Introduction

In this paper we consider nonstationary 1-D flow of a compressible viscous and heat-conducting micropolar fluid, being in a thermodynamical sense perfect and polytropic. In [3] and [4] we considered the problem with homogeneous boundary conditions.

Here we study, as in [5], the case of non-homogeneous boundary conditions for velocity and microrotation (”piston problem”, see [6] for classical fluid) and prove a global-in-time existence of generalized solution. The proof is based on a local existence theorem obtained in the previous paper [5]. Also we use some ideas of S. N. Antontsev, A. V. Kazhykhov and A. V. Monakhov ([1]) applied to the case of classical fluid and results from [4] and [5] as well.

## 2. Statement of the problem and the main result

Let $\rho, v, \omega$ and $\theta$ denote, respectively, the mass density, velocity, microrotation velocity and temperature of the fluid in the Lagrangean description. Then the problem which we consider has the formulation as follows ([3]):

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial v}{\partial x} = 0\,, \tag{2.1}$$

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left(\rho \frac{\partial v}{\partial x}\right) - K \frac{\partial}{\partial x}(\rho\theta)\,, \tag{2.2}$$

*Mathematics subject classification* (2000): 35K55, 35Q40, 76N10, 46E35.
*Keywords and phrases*: micropolar fluid, generalized solution, boundary value problem, global existence.


Paper MIA-12-49

$$\rho\frac{\partial\omega}{\partial t}=A\left[\rho\frac{\partial}{\partial x}\left(\rho\frac{\partial\omega}{\partial x}\right)-\omega\right], \tag{2.3}$$

$$\rho\frac{\partial\theta}{\partial t}=-K\rho^2\theta\frac{\partial v}{\partial x}+\rho^2\left(\frac{\partial v}{\partial x}\right)^2+\rho^2\left(\frac{\partial\omega}{\partial x}\right)^2+\omega^2+D\rho\frac{\partial}{\partial x}\left(\rho\frac{\partial\theta}{\partial x}\right) \tag{2.4}$$

in $]0,1[\times]0,T[,\ T>0$, where $K,A$ and $D$ are positive constants. Equations (2.1)–(2.4) are, respectively, local forms of the conservations laws for the mass, momentum, momentum moment and energy. We take the following non-homogeneous initial and boundary conditions:

$$\rho(x,0)=\rho_0(x), \tag{2.5}$$

$$v(x,0)=v_0(x), \tag{2.6}$$

$$\omega(x,0)=\omega_0(x), \tag{2.7}$$

$$\theta(x,0)=\theta_0(x), \tag{2.8}$$

$$v(0,t)=\mu_0(t),\quad v(1,t)=\mu_1(t), \tag{2.9}$$

$$\omega(0,t)=\nu_0(t),\quad \omega(1,t)=\nu_1(t), \tag{2.10}$$

$$\frac{\partial\theta}{\partial x}(0,t)=\frac{\partial\theta}{\partial x}(1,t)=0 \tag{2.11}$$

for $x\in]0,1[,\ t\in]0,T[$. Here $\rho_0,\ v_0,\ \omega_0,\ \theta_0,\ \mu_0,\ \mu_1,\ \nu_0$ and $\nu_1$ are given functions. We assume the compatibility conditions

$$v_0(0)=\mu_0(0),\quad v_0(1)=\mu_1(0), \tag{2.12}$$

$$\omega_0(0)=\nu_0(0),\quad \omega_0(1)=\nu_1(0) \tag{2.13}$$

and the inequalities

$$0<m\leqslant\rho_0(x)\leqslant M,\quad m\leqslant\theta_0(x)\leqslant M\quad\text{for } x\in]0,1[, \tag{2.14}$$

where $m,M\in R^+$. We assume also that there exists a constant $\delta>0$ such that

$$l(t)=\int_0^1\frac{1}{\rho_0(x)}dx+\int_0^t[\mu_1(\tau)-\mu_0(\tau)]d\tau\geqslant\delta,\ \ t\in]0,T[. \tag{2.15}$$

Conditions (2.14) and (2.15) are assumed to be physically reasonable. The requirement (2.15) means that the distance between moving domain boundaries of flow in Euler variables is highly more than zero.

DEFINITION 2.1. A generalized solution of the problem (2.1)–(2.11) in the domain $Q_T=]0,1[\times]0,T[$ is a function

$$(x,t)\to(\rho,v,\omega,\theta)(x,t),\quad (x,t)\in Q_T, \tag{2.16}$$

where

$$\rho\in L^\infty(0,T;H^1(]0,1[))\cap H^1(Q_T),\ \ \rho>0\text{ a.e. in }Q_T, \tag{2.17}$$

$$v,\omega,\theta \in L^{\infty}(0,T;H^1(]0,1[))\cap H^1(Q_T)\cap L^2(0,T;H^2(]0,1[)), \tag{2.18}$$

that satisfies equations (2.1)–(2.4) a.e. in $Q_T$ and conditions (2.5)–(2.11) in the sense of traces.

REMARK 2.1. From embedding and interpolation theorems (e.g. [2]) one can conclude that what follows from (2.17) and (2.18) is:

$$\rho \in L^{\infty}(0,T;C([0,1]))\cap C([0,T],L^2(]0,1[))\ , \tag{2.19}$$

$$v,\omega,\theta \in L^2(0,T;C^{(1)}([0,1]))\cap C([0,T],H^1(]0,1[)), \tag{2.20}$$

$$v,\omega,\theta \in C(\bar{Q}_T). \tag{2.21}$$

We can see later that from formula (4.4) follows continuity of $\rho$ on $\bar{Q}_T$ also.

In the same way as in [3] we can prove that the problem (2.1)–(2.11) has at most one generalized solution in $Q_T$.

In this paper we shall prove the following result.

THEOREM 2.1. *Let $T \in \mathbf{R}^+$ and let the functions*

$$\mu_0,\mu_1,\nu_0,\nu_1 \in H^2(]0,T[), \tag{2.22}$$

$$\rho_0,v_0,\omega_0,\theta_0 \in H^1(]0,1[) \tag{2.23}$$

*satisfy conditions (2.12)–(2.15). Then the problem (2.1)–(2.11) has a generalized solution in $Q_T$, having the property*

$$\theta > 0 \ \ in \ \ \bar{Q}_T. \tag{2.24}$$

## 3. An equivalent setting of the problem (2.1)–(2.11)

Instead of the velocity $v$ and microrotation $\omega$ we introduce new functions $V$ and $W$ in order to obtain a problem with the homogeneous boundary conditions.

Notice that using (2.9) from (2.1) we get

$$\int_0^1 \frac{dx}{\rho(x,t)} = l(t),\ \ t \in ]0,T[, \tag{3.1}$$

where the function $l$ is defined by (2.15). Let be

$$v_1(x,t) = \frac{\mu(t)}{l(t)}\int_0^x \frac{d\xi}{\rho(\xi,t)} + \mu_0(t), \tag{3.2}$$

$$\omega_1(x,t) = \frac{\nu(t)}{l(t)}\int_0^x \frac{d\xi}{\rho(\xi,t)} + \nu_0(t) \ \text{ on } Q_T, \tag{3.3}$$

where $\mu(t) = \mu_1(t) - \mu_0(t)$ and $\nu(t) = \nu_1(t) - \nu_0(t)$. It is evident that

$$v_1(0,t) = \mu_0(t)\ , \quad v_1(1,t) = \mu_1(t), \tag{3.4}$$

$$\omega_1(0,t) = v_0(t)\,, \quad \omega_1(1,t) = v_1(t),\ t \in ]0,T[. \tag{3.5}$$

Inserting

$$V(x,t) = v(x,t) - v_1(x,t)\,, \quad W(x,t) = \omega(x,t) - \omega_1(x,t) \tag{3.6}$$

into (2.1)–(2.4) we get the following equivalent system:

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial V}{\partial x} + \frac{\mu}{l}\rho = 0\,, \tag{3.7}$$

$$\frac{\partial V}{\partial t} = \frac{\partial}{\partial x}\left(\rho \frac{\partial V}{\partial x}\right) - K\frac{\partial}{\partial x}(\rho\theta) - \frac{\partial v_1}{\partial t}\,, \tag{3.8}$$

$$\rho \frac{\partial W}{\partial t} = A\left[\rho \frac{\partial}{\partial x}\left(\rho \frac{\partial W}{\partial x}\right) - \omega_1 - W\right] - \rho \frac{\partial \omega_1}{\partial t}\,, \tag{3.9}$$

$$\begin{aligned}\rho \frac{\partial \theta}{\partial t} &= -K\rho^2\theta \frac{\partial V}{\partial x} - K\rho\theta\frac{\mu}{l} + \rho^2\left(\frac{\partial V}{\partial x}\right)^2 + 2\rho\frac{\partial V}{\partial x}\frac{\mu}{l} + \left(\frac{\mu}{l}\right)^2 \\ &\quad + \rho^2\left(\frac{\partial W}{\partial x}\right)^2 + 2\rho\frac{\partial W}{\partial x}\frac{v}{l} + \left(\frac{v}{l}\right)^2 + (W+\omega_1)^2 + D\rho\frac{\partial}{\partial x}\left(\rho\frac{\partial \theta}{\partial x}\right),\end{aligned} \tag{3.10}$$

with the homogeneous boundary conditions

$$V(0,t) = V(1,t) = 0\,, \quad W(0,t) = W(1,t) = 0, \tag{3.11}$$

$$\frac{\partial \theta}{\partial x}(0,t) = \frac{\partial \theta}{\partial x}(1,t) = 0 \tag{3.12}$$

for $t \in ]0,T[$ and initial conditions

$$\rho(x,0) = \rho_0(x), \quad V(x,0) = V_0(x), \tag{3.13}$$

$$W(x,0) = W_0(x), \quad \theta(x,0) = \theta_0(x) \tag{3.14}$$

for $x \in ]0,1[$, where

$$V_0(x) = v_0(x) - \frac{\mu(0)}{l(0)}\int_0^x \frac{1}{\rho_0(\xi)}d\xi - \mu_0(0)\,, \tag{3.15}$$

$$W_0(x) = \omega_0(x) - \frac{v(0)}{l(0)}\int_0^x \frac{1}{\rho_0(\xi)}d\xi - v_0(0) \tag{3.16}$$

are known functions. Notice that because of (2.12), (2.13), (2.22) and (2.23) we have

$$V_0, W_0 \in H_0^1(]0,1[). \tag{3.17}$$

In the article [5] we proved the following local existence theorem: there exists $T_0$, $0 < T_0 \leqslant T$, such that the problem (3.7)–(3.14) and the problem (2.1)–(2.11) as well, have a generalized solution in the domain $Q_{T_0} = ]0,1[\times]0,T_0[$ with the property

$$\theta > 0 \ \text{ in } \bar{Q}_{T_0}. \tag{3.18}$$

With the use of that theorem, in this paper we shall prove the following result first.

THEOREM 3.1. *Let $T \in \mathbf{R}^+$ be same as in Theorem 2.1. Under the assumptions of Theorem 2.1, the problem (3.7)–(3.14) has a generalized solution $(\rho, V, W, \theta)$ in $Q_T$. Moreover,*

$$\theta > 0 \text{ in } \bar{Q}_T. \tag{3.19}$$

Theorem 2.1. is an immediate consequence of this result. In the proof of Theorem 3.1. we apply, as in [4], the method of the book [1], where Theorem 2.1. was proved for the classical fluid ($\omega = 0$) with the homogeneous boundary conditions.

## 4. The proofs of Theorems 2.1 and 3.1

The proof of Theorem 3.1 is very similar to that of Theorem 1.1 in [4]. Because of the local existence result, Theorem 3.1 just like Theorem 2.1 is an immediate consequence of the following statement.

PROPOSITION 4.1. *Let $T \in \mathbf{R}^+$ and let a function*

$$(x,t) \to (\rho, V, W, \theta)(x,t) \quad (x,t) \in Q_T \tag{4.1}$$

*satisfy the condition:*

*for each $T' \in ]0, T[$, (4.1) is a generalized solution of the problem (3.7)–(3.14) in the domain $Q_{T'} = ]0,1[\times]0,T'[$ and the inequality $\theta > 0$ in $\bar{Q}_{T'}$ holds true.*

*Then (4.1) is a generalized solution of the same problem in the domain $Q_T$ and inequality $\theta > 0$ in $\bar{Q}_T$ holds true.*

The above statement is a consequence of the results below. In that what follows we assume that the function (4.1) satisfies the condition of Proposition 4.1. By $C > 0$ or $C_i > 0 (i = 1,2,..)$ we denote a generic constant, having possibly different values at different places; we also use the notation $\|f\| = \|f\|_{L^2(]0,1[)}$. Some of our considerations are very similar or identical to that of [1] or [4]. In these cases we omit proofs or details of proofs, making reference to corresponding pages of the book [1] or article [4].

First, we shall need some properties of the function $\rho$ on $Q_{T'}$ which we here introduce.

From (3.7) we get

$$\rho(x,t) = \frac{\rho_0(x)}{\exp\{\int_0^t \frac{\mu}{l} d\tau\}(1 + \rho_0(x) \int_0^t \frac{\partial V}{\partial x} \exp\{-\int_0^\tau \frac{\mu}{l} ds\} d\tau)} \tag{4.2}$$

and because of (2.20), (2.22) and (2.23) we conclude that $\rho$ is a continuous function on $\bar{Q}_{T'}$. Therefore from identity (3.1) it follows that there exists a function $r : [0, T'] \to [0,1]$ such that

$$\rho(r(t),t) = (l(t))^{-1}, \; t \in [0, T']. \tag{4.3}$$

Also, in the same way as in [1, pp. 44-45] we get that the function $\rho$ satisfies the equality

$$\rho(x,t) = \rho_0(x)Y(t)B(x,t)(1+K\rho_0\int_0^t \theta(x,t)Y(\tau)B(x,\tau)d\tau)^{-1},\ (x,t)\in Q_{T'}, \tag{4.4}$$

where

$$Y(t) = \frac{\exp\left\{K\int_0^t \rho(r(t),\tau)\theta(r(t),\tau)d\tau\right\}}{l(t)\rho_0(r(t))}, \tag{4.5}$$

$$B(x,t) = \exp\left\{\int_{r(t)}^x [v_0(\xi)-v_1(\xi,t)-V(\xi,t)]d\xi\right\}. \tag{4.6}$$

LEMMA 4.1. *For each $(x,t)\in Q_T$ the inequalities*

$$\left|\frac{\partial v_1}{\partial t}(x,t)\right| \leqslant C(1+|V(x,t)|), \tag{4.7}$$

$$\left|\frac{\partial \omega_1}{\partial t}(x,t)\right| \leqslant C(1+|V(x,t)|), \tag{4.8}$$

*are satisfied. Moreover,*

$$v_1,\omega_1 \in L^\infty(Q_T). \tag{4.9}$$

*Proof.* Taking into account (3.7) from (3.2) and (3.3) we find that

$$\frac{\partial v_1}{\partial t} = \left[\left(\frac{\mu}{l}\right)' + \left(\frac{\mu}{l}\right)^2\right]\int_0^x \frac{1}{\rho}d\xi + \frac{\mu}{l}V + \mu_0',$$

$$\frac{\partial \omega_1}{\partial t} = \left[\left(\frac{\nu}{l}\right)' + \frac{\mu\nu}{l^2}\right]\int_0^x \frac{1}{\rho}d\xi + \frac{\nu}{l}V + \nu_0'.$$

Because of (2.22) and (3.1) we easily get (4.7) and (4.8). The conclusion (4.9) follows directly from (3.2) and (3.3).

LEMMA 4.2. *It holds*

$$V,W \in L^\infty(0,T;L^2(]0,1[)), \tag{4.10}$$

$$\theta \in L^\infty(0,T;L^1(]0,1[)), \tag{4.11}$$

$$\frac{\partial v_1}{\partial t},\frac{\partial \omega_1}{\partial t} \in L^\infty(0,T;L^2(]0,1[)). \tag{4.12}$$

*Proof.* Multiplying equations (3.8), (3.9) and (3.10) respectively by $V, 2A^{-1}\rho^{-1}W$ and $\rho^{-1}$, integrating over $]0,1[$ and making use of (3.11), (3.12) and (3.1), after addition of the obtained equalities we find that

$$\frac{d}{dt}\int_0^1\left(\frac{1}{2}V^2+\frac{1}{A}W^2+\theta\right)dx + \int_0^1\left[\rho\left(\frac{\partial W}{\partial x}\right)^2 + \frac{W^2}{\rho}\right]dx \tag{4.13}$$
$$= -\int_0^1 \frac{\partial v_1}{\partial t}V dx - \frac{2}{A}\int_0^1 \frac{\partial \omega_1}{\partial t}W dx - \frac{K\mu}{l}\int_0^1 \theta dx + \int_0^1 \frac{\omega_1^2}{\rho}dx + \frac{\mu^2+\nu^2}{l} \text{ on } ]0,T[.$$

Integrating over $]0,t[, t \in ]0,T[$, and applying the Young inequality and the results from Lemma 4.1 on the right-hand side of (4.13), we obtain

$$\int_0^1 \left(\frac{1}{2}V^2 + \frac{1}{A}W^2 + \theta\right) dx + \int_0^t \int_0^1 \left[\rho \left(\frac{\partial W}{\partial x}\right)^2 + \frac{W^2}{\rho}\right] dx d\tau \tag{4.14}$$

$$\leqslant C\left\{1 + \int_0^t \left[\int_0^1 \left(\frac{1}{2}V^2 + \frac{1}{A}W^2 + \theta\right) dx + \int_0^\tau \int_0^1 \left(\rho \left(\frac{\partial W}{\partial x}\right)^2 + \frac{W^2}{\rho}\right) dx ds\right] d\tau\right\}.$$

Application of Gronwell's inequality to (4.14) gives

$$\frac{1}{2}\|V(t)\|^2 + \frac{1}{A}\|W(t)\|^2 + \int_0^1 \theta dx + \int_0^t \int_0^1 \left[\rho \left(\frac{\partial W}{\partial x}\right)^2 + \frac{W^2}{\rho}\right] dx d\tau \leqslant C$$

and we immediately get (4.10) and (4.11). Notice that because of (4.10) from (4.7) and (4.8) follows (4.12).

Let

$$M_\theta(t) = \max_{x\in[0,1]} \theta(x,t), \quad m_\rho(t) = \min_{x\in[0,1]} \rho(x,t) \tag{4.15}$$

and

$$I_1(t) = \int_0^1 \rho(x,t) \left(\frac{\partial \theta}{\partial x}(x,t)\right)^2 dx, \quad I_2(t) = \int_0^t I_1(\tau) d\tau. \tag{4.16}$$

LEMMA 4.3. *There exists $C \in \mathbf{R}^+$ and (for each $\varepsilon > 0$) $C_\varepsilon \in \mathbf{R}^+$ such that for each $t \in ]0,T[$ the inequalities*

$$m_\rho(t) \geqslant C(1 + \int_0^t M_\theta(\tau) d\tau)^{-1}, \tag{4.17}$$

$$M_\theta^2(t) \leqslant \varepsilon I_1(t) + C_\varepsilon(1 + I_2(t)) \tag{4.18}$$

*hold true.*

*Proof.* Using (4.9)–(4.11) in the same way as in [1, pp. 45-46] from (4.4) we get (4.17). The proof of (4.18) is identical to that of Lemma 2.4 in [1].

LEMMA 4.4. *([4, Lemma 2.3] and [1, pp. 48-52]) It holds*

$$\inf_{Q_T} \theta > 0, \tag{4.19}$$

$$\rho \in L^\infty(Q_T). \tag{4.20}$$

LEMMA 4.5. *It holds*

$$M_\theta \in L^2(]0,T[), \tag{4.21}$$

$$\inf_{Q_T} \rho > 0, \tag{4.22}$$

$$\theta \in L^\infty(0,T;L^2(]0,1[)) \cap L^2(0,T : H^1(]0,1[)). \tag{4.23}$$

*Proof.* Let

$$\phi=\frac{1}{2}V^2+\frac{1}{A}W^2+\theta. \tag{4.24}$$

It is evident that for $t\in]0,T[$ we have $\int_0^t\phi(x,\tau)d\tau\leqslant C$. Multiplying (3.8), (3.9) and (3.10) respectively by $\phi V, 2A^{-1}\rho^{-1}\phi W$ and $\rho^{-1}\phi$, integrating over $]0,1[$ and making use of (3.11)–(3.12), after addition of the obtained equations, we find that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1\phi^2dx&+\int_0^1\rho\left(\frac{\partial W}{\partial x}\right)^2\phi dx+\int_0^1\frac{W^2}{\rho}\phi dx+\int_0^1\rho\left(\frac{\partial\phi}{\partial x}\right)^2dx\\ &=\left(\frac{2}{A}-2\right)\int_0^1\rho W\frac{\partial W}{\partial x}\frac{\partial\phi}{\partial x}dx+(1-D)\int_0^1\rho\frac{\partial\theta}{\partial x}\frac{\partial\phi}{\partial x}dx+K\int_0^1\rho\theta V\frac{\partial\phi}{\partial x}dx\\ &\quad-\int_0^1\frac{\partial v_1}{\partial t}V\phi dx-\frac{2}{A}\int_0^1\frac{\partial\omega_1}{\partial t}W\phi dx-K\frac{\mu}{l}\int_0^1\theta\phi dx-\frac{2\mu}{l}\int_0^1V\frac{\partial\phi}{\partial x}dx\\ &\quad+\frac{\mu^2+\nu^2}{l^2}\int_0^1\frac{\phi}{\rho}dx-\frac{2\nu}{l}\int_0^1W\frac{\partial\phi}{\partial x}dx+\int_0^1\frac{\omega_1^2}{\rho}\phi dx\quad\text{on }]0,T[.\end{aligned} \tag{4.25}$$

Applying on the right-hand side the Young inequality with a parameter $\delta>0$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1\phi^2dx&+\int_0^1\rho\left(\frac{\partial W}{\partial x}\right)^2\phi dx+\int_0^1\frac{W^2}{\rho}\phi dx\\ &+(1-4\delta)\int_0^1\rho\left(\frac{\partial\phi}{\partial x}\right)^2dx+(D-1)\int_0^1\rho\frac{\partial\theta}{\partial x}\frac{\partial\phi}{\partial x}dx\\ \leqslant C_1&\int_0^1\rho\left[W^2\left(\frac{\partial W}{\partial x}\right)^2+\theta^2V^2\right]dx\\ &+C_2\left(\left|\int_0^1\frac{\partial v_1}{\partial t}V\phi dx\right|+\left|\int_0^1\frac{\partial\omega_1}{\partial t}W\phi dx\right|+\int_0^1\theta\phi dx\right.\\ &\left.+\int_0^1\frac{V^2}{\rho}dx+\int_0^1\frac{\phi}{\rho}dx+\int_0^1\frac{W^2}{\rho}dx+\int_0^1\frac{\omega_1^2}{\rho}\phi dx\right).\end{aligned} \tag{4.26}$$

One can easily see that the following inequality holds true

$$\begin{aligned}&(1-4\delta)\left(\frac{\partial\phi}{\partial x}\right)^2+(D-1)\frac{\partial\theta}{\partial x}\frac{\partial\phi}{\partial x}\\ &\geqslant(D-16\delta)\left(\frac{\partial\theta}{\partial x}\right)^2-V^2\left(\frac{\partial V}{\partial x}\right)^2\frac{8\delta+(1-4\delta)^2(\delta+\delta^{-1})+\delta(D-1)^2}{2}\\ &\quad-W^2\left(\frac{\partial W}{\partial x}\right)^2\frac{16(\delta^2A+1)+(1-4\delta)^2+(D-1)^2}{2\delta A^2}.\end{aligned} \tag{4.27}$$

Let $\delta=\frac{1}{32}\min\{1,D\}$. From (4.26) and (4.27) follows the inequality

$$\frac{d}{dt}\int_0^1\phi^2dx+D\int_0^1\rho\left(\frac{\partial\theta}{\partial x}\right)^2dx$$

$$\leqslant C_1 \int_0^1 \rho \left[ W^2 \left( \frac{\partial W}{\partial x} \right)^2 + V^2 \left( \frac{\partial V}{\partial x} \right)^2 \right] dx$$
$$+ C_2 \left( \int_0^1 \rho \theta^2 V^2 dx + \left| \int_0^1 \frac{\partial v_1}{\partial t} V \phi dx \right| + \left| \int_0^1 \frac{\partial \omega_1}{\partial t} W \phi dx \right| + \int_0^1 \theta \phi dx \right.$$
$$\left. + \int_0^1 \frac{V^2}{\rho} dx + \int_0^1 \frac{\phi}{\rho} dx + \int_0^1 \frac{W^2}{\rho} dx + \int_0^1 \frac{\omega_1^2}{\rho} \phi dx \right) \text{ on } ]0,T[, \tag{4.28}$$

Multiplying (3.8) and (3.9) respectively by $V^3$ and $\rho^{-1} W^3$, integrating over $]0,1[$ and making use of (3.11), (3.1) and (4.9), after applying the Young inequality we obtain the inequalities

$$\frac{d}{dt} \int_0^1 V^4 dx + \int_0^1 \rho \left( \frac{\partial V}{\partial x} \right)^2 V^2 dx \leqslant C \left( \int_0^1 \rho \theta^2 V^2 dx + \left| \int_0^1 \frac{\partial v_1}{\partial t} V^3 dx \right| \right) \text{ on } ]0,T[, \tag{4.29}$$
$$\frac{d}{dt} \int_0^1 W^4 dx + A \int_0^1 \rho \left( \frac{\partial W}{\partial x} \right)^2 W^2 dx \leqslant C \left( 1 + \left| \int_0^1 \frac{\partial \omega_1}{\partial t} W^3 dx \right| \right) \text{ on } ]0,T[. \tag{4.30}$$

Multiplying (4.29) by $C_1$ and (4.30) by $C_1 A^{-1}$, after addition of the obtained inequalities with (4.28), we find that

$$\frac{d}{dt} \int_0^1 (\phi^2 + C_1 V^4 + C_1 A^{-1} W^4) dx + D \int_0^1 \rho \left( \frac{\partial \theta}{\partial x} \right)^2 dx$$
$$\leqslant C \left( \int_0^1 \rho \theta^2 V^2 dx + \left| \int_0^1 \frac{\partial v_1}{\partial t} V^3 dx \right| + 1 + \left| \int_0^1 \frac{\partial \omega_1}{\partial t} W^3 dx \right| \right.$$
$$+ \left| \int_0^1 \frac{\partial v_1}{\partial t} V \phi dx \right| + \left| \int_0^1 \frac{\partial \omega_1}{\partial t} W \phi dx \right| + \int_0^1 \theta \phi dx + \int_0^1 \frac{V^2}{\rho} dx$$
$$\left. + \int_0^1 \frac{\phi}{\rho} dx + \int_0^1 \frac{W^2}{\rho} dx + \int_0^1 \frac{\omega_1^2}{\rho} \phi dx \right) \text{ on } ]0,T[. \tag{4.31}$$

With the help of (4.20), (4.10), (4.17), (4.18) and using the Young inequality for the terms on the right-hand side of (4.31) we find estimates on $]0,T[$ as follows:

$$\int_0^1 \rho \theta^2 V^2 dx \leqslant C M_\theta^2 \|V\|^2 \leqslant C M_\theta^2 \leqslant \varepsilon I_1 + C_\varepsilon (1 + I_2), \tag{4.32}$$

$$\left| \int_0^1 \frac{\partial v_1}{\partial t} V^3 dx \right| \leqslant C \int_0^1 (1 + |V|) |V|^3 \leqslant C \left( \int_0^1 V^4 dx + 1 \right), \tag{4.33}$$

$$\left| \int_0^1 \frac{\partial \omega_1}{\partial t} W^3 dx \right| \leqslant C \int_0^1 (1 + |V|) |W|^3 \leqslant C \left( 1 + \int_0^1 W^4 dx + \int_0^1 V^4 dx \right), \tag{4.34}$$

$$\left| \int_0^1 \frac{\partial v_1}{\partial t} V \phi dx \right| \leqslant C \left( \int_0^1 \phi^2 dx + \int_0^1 \left( \frac{\partial v_1}{\partial t} \right)^2 V^2 dx \right)$$

$$\leqslant C\left(\int_0^1 \phi^2 dx + \int_0^1 (1+|V|^2)V^2 dx\right)$$
$$\leqslant C\left(1 + \int_0^1 \phi^2 dx + \int_0^1 V^4 dx\right), \tag{4.35}$$

$$\begin{aligned}\left|\int_0^1 \frac{\partial \omega_1}{\partial t} W\phi dx\right| &\leqslant C\left(\int_0^1 \phi^2 dx + \int_0^1 \left(\frac{\partial \omega_1}{\partial t}\right)^2 W^2 dx\right)\\ &\leqslant C\left(\int_0^1 \phi^2 dx + \int_0^1 (1+|V|^2)W^2 dx\right)\\ &\leqslant C\left(1 + \int_0^1 \phi^2 dx + \int_0^1 V^4 dx + \int_0^1 W^4 dx\right),\end{aligned} \tag{4.36}$$

$$\int_0^1 \theta\phi dx \leqslant M_\theta \int_0^1 \phi dx \leqslant M_\theta \leqslant (1+M_\theta^2) \leqslant \varepsilon I_1 + C(1+I_2), \tag{4.37}$$

$$\int_0^1 \frac{V^2}{\rho} dx \leqslant \frac{C}{m_\rho}\|V\|^2 \leqslant C\left(1 + \int_0^t M_\theta^2(\tau)d\tau\right) \leqslant C(1+I_2), \tag{4.38}$$

$$\int_0^1 \frac{\phi}{\rho} dx \leqslant \frac{1}{m_\rho}\int_0^1 \phi dx \leqslant C\left(1 + \int_0^t M_\theta^2(\tau)d\tau\right) \leqslant C(1+I_2), \tag{4.39}$$

$$\int_0^1 \frac{W^2}{\rho} dx \leqslant \frac{C}{m_\rho}\|W\|^2 \leqslant C\left(1 + \int_0^t M_\theta^2(\tau)d\tau\right) \leqslant C(1+I_2), \tag{4.40}$$

$$\int_0^1 \frac{\omega_1^2}{\rho}\phi dx \leqslant C\int_0^1 \frac{\phi}{\rho} dx \leqslant C(1+I_2). \tag{4.41}$$

Inserting (4.32)–(4.41) in (4.31) we find that

$$\frac{d}{dt}\left[\int_0^1 (\phi^2 + C_1 V^4 + C_1 A^{-1} W^4)dx + DI_2\right]$$
$$\leqslant C\left[1 + \int_0^1 (\phi^2 + C_1 V^4 + C_1 A^{-1} W^4)dx + DI_2\right] \text{ on } ]0,T[. \tag{4.42}$$

From (4.42) we conclude that

$$\int_0^1 (\phi^2 + C_1 V^4 + C_1 A^{-1} W^4)dx + DI_2 \leqslant C \tag{4.43}$$

and therefore it holds

$$I_2 \in L^\infty(]0,T[), \tag{4.44}$$
$$\phi \in L^\infty(0,T;L^2(]0,1[)). \tag{4.45}$$

From (4.44) and (4.18) we conclude that (4.21) holds true. The inequality (4.22) now follows from (4.21) and (4.17). The inclusion (4.23) follows from (4.45), (4.22) and (4.44).

LEMMA 4.6. *It holds*

$$\rho \in L^\infty(0,T;H^1(]0,1[)). \tag{4.46}$$

*Proof.* Using (4.20), (4.23), (4.10) and (4.9), in the same way as in [1, p. 53], from (4.4) we get (4.46).

LEMMA 4.7. *The following inclusions hold true:*

$$V \in L^\infty(0,T;H^1(]0,1[)) \cap L^2(0,T;H^2(]0,1[)), \tag{4.47}$$

$$v_1 \in L^\infty(0,T;H^2(]0,1[)). \tag{4.48}$$

*Proof.* Taking into account (2.22) and (4.46) from the equalities

$$\frac{\partial v_1}{\partial x} = \frac{\mu}{l}\frac{1}{\rho}, \quad \frac{\partial^2 v_1}{\partial x^2} = -\frac{\mu}{l}\frac{1}{\rho^2}\frac{\partial \rho}{\partial x}$$

we easily get (4.48). The proof of (4.47) is identical to that of [1, pp. 52-54].

LEMMA 4.8. *It holds*

$$\omega_1 \in L^\infty(0,T;H^2(]0,1[)), \tag{4.49}$$

$$W \in L^\infty(0,T;H^1(]0,1[)) \cap L^2(0,T;H^2(]0,1[)). \tag{4.50}$$

*Proof.* The conclusion (4.49) follows directly from the equalities

$$\frac{\partial \omega_1}{\partial x} = \frac{\nu}{l}\frac{1}{\rho}, \quad \frac{\partial^2 \omega_1}{\partial x^2} = -\frac{\nu}{l}\frac{1}{\rho^2}\frac{\partial \rho}{\partial x}.$$

Multiplying equation (3.8) by $A^{-1}\rho^{-1}W$, integrating over $]0,1[$ and applying the Young inequality we obtain

$$\frac{1}{2A}\frac{d}{dt}\int_0^1 W^2 dx + \int_0^1 \left[\rho\left(\frac{\partial W}{\partial x}\right)^2 + \frac{W^2}{\rho}\right] dx$$

$$\leqslant C\left(\left\|\frac{\partial \omega_1}{\partial t}\right\|^2 + \|W\|^2 + \|\omega_1\|^2 + \left\|\frac{W}{\rho}\right\|^2\right). \tag{4.51}$$

Taking into account (4.12), (4.10), (4.9) and (4.22) from (4.51) we conclude that

$$W \in L^2(0,T;H^1(]0,1[)).$$

Also, multiplying (3.8) by $A^{-1}\rho^{-1}\frac{\partial^2 W}{\partial x^2}$ and integrating over $]0,1[$, after integration by parts on the left-hand side and making use of (3.11), we find that

$$\frac{1}{2A}\frac{d}{dt}\left\|\frac{\partial W}{\partial x}\right\|^2 + \int_0^1 \rho\left(\frac{\partial W}{\partial x}\right)^2 dx = -\int_0^1 \frac{\partial \rho}{\partial x}\frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial x^2} dx + \int_0^1 \frac{\omega_1}{\rho}\frac{\partial^2 W}{\partial x^2} dx$$

$$+\int_0^1 \frac{W^2}{\rho}\frac{\partial^2 W}{\partial x^2}dx+\frac{1}{A}\int_0^1 \frac{\partial \omega_1}{\partial t}\frac{\partial^2 W}{\partial x^2}dx. \tag{4.52}$$

Applying the Young inequality for the terms on the right-hand side of (4.52), in the similar way as in [4, Lemma 2.7] we get the following estimate

$$\left\|\frac{\partial W}{\partial x}(t)\right\|^2+\int_0^t \left\|\frac{\partial^2 W}{\partial x^2}(\tau)\right\|^2 d\tau \leqslant C,\ \ t\in ]0,T[. \tag{4.53}$$

LEMMA 4.9.

$$\rho, V, W \in H^1(Q_T). \tag{4.54}$$

*Proof.* Squaring equations (3.7), (3.8) and (3.9), integrating over $]0,1[$ and $]0,t[$ and using the Young inequality and the results of the above lemmas, in the same way as in [1, pp. 53-54] and in [4, Lemma 2.7] we get (4.54).

LEMMA 4.10. *It holds*

$$\theta \in L^\infty(0,T;H^1(]0,1[))\cap L^2(0,T;H^2(]0,1[))\cap H^1(Q_T). \tag{4.55}$$

*Proof.* Using the obtained estimates for the functions $v_1$ and $\omega_1$ in a similar way as in [4, Lemma 2.8] we get (4.55).

Proposition 4.1 follows immediately from (4.19), (4.22), (4.46), (4.47), (4.50), (4.54) and (4.55). Also, Theorem 2.1 is proved.

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(Received September 15, 2007)

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