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A LOOK AT THE WORK OF JOSIP PEČARIĆ

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Abstract. This paper is a version of an address to the conference Mathematical Inequalities and Applications 2008 held in Trogir, Croatia June 8–14 to honour Professor J. Pečarić's 60th birthday.

1. Introduction

In 1967 I began a long association with Professor Mitrinović and his school of inequalities. Some time during these years I began to receive preprints and copies of papers from a Josip Pečarić. As you know Professor Pečarić gives a topic he takes up a very thorough treatment and papers arrived with titles like: *The Jensen Inequality, An Inequality of Jensen, A Reverse Jensen Inequality, A Reverse Jensen Inequality II* and so on. I was impressed by these papers and wrote to Professor Mitrinović for information on the author. Professor Mitrinović replied that Josip Pečarić was a young, ambitious and very talented mathematician. I followed the work of this young correspondent over the years and finally in 1997 wrote a joint paper with him. As almost everyone has written a joint paper with Professor Pečarić both he and I were surprised that it had taken so long.

1.1. References. [8].

2. Administrative Accomplishments

I now mention the first of Professor Pečarić's many accomplishments that I call, for want of a better word, administrative. The turmoil of the war years sent him travelling and for some time I lost track and at one time even thought he had settled permanently in Australia. I was sending e-mails to Adelaide when he was long back in Croatia. In Croatia he has been instrumental in construction of a school of inequalities that rivals and even surpasses the one that had existed in Belgrade. The deaths in the late nineties of both Mitrinović and Vasić dealt a blow to that school. There were plenty of possible successors but the possibility of continuing was completely destroyed by the war. In Zagreb the school was reborn. In addition Professor Pečarić has created a very strong graduate school as can be seen by the many ex-students that are now populating the departments of mathematics throughout Croatia and who are co-authors of many of his papers. Further this school publishes an excellent journal, *Mathematical*

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Inequalities and Applications, now in its eleventh year. To have obtained the publication of an internationally recognised, reviewed journal as well as a school of inequalities is a very significant achievement at any time and given the circumstances remarkable. By itself this achievement would justify a conference in his honour. Furthermore two more journals also of excellent quality have recenyly begun publication; *Journal of Mathematical Inequalities* and *Operators and Matrices* in their first and second years respectively.

At the same time as setting up and encouraging the group of mathematicians in Zagreb and arranging for the publication of the journal mentioned above Professor Pečarić has produced a steady stream of research papers both on his own and in collaboration with mathematicians from all over the world. In fact stream is too placid a word to describe the torrent of research that comes out every year from this mathematician. It is to this accomplishment that I now turn.

3. Mathematical Work

A cursory glance at the 1148 publications with Professor Pečarić as an author listed in the Mathematical Reviews, covering the years 1979–2007, suggests that it is foolhardy to categorise his work. I will try to concentrate on what I see as three main areas.

Incidentally 1148 is a lower bound as not all journals are reviewed by the Mathematical Reviews. I am unable to add the ones in the Zentralblatt so 1148 will have to be my estimate.

3.2. Collegiality. One of the most remarkable facets of Professor Pečarić's work is his collaboration with other mathematicians. These come from all over the world, Australia, Canada, England, Europe, Iran, and of course from Zagreb. Coupled with this is the extraordinary range of journals in which his papers have appeared. Of course if you are publishing about 35 papers a year over 30 years there is a need to find all the journals possible.

While many of the choices of collaborators are obvious — with Mitrinović and his colleagues while still a student, with his students and colleagues in Zagreb today, and with colleagues wherever he worked during the years of travel, many are less obvious. Their variety show on Pečarić's part a deep interest in all that is unequal and a mind full of ideas that is stimulated by everything written having a connection however tenuous with inequalities.

As an example of one of the less obvious choices for a collaborator I will use his work with the mathematical physicist Peter Landsberg.

Here as in later examples I will only cover the simplest case as this is not the place to try and cover the most general results and in any case full details can be found in the references.

In 1978 Landsberg gave a proof of the *Geometric-Arithmetic Mean Inequality*. This proof caused some discussion at the time and goes as follows.

Consider a physical system that consists of two heat reservoirs with temperatures T_1, T_2 and heat capacities C_1, C_2 respectively. Let the system come to a state of equilibrium at a temperature T. This process is covered by two basic laws of physics.

(i) *The first law of thermodynamics; the law of conservation of energy*. This law implies that:

$$C_1(T-T_1)+C_2(T-T_2)=0. \tag{1}$$

(ii) *The second law of thermodynamics*. This law says that there is a gain in entropy, that is:

$$C_1(\log T-\log T_1)+C_2(\log T-\log T_2)\geqslant 0. \tag{2}$$

From (1) and (2) it is easy to deduce that if $C=C_1+C_2$ denotes the total heat capacity of the system:

(a) $T=\big(C_1T_1+C_2T_2\big)/C$, the arithmetic mean of the original temperatures with weights the heat capacities;

(b) $\log\big(T/(T_1^{C_1}T_2^{C_2})^{1/C}\big)\geqslant 0$.

From (a) and (b) we get the inequality

$$\frac{C_1T_1+C_2T_2}{C}\geqslant (T_1^{C_1}T_2^{C_2})^{1/C}, \tag{GA}$$

together with the case of equality.

The extension to the case of n heat reservoirs, $n>2$, is easily made, giving a quick proof of the Geometric-Arithmetic mean inequality,

Landsberg pointed out that one could assume the heat capacities were temperature dependent when the equilibrium temperature is given by the following generalizations of (1) and (2):

$$\int_{T_1}^{T} C_1(t)\,\mathrm{d}t+\int_{T_2}^{T} C_2(t)\,\mathrm{d}t=0,$$
$$\int_{T_1}^{T} C_1(t)/t\,\mathrm{d}t+\int_{T_2}^{T} C_2(t)/t\,\mathrm{d}t\geqslant 0.$$

This led Landsberg to introduce a new general mean in what he considered as either energy or entropy conserving systems.

Let C_1,C_2,ϕ be positive functions, with ϕ increasing; then the *ϕ-Landsberg mean with weight functions C_1,C_2 of the two positive numbers T_1 and T_2*, denoted by $L_\phi(T_1,T_2;C_1,C_2)$ is T defined by:

$$\int_{T_1}^{T} C_1(t)/\phi(t)\,\mathrm{d}t+\int_{T_2}^{T} C_2(t)/\phi(t)\,\mathrm{d}t=0.$$

Different choices of the functions C_1,C_2,ϕ give the various power means; and the fairly simply proved inequality

$$L_1(T_1,T_2;C_1,C_2)\geqslant L_\phi(T_1,T_2;C_1,C_2), \tag{3}$$

has as a particular case the inequality between the power means.

One would have thought the matter was now closed. However a question can be asked and was asked by Pečarić: what happens if negative weights are allowed?

It is known, and is easy to prove that one of the weights is negative, with $C \neq 0$, then Geometric-Arithmetic Mean Inequality is reversed:

$$\frac{C_1T_1+C_2T_2}{C} \leqslant (T_1^{C_1}T_2^{C_2})^{1/C}, \tag{$\sim$ GA}$$

and Professor Pečarić asked Landsberg if there was a physical proof of this case as well?

It turns out that there this is such a proof. However the physical situation is much more complicated than the mathematical one. There are 8 cases to be considered. If for instance $C_1 < 0 < C_2$ then: either the larger of the two initial temperatures can be associated with the positive capacity or the other way around; also we can either have $C > 0$ or $C < 0$. However Landsberg found physical interpretations for all eight cases and his full result gives a physical proof of the complete inequality that agrees with the two mathematical cases.

The possibility of negative weights was considered in the Landsberg mean and it was shown that (3) holds if the weight functions are of the same sign but the reverse holds if the weight functions have opposite signs.

Clearly the Landsberg mean can be defined for the case of n temperatures and weights, $n > 2$, and this is pursued in the joint paper where the possibility of negative weights is discussed. This general case is more complicated than the $n = 2$ case as we will see in the next section.

In all this was an excellent paper that clearly arose from the joint input of two talented mathematicians and illustrates Professor Pečarić at his best as a collaborator.

3.2.1. References [4], [Bul pp.420–422].

3.3. Convexity. One of the constant themes of Professor Pečarić's work is the concept of convex functions. From his earliest work up to the present day he has added to our knowledge of convexity. Many other authors have also contributed to this area but Professor Pečaric is distinguished by the usefulness of the results that he has obtained.

The number of papers by Professor Pečarić dealing with convexity is too large to consider as a unit but one interest throughout his mathematical career has been what happens when negative weights occur in the various convexity inequalities. I will illustrate this by two results, one from his early work and one so recent it is not in the paper count given above.

A function f being convex on the interval I means that if $x, y \in I$ then if $0 \leqslant t \leqslant 1$,

$$D(t) = f\big((1-t)x+ty\big) - \big((1-t)f(x)+tf(y)\big) \leqslant 0. \tag{4}$$

If we assume, without loss in generality, that f is twice differentiable then it is easy to check that if either $t < 0$ or $1-t < 0$, equivalently $t > 1$, then D is non-negative[1]. That is if one of the weights is negative the reverse inequality to (4) holds:

$$D(t) = f\big((1-t)x+ty\big) - \big((1-t)f(x)+tf(y)\big) \geqslant 0. \tag{$\sim$ 4}$$

[1]There are two simple graphical ways of seeing this. (a) Look at the graph of f; (ii) look at the graph of D, a function that is defined on $\mathbb{R}$ and is easily seen to be convex on that domain.

The first inequality (4) extends, by induction, to *Jensen's inequality*, (J) below; however if $n \geqslant 3$ continuity arguments assure us that the inequality (J) holds not only for positive weights but for a larger set of weights.

Consider the case of $n = 3$ when the function D of (4) becomes

$$D(s,t) = f\big((1-s-t)x + sy + tz\big) - \big((1-s-t)f(x) + sf(y) + tf(z)\big), \quad (s,t) \in \mathbb{R}^2. \quad (5)$$

Under very light differentiability conditions the function D can be shown to have no stationary points in $\mathbb{R}$. So if the weights are non-negative, that is if (s,t) lies in the closed triangle

$$T = \{(s,t); 0 \leqslant s \leqslant 1, 0 \leqslant t \leqslant 1, 0 \leqslant s+t \leqslant 1\},$$

the maximum of D on T occurs on the boundary where by (4) this function is non-positive. Hence $D \leqslant 0$ on T, which is just (J) in the case $n = 3$

Now by continuity D must be negative on a larger set than T, in contrast to the $n = 2$ case. Simple arguments show that the range of validity of (J) can be extended to parts of the larger triangle

$$T^\sharp = \{(s,t); -1 \leqslant s \leqslant 1, -1 \leqslant t \leqslant 1, 0 \leqslant s+t \leqslant 2\},$$

depending on the order of x, y, z.

Generally we have, with a slight change of notation, the following result due to Steffensen.

If f, I are as above and if $\mathbf{a} = (a_1, \ldots, a_n)$ is a monotonic n-tuple, $n > 2$, with elements in I and if $\mathbf{w} = (w_1, \ldots, w_n)$ is an n-tuple of non-zero real numbers satisfying

$$W_n = 1, \text{ and } 0 \leqslant W_i \leqslant 1,\ 1 \leqslant i \leqslant n, \quad (6)$$

where $W_k = \sum_{i=1}^k w_i$, $1 \leqslant k \leqslant n$, then Jensen's inequality holds. That is in the present notation:

$$f\Big(\sum_{i=1}^n w_i a_i\Big) \leqslant \sum_{i=1}^n w_i f(a_i). \quad \text{(J)}$$

Further if f is strictly convex the inequality holds strictly unless $\mathbf{a}$ is essentially constant.

The question of extending the reverse inequality, (~ 4) above, to the case of $n > 2$ was not studied until the work of Pečarić. This important work of Pečarić seems to have been missed in the various books on inequalities.

For the case $n = 3$ the above function (5) can suggest what might be true since on certain regions the boundary values are non-negative and so as there are no stationary points the function should be non-negative on that region. Unfortunately the regions are unbounded and little more work is needed but the suggestions are in fact correct under some conditions. The regions are the unbounded regions bounded by the extended sides of the triangle T:

(i) $s \leqslant 0, t \leqslant 0$,

(ii) $s \geqslant 1, t \leqslant 0$,

(iii) $s+t \geqslant 1, t \geqslant 1, s \leqslant 0$.

In general Pečarić obtained the following result.

Let $\mathbf{a}, I, n$, *be as above with* $\mathbf{w}$ *an* n*-tuple with positive or negative elements and with* $W_n = 1$, *then the reverse Jensen inequality holds for all functions* f *convex on* I *and for every monotonic* $\mathbf{a}$ *if and only if for some* $m, 1 \leqslant m \leqslant n, W_k \leqslant 0, 1 \leqslant k < m$, *and* $W_k \geqslant 1, m \leqslant k \leqslant n$.

The proof of this important result appears only in a journal that is not readily available and is not given in either of Pečarić's books or my book. This is certainly an oversight on my part and while this is not the place to give the proof I will show the method by using Pečarić's idea to prove Jensen's inequality, (J), in the case of positive weights.

□ There is no loss in generality in assuming that the n-tuple $\mathbf{a}$ is decreasing and define r, $1 \leqslant r \leqslant n$ by

$$a_1 \geqslant \cdots \geqslant a_r \geqslant \overline{a} \geqslant a_{r+1} \geqslant \cdots \geqslant a_n,$$

where $\overline{a}$ is the weighted arithmetic mean of the given n-tuple.

Consider then the three $n+1$-tuples $\mathbf{x}, \mathbf{y}, \mathbf{q}$:

$$\begin{aligned} x_i &= a_i, \quad & q_i &= w_i, \quad & 1 &\leqslant i \leqslant r; \\ x_{r+1} &= \overline{a}, \quad & q_{r+1} &= -1; \\ x_i &= a_{i-1}, \quad & q_i &= w_{i-1}, \quad & r+2 &\leqslant i \leqslant n+1; \\ y_i &= \overline{a}, \quad & 1 &\leqslant i \leqslant n+1. \end{aligned}$$

Simple calculations show that:

$$\sum_{i=1}^{k} q_i x_i \geqslant \sum_{i=1}^{k} q_i y_i, \quad 1 \leqslant k \leqslant n; \qquad \sum_{i=1}^{n+1} q_i x_i = \sum_{i=1}^{n+1} q_i y_i.$$

Hence by Karamata's extension of the Hardy, Littlewood & Pólya majorization theorem if f is convex then

$$\sum_{i=1}^{n+1} q_i f(x_i) \geqslant \sum_{i=1}^{n+1} q_i f(y_i) = 0,$$

which is just (J). □

Inspection of this proof shows that no use is made of the positive nature of the weights, only of the positivity of their partial sums, and so this argument will also prove the Jensen-Steffensen inequality. A further close inspection of the proof gives a proof of the Pečarić result.

The importance of the Pečarić result, other than being a property of convex functions, is that a similar result is implied for many mean inequalities that are particular cases of Jensen's inequality.

One of the applications of convexity is to the comparison of means. If ϕ is a strictly increasing function then we can define a quasi-arithmetic mean as follows

$$M_\phi(\mathbf{a}; \mathbf{w}) = \phi^{-1}\Big(\frac{1}{W_n} \sum_{i=1}^{n} w_i \phi(a_i)\Big)$$

An important question is when two such means are comparable that is when is it always true that:

$$M_\phi(\mathbf{a};\mathbf{w}) \leqslant M_\psi(\mathbf{a};\mathbf{w})$$

Writing $\phi(a_i) = b_i$ this last inequality :

$$\psi\circ\phi^{-1}\Big(\frac{1}{W_n}\sum_{i=1}^{n} w_i b_i\Big) \leqslant \frac{1}{W_n}\sum_{i=1}^{n} w_i \psi\circ\phi^{-1}(b_i),$$

showing, from Jensen's inequality (J), that the means are comparable exactly when $\psi\circ\phi^{-1}$ is convex.

Daróczy & Páles have defined a class of general means that they called *L-conjugate means* and proved a comparison theorem for such means. A recent result of Pečarić has given a property of convex functions that plays the same role for these means as as Jensen inequality plays for quasi-arithmetic means.

Consider the case of two variables x,y, a convex function ϕ and a mean M then the L-conjugate mean is:

$$L_\phi^M(x,y) = \phi^{-1}\Bigg(\phi(x)+\phi(y)-\phi\circ M(x,y)\Bigg). \tag{7}$$

Now suppose we wish to compare two L-conjugate means:

$$L_\phi^M(x,y) \leqslant L_\psi^M(x,y), \tag{8}$$

Then putting $u = \phi(x), V = \phi(y)$ and $N = \phi\circ M$ (8) becomes

$$\psi\circ\phi^{-1}\Big(u+v-N(u,v)\Big) \leqslant \psi\circ\phi^{-1}(u)+\psi\circ\phi^{-1}(v)-\psi\circ\phi^{-1}\circ N(u,v)$$

which from the Steffensen result holds if $\psi\circ\phi^{-1}$ is convex, as for the quasi- arithmetic means.

However it can be noticed, as it was by Professor Pečarić, that this result uses a much weaker form of the Steffensen result. In this situation we have a condition that enables us to ignore the order requirements of that theorem: this happens if

(i) the weights of the biggest and smallest terms are positive;

(ii) each positive weight at least balances the total of all the negative weights.

More importantly this weaker form of the Steffensen theorem can be given an independent proof that generalizes to convex functions of several variables where the order condition cannot be used.

Consider the numbers x,y,z with z between x and y together with three positive number p,q,r with $p \geqslant r, q \geqslant r$. Then for some $t, 0 \leqslant t \leqslant 1$,

$$\frac{px+qy-rz}{p+q-r} = \frac{px+qy-\big((1-t)x+ty\big)}{p+q-r} = \frac{\big(p-r(1-t)\big)x+(q-rt)y}{p+q-r}.$$

The coefficients of x and y in the last expression are positive and have sum 1 so since f is convex we have:

$$f\left(\frac{px+qy-rz}{p+q-r}\right) \leqslant \frac{\big(p-r(1-t)\big)f(x)+(q-rt)f(y)}{p+q-r} = \frac{pf(x)+qf(y)-r\big((1-t)f(x)+tf(y)\big)}{p+q-r} \leqslant \frac{pf(x)+qf(y)-rf(z)}{p+q-r}. \tag{9}$$

Inequality (9) is the simplest case of Pečarić's extension of Jensen's inequality which in general is as follows.

Let U be a convex set in $\mathbb{R}^k$, $\mathbf{a}_i \in U, 1 \leqslant i \leqslant n$, and let $w_i, 1 \leqslant i \leqslant n$, be non-zero real numbers with $W_n = 1$ and $I_- = \{i; 1 \leqslant i \leqslant n \wedge w_i < 0\}, I_+ = \{i; 1 \leqslant i \leqslant n \wedge w_i > 0\}$. Further assume that $\forall i,\ i \in I_-$, $\mathbf{a_i}$ lies in the convex hull of the set $\{\mathbf{a}_i; i \in I_+\}$ and that $\forall j,\ j \in I_+,\ w_j + \sum_{i \in I_-} w_i \geqslant 0$. If $f\colon U \mapsto \mathbb{R}$ is convex then (J) holds.

In this more general form the result can be used to prove the comparison theorem for the general L-conjugate means and has many other uses.

3.3.1. References [1], [5 pp. 83–84], [7 p. 6], [12], [13], [Bul pp. 30, 32–33, 39–44, 273–278], [D & P].

3.4. Positive Linear Functionals. Finally I want to mention the topic of isotonic functionals that again has been a constant theme throughout Pečarić's work, being even the main concept in his book, in English, on convex functions. The idea goes back to the work of Jessen and McShane in the thirties of the last century but for Pečarić it has been a constant theme and extensive generalizations of many inequalities have been given using this method.

$\mathscr{L} = \{f;\ f\colon X \mapsto \mathbb{R}\}$ satisfying:

L1: if $\alpha, \beta \in \mathbb{R} \wedge f, g \in \mathscr{L}$ then $\alpha f + \beta g \in \mathscr{L}$;

L2: $1 \in \mathscr{L}$.

$A\colon \mathscr{L} \mapsto \mathbb{R}$ satisfies:

A1: if $\alpha, \beta \in \mathbb{R} \wedge f, g \in \mathscr{L}$ then $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$;

A2: if $f \in \mathscr{L}$ and $f \geqslant 0$ then $Af \geqslant 0$.

That is to say A is an isotonic, or a positive linear, functional on the linear space $\mathscr{L}$. There are many examples of such and the following is a version of Jensen's inequality due to Pečarić.

If $f \in \mathscr{L}, f \geqslant 0$ with $Af > 0$, and if $I \subseteq \mathbb{R}$ with $\phi\colon I \mapsto \mathbb{R}$ continuous and convex then: if $g\colon X \mapsto I$ is such that $fg, f(\phi \circ g) \in \mathscr{L}$ and $A(fg)/A(f) \in I$ it follows that

$$\phi\left(\frac{A(fg)}{A(f)}\right) \leqslant \frac{A\big(f(\phi \circ g)\big)}{A(f)}. \tag{11}$$

In particular if A satisfies

A3: $A(1) = 1$

we get:

$$\phi\big(A(g)\big) \leqslant A\big(\phi\circ g)\ \big). \tag{12}$$

The impressive nature of Pečarić's work is the ability to derive in this general setting analogues of most of the classical inequalities. Here we show how an analogue of Hölder's inequality is obtained using (11).

$$\begin{aligned} A\big(f^{1/p}g^{1/q}\big) &= A\big(f(g/f)^{1/q}\big) \\ &\leqslant A(f)\left(\frac{A(g)}{A(f)}\right)^{1/q} = A(f)^{1/p}A(g)^{1/q}, \end{aligned}$$

provided $f,g\in\mathscr{L}$ and are positive, $p,q>0, 1/p+1/q=1$ and $A(f),A(g)>0$.

This method can be extended to give

$$A\Big(\prod_{i=1}^{n} f_i^{1/p_i}\Big) \leqslant \prod_{i=1}^{n} A(f_i)^{1/p_i}$$

under very easily stated conditions.

While I am basically uncomfortable with abstract situations I cannot but think that this last result is elegant when compared with the more concrete result:

$$\left(\sum_{j=1}^{n}\Big(\prod_{i=1}^{m} a_{ij}\Big)^{\rho_m}\right)^{1/\rho_m} \leqslant \prod_{i=1}^{m}\Big(\sum_{j=1}^{n} a_{ij}^{r_i}\Big)^{1/r_i},$$

the notations being almost self explanatory.

A simple deduction from the properties of A is: assume $I=[m,M]\subset\mathbb{R}$ then,

$$A\big(f\phi\circ g\big) \leqslant \frac{\big(MA(f)-A(fg)\big)\phi(m)+\big(A(fg)-mA(f)\big)\phi(M)}{M-m}. \tag{13}$$

A further very simple deduction from (12) and (13) is:

$$\phi\Big(\frac{pm+qM}{p+q}\Big) \leqslant A\big(\phi\circ g\big) \leqslant \frac{p\phi(m)+q\phi(M)}{p+q}$$

where $p,q\geqslant 0, p+q>0$ are defined by $A(g)=(pm+qM)\big/(p+q)$.

This result includes as a special case the well known Hadamard-Hermite inequality:

$$\phi(\gamma) \leqslant \frac{1}{2y}\int_{\gamma-y}^{\gamma+y}\phi \leqslant \frac{p\phi(a)+q\phi(b)}{p+q}$$

where $p>0, q>0$, $\gamma=(pa+qb)\big/(p+q)$ and $0\leqslant y\leqslant \big((b-a)/(p+q)\big)\min\{p,q\}$

All this illustrates the power and elegance of this approach to inequalities and this is merely scratching the surface of a large volume of results produced by Professor Pečarić using this method.

3.4.1. References [2], [3], [5], [9], [11], [Je], [Mc].

4. Conclusion

In this survey I have of course missed many topics. There are the insightful histories of various inequalities, the inequalities of Bernoulli and Carleman in particular. Then there is the more recent extensive study of quadrature rules and of course the many articles on particular inequalities.

All in all a body of work to be proud of and I for one am happy to have been associated with Professor Pečarić over these many years.

A SHORT LIST OF SOURCES

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